Factorizations of Algebraic Integers, Block Monoids, and Additive Number Theory

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Prologue

This talk is based the paper:


More information and background on this area can be found in:


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Let $\mathcal{O}_K = \{ \alpha \in K \mid f(\alpha) = 0 \text{ for some monic } f(X) \in \mathbb{Z}[X] \}$ be the ring of integers of $K$.

Let $\mathcal{I}(\mathcal{O}_K)$ represent the set of nonzero ideals of $\mathcal{O}_K$ and $\mathcal{P}(\mathcal{O}_K)$ its associated subset of nonzero principal ideals.

**Fundamental Question**

If $\alpha \in \mathcal{O}_K$, then how does $\alpha$ factor into irreducible elements of $\mathcal{O}_K$? When do the elements of $\mathcal{O}_K$ have unique factorization like in $\mathbb{Z}$?

**Answer:** The factorizations of $\alpha$ depend on the factorization of the ideal $(\alpha)$ into the prime ideals of $\mathcal{I}(\mathcal{O}_K)$. $\mathcal{O}_K$ is a unique factorization domain exactly when $\mathcal{I}(\mathcal{O}_K) = \mathcal{P}(\mathcal{O}_K)$. 
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Chapman (Sam Houston State University)
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\[ 6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}) \]  

in the algebraic number ring \( \mathbb{Z}[\sqrt{-5}] \).

The actual argument to complete this observation involves showing two things:

(i) 2, 3, \(1 + \sqrt{-5}\) and \(1 - \sqrt{-5}\) are all irreducible, and

(ii) 2 (resp. 3) is neither an associate of \((1 + \sqrt{-5})\) nor of \((1 - \sqrt{-5})\) (this is clear once \(\pm 1\) are established as the only units of \(\mathbb{Z}[\sqrt{-5}]\)).
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Motivation

Most books fail to point out to the readers that while $\mathbb{Z}[\sqrt{-5}]$ is not a UFD, it does have a rather nice factorization property.

Specifically, if $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ are irreducible elements of $\mathbb{Z}[\sqrt{-5}]$ with

$$\alpha_1 \cdots \alpha_n = \beta_1 \cdots \beta_m,$$

then $n = m$.

In general, an integral domain with this property is known as a \textit{half-factorial domain} (HFD).
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Using the ideal class group (and, more generally, the class number), one can construct a very simple proof of this fact for $\mathbb{Z}[\sqrt{-5}]$.

Carlitz first illustrated this argument in *PAMS* 11(1960), 391-392. His proof (while short) leads to a deeper understanding of how elements factor in an algebraic ring of integers.
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The purpose of this talk is to develop this understanding by using a structure, known as a block monoid, that is associated to the class group. In fact, block monoids have greater utility and we shall show that they can be used in a similar line of analysis in more general classes of integral domains, such as Dedekind domains and Krull domains. Our work will involve a close study of the combinatorial properties of block monoids and lead to an examination of an actively researched concept from Additive Number Theory known as Davenport’s constant.
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**Proposition**

Let $I$ be an ideal of $\mathcal{O}_K$ and $\mathcal{I}(\mathcal{O}_K)$ and $\mathcal{P}(\mathcal{O}_K)$ be as above.

1. $\mathcal{O}_K$ is a Dedekind domain. Moreover, there exists elements $\alpha$ and $\beta$ in $\mathcal{O}_K$ such that $I = (\alpha, \beta)$.

2. The factor monoid $\mathcal{C}(\mathcal{O}_K) = \mathcal{I}(\mathcal{O}_K) / \mathcal{P}(\mathcal{O}_K)$ forms a finite abelian group.

3. Let $[I]$ represent the image of the ideal $I$ in $\mathcal{C}(\mathcal{O}_K)$. Then, for each $g \in \mathcal{C}(\mathcal{O}_K)$ there exists a prime ideal $P$ of $\mathcal{O}_K$ such that $[P] = g$. 

**Definitions**
A Classic Theorem

The group $\mathcal{C}(\mathcal{O}_K)$ is known as the class group of $\mathcal{O}_K$ and its order $|\mathcal{C}(\mathcal{O}_K)|$ is the class number of $\mathcal{O}_K$.

The class number gives a classic answer to the question of when a ring of algebraic integers admits unique factorization.

**Theorem**

The ring of integers $\mathcal{O}_K$ in an algebraic number field $K$ is a unique factorization domain if and only if the class number of $\mathcal{O}_K$ is 1.

In fact, the size of the class group of $\mathcal{O}_K$ was generally assumed to be a measure of how far a ring of integers was from being a UFD.
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The Connection Between Ideals and Factorizations

Proposition

Let $D$ be a Dedekind domain and $x \in D$ a nonzero nonunit. Suppose in $D$ that

$$(x) = P_1 \cdots P_k$$

where $k \geq 1$ and the $P_1, \cdots P_k$ are not necessarily distinct prime ideals of $D$. Then

1. In $\mathcal{C}(D)$, $[P_1] + \cdots + [P_k] = 0$.
2. The element $x$ is prime in $D$ if and only if $k = 1$.
3. The element $x$ is irreducible in $D$ if and only if for every nonempty proper subset $T \subset \{1, \ldots, k\}$, $\sum_{i \in T} [P_i] \neq 0$. 
Proof of (3)

We prove (3) by contrapositive. \((\Rightarrow)\) Suppose for some proper subset \(T\) that \(\sum_{i \in T} \mathbb{P}_i = 0\). Then \(\prod_{i \in T} P_i = (y)\) for some nonzero nonunit \(y \in D\). By (1) we have \([P_1] + \cdots + [P_k] = 0\), so \(\sum_{i \in T} [P_i] = 0\) also. Thus, \(\prod_{i \in T} P_i = (z)\) for some nonzero nonunit \(z \in D\). Hence \((x) = (y)(z)\) implies that \(x = uyz\) where \(u\) is a unit of \(D\) and so \(x\) is reducible. \((\Leftarrow)\) Suppose that \(x\) is reducible in \(D\), i.e. \(x = yz\) for nonunits \(y\) and \(z\) in \(D\). By the Fundamental Theorem, there is a proper nonempty subset \(T \subset \{1, \ldots, k\}\) such that \((y) = \prod_{i \in T} P_i\). By (1), in \(C(D)\), \(\sum_{i \in T} [P_i] = 0\).
What happened in $\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$?

The only units of $\mathcal{O}_K$ are $\pm 1$ and it is well known that the class number of $\mathcal{O}_K$ is 2 (hence $\mathcal{C}(\mathcal{O}_K) \cong \mathbb{Z}_2$).

Let’s reconsider

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}) \quad (3)$$

in $\mathbb{Z}[\sqrt{-5}]$.

The prime ideal decompositions of (2) and (3) in $\mathbb{Z}[\sqrt{-5}]$ are

$$(2) = (2, 1 + \sqrt{-5})^2 \quad \text{and} \quad (3) = (3, 1 + \sqrt{-5})(3, 1 - \sqrt{-5}).$$
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\[(6) = (2)(3) = (2, 1 + \sqrt{-5})^2 (3, 1 + \sqrt{-5})(3, 1 - \sqrt{-5}). \quad (4)\]

The second factorization in Eq. 3 is obtained by rearranging the product in Eq. 4,

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Moreover, since the class group of \( \mathbb{Z}[^5] \) requires a product of two nonprincipal prime ideals to obtain a principal ideal, these are the only two factorizations of 6 in \( \mathbb{Z}[^5] \) up to associates.
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Let $G$ be an abelian group. If $A \subseteq G$, then let $\langle A \rangle$ represent the subgroup generated by $A$.

Further, let $\mathcal{F}(G)$ represent the free abelian monoid on $G$. We write the elements of $\mathcal{F}(G)$ as $C = \prod_{g \in G} g^{v_g(C)}$ where $v_g(C)$ is a nonnegative integer.

**Definition**

Let $G$ be an abelian group. The set

$$\mathcal{B}(G) = \left\{ C \mid C = \prod_{g \in G} g^{v_g(C)} \text{ with } \sum_{g \in G} v_g(C)g = 0 \right\}$$

forms a submonoid of $\mathcal{F}(G)$ known as the *block monoid of $G$*. 
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Block Monoids

Definition

If $S$ is a nonempty subset of $G$, then the set

$$\mathcal{B}(G, S) = \left\{ C \mid C = \prod_{g \in G} g^{v_g(C)} \text{ with } \sum_{g \in G} v_g(C)g = 0 \text{ and } v_g(C) = 0 \text{ if } g \notin S \right\}$$

is a submonoid of $\mathcal{B}(G)$ known as the block monoid of $G$ restricted to $S$.

We call the identity of $\mathcal{B}(G, S)$, $E = \prod_{g \in G} g^0$, the empty block.

A block $B$ divides a block $C$, denoted $B \mid C$ if there is a block $T$ such that $C = BT$. 
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A block $B$ divides a block $C$, denoted $B \mid C$ if there is a block $T$ such that $C = BT$. 
A block $B \neq E$ is **irreducible** if $B = CT$ for $C$, $T$ in $\mathcal{B}(G, S)$ implies that either $C = E$ or $T = E$.

A block $B \neq E$ is **prime** if whenever $B \mid CT$ then either $B \mid C$ or $B \mid T$.

As with the usual theory of factorization in an integral domain, a prime block $B$ is irreducible, but not conversely.

For the block $C = \prod_{g \in G} g^{v_g(C)}$, we set $|C| = \sum_{g \in G} v_g(C)$ to be the **size** of $C$. 
A block $B \neq E$ is \textit{irreducible} if $B = CT$ for $C, T$ in $B(G, S)$ implies that either $C = E$ or $T = E$.

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For the block $C = \prod_{g \in G} g^{\nu_g(C)}$, we set $|C| = \sum_{g \in G} \nu_g(C)$ to be the *size* of $C$. 
We compile a few facts about block monoids.

**Proposition**

Let $G$ be an abelian group and $S$ a nonempty subset of $G$.

1. The block $B = \prod_{g \in S} g^{v_g(B)} \neq E$ is irreducible in $\mathcal{B}(G, S)$ if and only if for each nonempty subset $T$ of $S$ we have $\sum_{g \in T} v'_g(B)g \neq 0$ for any integers $v'_g(B)$ with $0 \leq v'_g(B) \leq v_g(B)$ where at least one $v'_g(B) \neq 0$ and at least one $v'_g(B) < v_g(B)$.

2. If $B \neq E$ in $\mathcal{B}(G, S)$, then $B$ can be written as a product of irreducible blocks in $\mathcal{B}(G, S)$.

3. If $0 \in S$, then the block $0^1$ is prime in $\mathcal{B}(G, S)$.

4. If $G$ is finite, then $\mathcal{B}(G, S)$ contains finitely many irreducible blocks.
Basic Facts About Block Monoids

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Proposition

Let $G$ be an abelian group and $S$ a nonempty subset of $G$.

1. The block $B = \prod_{g \in S} g^{v_g(B)} \neq E$ is irreducible in $\mathcal{B}(G, S)$ if and only if for each nonempty subset $T$ of $S$ we have $\sum_{g \in T} v'_g(B)g \neq 0$ for any integers $v'_g(B)$ with $0 \leq v'_g(B) \leq v_g(B)$ where at least one $v'_g(B) \neq 0$ and at least one $v'_g(B) < v_g(B)$.

2. If $B \neq E$ in $\mathcal{B}(G, S)$, then $B$ can be written as a product of irreducible blocks in $\mathcal{B}(G, S)$.

3. If $0 \in S$, then the block $0^1$ is prime in $\mathcal{B}(G, S)$.

4. If $G$ is finite, then $\mathcal{B}(G, S)$ contains finitely many irreducible blocks.
Let $G = \mathbb{Z}_4$. Here

$$B(\mathbb{Z}_4) = \{0^{x_0}1^{x_1}2^{x_2}3^{x_3} \mid \text{each } x_i \geq 0 \text{ and } x_1 + 2x_2 + 3x_3 \equiv 0 \pmod{4}\}.$$  

Notice that the non-prime irreducible blocks of $B(\mathbb{Z}_4)$ are as follows:

$$1^4, 2^2, 3^4, 1^22^1, 1^13^1, \text{ and } 2^13^2.$$

In this monoid it is easy to produce factorizations of blocks into irreducible blocks which differ in length. For instance

$$B = (1^4)(3^4) = (1^13^1)^4$$

is a factorization of $B$ into 2 and 4 irreducible blocks respectively.
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Notice that the non-prime irreducible blocks of $\mathcal{B}(\mathbb{Z}_4)$ are as follows:

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Factorial vs. Half-Factorial

Proposition

Let $G$ be an abelian group. The following statements are equivalent.

1. $\mathcal{B}(G)$ is factorial.
2. $\mathcal{B}(G)$ is half-factorial.
3. $|G| \leq 2$.

Proof.

(2) $\Rightarrow$ (3) Suppose $\mathcal{B}(G)$ is half-factorial and that $|G| > 3$. Then $G$ has two distinct nonzero elements $g_1$ and $g_2$ with $g_3 = g_1 + g_2 \neq 0$ and $g_3 \neq g_1, g_2$. The blocks $A_1 = (-g_3)^1 g_1^1 g_2^1$, $A_2 = g_3^1 (-g_1)^1 (-g_2)^1$, $B_1 = g_1^1 (-g_1)^1$, $B_2 = g_2^1 (-g_2)^1$ and $B_3 = g_3^1 (-g_3)^1$ are all irreducibles of $\mathcal{B}(G)$. But $A_1 A_2 = B_1 B_2 B_3$, so $\mathcal{B}(G)$ is not half factorial, a contradiction. Hence $|G| \leq 3$. If $|G| = 3$, then $G \cong \mathbb{Z}_3$. If $A = 1^3$, $B = 2^3$ and $C = 1^1 2^1$, then $AB = C^3$ and $\mathcal{B}(\mathbb{Z}_3)$ is not half-factorial. Hence, we conclude that $|G| \leq 2$. 

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Hence $|G| \leq 3$. If $|G| = 3$, then $G \cong \mathbb{Z}_3$. If $A = 1^3$, $B = 2^3$ and $C = 1^1 2^1$, then $AB = C^3$ and $\mathcal{B}(\mathbb{Z}_3)$ is not half-factorial. Hence, we conclude that $|G| \leq 2$. 

Definition

Let $G$ be an abelian group. The *Davenport constant* of $G$ is defined as

$$D(G) = \sup \{|B| \mid B \text{ is an irreducible element of } \mathcal{B}(G)\}.$$ 

If $S$ is a nonempty subset of $G$, then

$$D(G, S) = \sup \{|B| \mid B \text{ is an irreducible element of } \mathcal{B}(G, S)\}$$

is known as the Davenport constant of $G$ relative to $S$.

No closed formula for the computation of the Davenport constant is known.

Davenport’s constant arises in several unexpected areas. Alford, Granville and Pomerance used the bound $D(G) \leq \exp(G)(1 + \log(|G|/\exp(G))$ to prove there are infinitely many Carmichael numbers.
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A Little Additive Number Theory

If $G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$ is a finite abelian group with $n_i \mid n_{i+1}$ for each $1 \leq i < k$, then set

$$M(G) = \left\lfloor \sum_{i+1}^k (n_i - 1) \right\rfloor + 1.$$

**Proposition**

Let $G$ be an abelian group.

1. If $|G| = \infty$, then $D(G) = \infty$.
2. If $|G| < \infty$, then $M(G) \leq D(G) \leq |G|$.
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Davenport Facts

It is possible for the upper inequality in Proposition 10 (2) to be strict. Erdős conjectured in the mid-sixties that $D(G) = M(G)$. It was not until 1969 that this conjecture was disproved. The group of smallest order that is a counterexample is

$$G_1 = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_6.$$

If $G$ is of rank less than or equal to 2, then $D(G) = M(G)$. It is unknown whether there is a counterexample of rank 3, and this, in fact, is an active area of research.
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A Little More Terminology

Let $M$ be a commutative cancellative monoid in which each nonunit can be written as product of irreducible elements (such a monoid is called *atomic*).

Let $A(M)$ represent the set of irreducible elements of $M$ and $M^\times$ its set of units.

For $x \in M \setminus M^\times$, set

$$L(x) = \{ n \mid n \in \mathbb{N} \text{ and there exist } x_1, \ldots, x_n \in A(M) \text{ with } x = x_1 \cdots x_n \}.$$  

We will refer to $L(x)$ as the *set of lengths of $x$ in $M$*. 
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We can extend $\mathcal{L}(x)$ to a global descriptor by setting

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There is another popular invariant which describes the variance in length of the factorizations of an element.

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The *elasticity of* $x$ is defined as
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\rho(x) = \frac{L(x)}{l(x)}.
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We can again extend this definition to all of $M$ by setting
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Questions

Obvious Questions:

(1) Which rings of algebraic integers $\mathcal{O}_K$ are half-factorial?

(2) What is the elasticity of a given ring $\mathcal{O}_K$ of integers?

HARDER QUESTIONS:

(3) What Dedekind domains are half-factorial?

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An Example

Example

To illustrate the above ideas, we can compute the sets of length for the block monoid $\mathcal{B}(\mathbb{Z}_3)$.

If $B = 0^x_1 1^x_2 2^x_3$ is in $\mathcal{B}(G)$, then $x_2 + 2x_3 \equiv 0 \pmod{3}$, so $x_2 \equiv x_3 \pmod{3}$.

Write $x_2 = 3q_2 + r$ and $x_3 = 3q_3 + r$, where $0 \leq r < 3$.

A calculation involving the irreducible blocks yields

$$\mathcal{L}(B) = \{x_1 + q_2 + q_3 + r + k \mid 0 \leq k \leq \min\{q_2, q_3\}\}$$

and so $\rho(B) = 1 + \min\{q_2, q_3\}/(x_1 + q_2 + q_3 + r)$.

This formula is maximized when $q_2 = q_3$ and $x_1 = r = 0$, so that $\rho(\mathcal{B}(\mathbb{Z}_3)) = 3/2$. 
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This formula is maximized when \( q_2 = q_3 \) and \( x_1 = r = 0 \), so that \( \rho(B(\mathbb{Z}_3)) = 3/2 \).
Geroldinger’s Theorem

Let $D$ be a Dedekind domain with divisor class group $G = \mathcal{C}(D)$, $D^*$ the multiplicative monoid of $D$ and $S$ be the set of divisor classes of $\mathcal{C}(D)$ containing prime ideals. Suppose further that for $x \in D^*$, we have $(x) = P_1 \cdots P_k$ for not necessary distinct prime ideals $P_1, \ldots, P_k$ of $D$. The function

$$\varphi : D^* \to \mathcal{B}(G, S)$$

is a well-defined monoid homomorphism that is surjective and preserves lengths of factorizations into irreducibles (i.e., $\mathcal{L}(x) = \mathcal{L}(\varphi(x))$ for each $x \in D^*$). Hence

$$\mathcal{L}(D) = \mathcal{L}(\mathcal{B}(G, S)),$$
Geroldinger’s Theorem can be extended to include the more general class of *Krull domains*.

When $D = \mathcal{O}_K$ is the ring of integers of a finite extension $K$ of the rationals, we earlier established that $S = G$, so Geroldinger’s Theorem establishes a correspondence between $\mathcal{O}_K$ and the full block monoid $B(G)$ over the class group. The following well-known theorem of Carlitz now follows as a corollary to Geroldinger’s Theorem.

**Carlitz’s Theorem**

Let $\mathcal{O}_K$ be the ring of integers in a finite extension of the rationals. Then $\mathcal{O}_K$ is half-factorial if and only if the class number of $\mathcal{O}_K$ is less than or equal to 2. Equivalently, $\mathcal{O}_K$ is half-factorial if and only if $|C(\mathcal{O}_K)| \leq 2$. 
Implications of Geroldinger’s Theorem

Geroldinger’s Theorem can be extended to include the more general class of *Krull domains*.

When $D = \mathcal{O}_K$ is the ring of integers of a finite extension $K$ of the rationals, we earlier established that $S = G$, so Geroldinger’s Theorem establishes a correspondence between $\mathcal{O}_K$ and the full block monoid $B(G)$ over the class group. The following well-known theorem of Carlitz now follows as a corollary to Geroldinger’s Theorem.

**Carlitz’s Theorem**

Let $\mathcal{O}_K$ be the ring of integers in a finite extension of the rationals. Then $\mathcal{O}_K$ is half-factorial if and only if the class number of $\mathcal{O}_K$ is less than or equal to 2. Equivalently, $\mathcal{O}_K$ is half-factorial if and only if $|C(\mathcal{O}_K)| \leq 2$. 

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Proposition

Let $D$ be a Dedekind domain with class group $G$ and $S$ defined as above. Assume further that $|G| < \infty$ and $G \neq \{0\}$.

1. If $S \neq \{0\}$, then $\rho(D) \leq \frac{D(G,S)}{2}$.
2. If $G = S$, then $\rho(D) = \frac{D(G)}{2}$. Moreover, in this case there is an $x \in D^*$ with $\rho(x) = \rho(D)$.

Sketch of Proof: By Geroldinger’s Theorem, we can pass to $B(G, S)$. If $B \in B(G, S)$, then write it as $B = g_1 \cdots g_n$. The shortest factorization of $B$ is greater than $n/D(G,S)$ and the longest less than $n/2$. Hence, $\rho(B(G, S)) \leq \frac{n/2}{n/D(G,S)} = \frac{D(G,S)}{2}$.
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On Elasticity

**Proposition**

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2. If $G = S$, then $\rho(D) = \frac{D(G)}{2}$. Moreover, in this case there is an $x \in D^*$ with $\rho(x) = \rho(D)$.

**Sketch of Proof:** By Geroldinger’s Theorem, we can pass to $\mathcal{B}(G, S)$. If $B \in \mathcal{B}(G, S)$, then write it as $B = g_1 \cdots g_n$. The shortest factorization of $B$ is greater than $n/D(G, S)$ and the longest less than $n/2$. Hence, $\rho(\mathcal{B}(G, S)) \leq \frac{n/2}{n/D(G, S)} = \frac{D(G, S)}{2}$. 
Valenza’s Theorem

The last result leads to an easy proof of a well-known extension of Carlitz’s Theorem by Valenza.

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Let $\mathcal{O}_K$ be the ring of integers in a finite extension of the rationals. Then

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\rho(\mathcal{O}_K) = \frac{D(C(\mathcal{O}_K))}{2}.
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