What Happens When the Division Algorithm “Almost” Works

Scott T. Chapman

Abstract. Let $K$ be any field. The division algorithm plays a key role in studying the basic algebraic structure of $K[X]$. While the division algorithm implies that all the ideals of $K[X]$ are principal, we show that subrings of $K[X]$ satisfying a slightly weaker version of the division algorithm produce ideals that while not principal, are still finitely generated. Our construction leads to an example for each positive integer $n$ of an integral domain with the $n$ generator property.

Dedicated to the Memory of Nick Vaughan

Central in a first abstract algebra course is the notion of the division algorithm. Indeed, a first abstraction for students studying ring theory is moving from the standard division algorithm over $\mathbb{Z}$ (the integers) to a similar statement for a polynomial ring over a field. The result below can be found in any standard abstract algebra text (such as [4] or [6]).

The Division Algorithm. Let $K$ be a field and $K[X]$ the polynomial ring over $K$. If $f(X)$ and $g(X)$ are in $K[X]$ with $g(X) \neq 0$, then there exist unique polynomials $q(X)$ and $r(X)$ in $K[X]$ such that

$$f(X) = g(X)q(X) + r(X)$$

and either $r(X) = 0$ or $\deg r(X) < \deg g(X)$.

A simple application of the division algorithm shows that ideals in $K[X]$ are principal (i.e., generated by one element). While many introductory textbooks give an example to show that not all ideals are principal (a popular one is $I = (2, X)$ in $\mathbb{Z}[X]$), most books do not go into great detail describing ideal generation problems. In this note, we consider a natural class of subrings of $K[X]$, namely those subrings $R$ with $K \subseteq R \subseteq K[X]$. We show that if such $R$ satisfy a weaker form of the division algorithm, then we can not only bound the number of generators of an ideal $I$ of $R$, but also offer examples of ideals that can be generated by $n$, but not $n - 1$ elements. We describe this weaker algorithm below.

Definition – The Almost Division Algorithm. A subring $R$ of $K[X]$, with $K \subseteq R$, has an almost division algorithm of index $m$ (where $m \in \mathbb{N}$) if it satisfies the following property. If $f(X)$ and $g(X)$ are in $R$ with $g(X) \neq 0$, then there exist polynomials $h(X)$ and $r(X)$ in $R$ such that

$$f(X) = h(X)g(X) + r(X)$$

where

(d1) $r(X) = 0$,
(d2) $\deg r(X) < \deg g(X)$, or

January 2014] AN ALMOST DIVISION ALGORITHM
(d3) \( \deg r(X) = \deg g(X) + i \) for \( 1 \leq i \leq m \).

A more general approach to rings and semirings satisfying an almost division algorithm can be found in [11] and [12].

Before proceeding, we note that various arguments can be used to show that the \( K \)-subalgebra \( R \) of \( K[X] \) is finitely generated and Noetherian (see for instance [13]). An in-depth look at computing generating sets for a particular \( R \) can be found in [1]. Also, we deal exclusively here with the one variable case, as with multiple variables (such as \( K \subseteq R \subseteq K[X, Y] \)), the subring \( R \) may not be Noetherian. The almost division algorithm leads directly to a proof of the following.

**Theorem 1.** Let \( R \) be a subring of \( K[X] \) with an almost division algorithm of index \( m \) and \( I \) a proper ideal of \( R \). There exist polynomials \( f_1(X), f_2(X), \ldots, f_{m+1}(X) \) such that

\[ I = (f_1(X), f_2(X), \ldots, f_{m+1}(X)). \]

Thus \( R \) has the \( m + 1 \) generator property on ideals.

**Proof.** Let \( I \) be a proper ideal of \( R \). If \( d \) is the minimal degree of a polynomial in \( I \), then for each \( i \) with \( 0 \leq i \leq m \), choose a polynomial \( t_{d+i}(X) \in I \) with \( \deg t_{d+i}(X) = d + i \). (If \( I \) does not contain a polynomial of such degree, then set \( t_{d+i}(X) = 0 \)). Setting

\[ J = (t_d(X), t_{d+1}(X), \ldots, t_{d+m}(X)), \]

we will prove that \( I = J \). Clearly \( J \subseteq I \). We prove the reverse containment.

Let \( f(X) \) be an arbitrary nonzero element of \( I \). Since \( S \) has an almost division algorithm of index \( m \),

\[ f(X) = h(X)t_d(X) + r(X) \]

where \( r(X) \) satisfies (d1), (d2), or (d3). Option (d2) cannot hold, as otherwise \( r(X) \in I \) contradicts the minimality of \( d \). If (d1) holds, then \( f(X) \in J \).

Now suppose (d3) holds. Then \( \deg r(X) = d + i \) for some \( 1 \leq i \leq m \). Now \( \deg t_{d+i}(X) = \deg r(X) \) and so there is a \( k \in K \) with \( r(X) = kt_{d+i}(X) + r_1(X) \) where either (d1) or (d2) holds. If (d1) holds, then \( f(X) = h(X)t_d(X) + kt_{d+i}(X) \in J \). If (d2) holds, then \( r_1(X) \in I \) with \( d \leq \deg r_1(X) < d + i \). Repeat this process on \( r_1(X) \) with the polynomial \( t_{\deg r_1(X)} \) and obtain the remainder term \( r_2(X) \). Since the degrees of the remainder terms are strictly descending \( (\deg r(X) > \deg r_1(X) > \deg r_2(X) > \cdots) \), this process must terminate and we have inductively constructed a finite sequence \( \{r_0(X) = r(X), r_1(X), \ldots, r_N(X)\} \) of remainders. Notice that \( f(X) = h(X)t_d(X) + \sum k_n t_{\deg r_n(X)}(X) \) where each \( k_n \in K \) and hence \( f(X) \in I \). Thus \( I \subseteq J \) and the proof is complete. ■

We apply Theorem 1 to a well-studied class of subrings of \( K[X] \). We will need the notion of a numerical semigroup to complete our work. Let \( \mathbb{N}_0 \) represent the nonnegative integers. An additive submonoid \( S \) of \( \mathbb{N}_0 \) is called a numerical monoid. Using elementary number theory, it is easy to show that there is a finite set of positive integers \( n_1, \ldots, n_k \) such that if \( s \in S \), then \( s = x_1n_1 + \cdots + x_kn_k \) where each \( x_i \) is a nonnegative integer. To represent that \( n_1, \ldots, n_k \) is a generating set for \( S \), we use the notation

\[ S = \langle n_1, \ldots, n_k \rangle = \{ x_1n_1 + \cdots + x_kn_k \mid x_i \in \mathbb{N}_0 \}. \]
If the generators \( n_1, \ldots, n_k \) are relatively prime, then \( S \) is called primitive. We shall need the following three facts concerning numerical semigroups. The proofs of all three can be found in [14] (part (a) is Proposition 1.2, (b) is Theorem 1.7, and (c) is a by-product of Lemma 1.1).

**Proposition 2.** Let \( S = \langle n_1, \ldots, n_k \rangle \) be a numerical semigroup.

(a) \( S \) is isomorphic to a primitive numerical semigroup \( S' \).

(b) \( S \) has a unique minimal cardinality generating set.

(c) If \( S \) is a primitive numerical semigroup, then there is a largest element \( F(S) \not\in S \) with the property that any \( s > F(S) \) is in \( S \).

Due to (a), we assume that \( S \) is primitive throughout the remainder of this work. The value \( F(S) \) is known as the Frobenius number of \( S \) and its computation remains a matter of current mathematical research. If \( S = \langle a, b \rangle \), then it is well known that \( F(S) = ab - a - b \) (see [15]), but for more than 2 generators, no general formula is known (see [14, Section 1.3] for more on Frobenius numbers).

Now, if \( K \) is a field and \( S \) a numerical semigroup, then set

\[
K[X; S] = \{ f(X) \mid f(X) \in K[X] \text{ and } f(X) = \sum_{\sigma \in S} a_{\sigma} X^{\sigma} \},
\]

where it is understood that the sum above is finite. The rings \( K[X; S] \) are known as semigroup rings, and [5] is a good general reference on the subject. Under our hypotheses, the rings \( K[X; S] \) consist of all polynomials with exponents coming from the numerical monoid \( S \). We illustrate this with some examples.

**Example 3.** Let \( S = \langle 3, 7, 11 \rangle \). A quick calculation shows that

\[
S = \{0, 3, 6, 7, 9, 10, 11, \ldots \}
\]

and \( F(S) = 8 \). Hence a typical element in \( K[X; \langle 3, 7, 11 \rangle] \) is of the form

\[
f(X) = a_0 + a_3 X^3 + a_6 X^6 + a_7 X^7 + \sum_{i=9}^{k} a_i X^i
\]

for some \( k \geq 9 \) with each \( a_i \) in \( K \).

**Example 4.** Let \( S = \langle 2, 3 \rangle \). Thus \( S = \{0, 2, 3, 4, 5, \ldots \} \) and a typical element of

\[
K[X; \langle 2, 3 \rangle] \]

is of the form \( f(X) = a_0 + \sum_{i=2}^{k} a_i X^i \) for some \( k \geq 2 \) with each \( a_i \) in \( K \). Thus, \( K[X; \langle 2, 3 \rangle] \) consists of all polynomials from \( K[X] \) which lack an \( X \) term. A version of Theorem 5 below specifically for \( K[X; \langle 2, 3 \rangle] \) can be found in [16].

We can generalize the last example as follows. Let \( n > 1 \) be a positive integer and set \( S = \langle n, n+1, \ldots, 2n-1 \rangle \). Notice that \( S \) consists of 0 along with all positive integers greater than or equal to \( n \). Thus, a typical element in \( K[X; \langle n, n+1, \ldots, 2n-1 \rangle] \) is of the form \( f(X) = a_0 + \sum_{i=n}^{k} a_i X^i \) where \( k \geq n \) and again each \( a_i \) is in \( K \).

As the last examples make clear, if \( S = \langle n_1, \ldots, n_k \rangle \) is a numerical semigroup, then the semigroup ring \( K[X; S] \) is equivalent to the extension of \( K \) by the monomial terms \( X^{n_1}, \ldots, X^{n_k} \) (i.e., \( K[X; S] \cong K[X^{n_1}, \ldots, X^{n_k}] \)).
Theorem 5. If $K$ is a field and $S$ a numerical semigroup, then $K[X; S]$ has an almost division algorithm of index $\mathcal{F}(S)$.

Proof. Let $f(X)$ and $g(X)$ be in $K[X; S]$ with $g(X) \neq 0$; we will divide $f(X)$ by $g(X)$ and verify that either (d1), (d2), or (d3) holds. If $\deg f(X) < \deg g(X)$, then the result is trivial. Hence, we assume $\deg f(X) \geq \deg g(X)$. By the regular division algorithm in $K[X]$, there exist $h(X)$ and $r(X)$ in $K[X]$ with

$$f(X) = h(X)g(X) + r(X)$$

where $r(X) = 0$ or $\deg r(X) < \deg g(X)$. If $h(X) \in K[X; S]$, then $r(X) \in K[X; S]$ and we are done. If not, then write

$$h(X) = \sum_{\gamma \in S} a_\gamma X^\gamma + \sum_{\sigma \in S} a_\sigma X^{\sigma}.$$ 

Setting $h^*(X) = \sum_{\gamma \in S} a_\gamma X^\gamma$ yields that $h^{**}(X) = h(X) - h^*(X)$ is in $K[X; S]$. If $r^*(X) = h^*(X)g(X) + r(X)$, then we have

$$f(X) = h(X)g(X) + r(X)$$

$$= [h(X) - h^*(X)]g(X) + [h^*(X)g(X) + r(X)]$$

$$= h^{**}(X)g(X) + r^*(X).$$

Since $f(X) - h^{**}(X)g(X) \in K[X; S]$, it follows that so too is $r^*(X)$. Since $\deg g(X) < \deg r^*(X) \leq \deg g(X) + \mathcal{F}(S)$, the proof is complete. $\blacksquare$

By a slight adjustment of $h^*(X)$ in the proof above, we see that the representation (d3) in the almost division algorithm may not be unique. For instance, returning to Example 4, if $S = \langle 2, 3 \rangle, f(X) = X^3$, and $g(X) = X^2$, then $X^3 = 0 \cdot X^2 + X^3$ and $X^3 = (-1) \cdot X^2 + (X^3 + X^2)$. The next corollary follows directly from Theorems 1 and 5.

Corollary 6. If $K$ is a field and $S$ a numerical semigroup, then the ideals of $K[X; S]$ require at most $\mathcal{F}(S) + 1$ generators.

A Noetherian integral domain in which the ideals can be $n$-generated is said to have the $n$-generator property. If an integral domain $D$ has the $n$-generator property for some $n \in \mathbb{N}$, then it has the $m$-generator property for some minimal value $m \in \mathbb{N}$. Dedekind domains (a very natural class of rings that are ubiquitous in algebraic number theory and algebraic geometry) are generally not principal ideal domains, but they always have the 2-generator property (a proof of this can be found in [8, Theorem 17]). While Corollary 6 shows that $K[X; S]$ has the $\mathcal{F}(S) + 1$ generator property, this value may not be minimal, and in fact is not sharp for all $S$. Using semigroup ideals, a precise minimal value can be found (the interested reader can construct examples for which our bound is not sharp by using [2, Corollary 7] or [10]). Further reading on rings with the $n$-generator property can be found in [3], [7], and [9].

We close by showing that the value of Corollary 6 is sharp for the numerical semigroups introduced in Example 4.

Proposition 7. Let $K$ be a field, $n > 1$ a positive integer, and $S = \langle n, n + 1, \ldots, 2n - 1 \rangle$ a numerical semigroup. The integral domain $K[X; S]$ has the $n$, but not the $n - 1$ generator property.
Proof. Since $F(S) = n - 1$, Corollary 6 implies that $K[X; S]$ has the $n$-generator property. We argue that the ideal

$$I = (X^n, X^{n+1}, \ldots, X^{2n-1})$$

requires $n$ generators. The argument will center around the $K$-vector space $V$ generated by $X^n, \ldots, X^{2n-1}$. Since the elements $X^n, \ldots, X^{2n-1}$ are linearly independent over $K$, $V$ has dimension $n$.

Suppose $I = (f_1(X), \ldots, f_k(X))$ where each $f_i(X) \in K[X; S]$ and $k < n$. Since $I$ contains no elements with nonzero constant terms, the constant terms on the $f_i(X)$’s are all zero. For each $i = 1, \ldots, k$ define $f_i'(X)$ by

$$f_i(X) = a_{i,1}X^n + \cdots + a_{i,k}X^{2n-1} + \sum_{j=2n}^{r_i} a_{j,1}X^j = f_i'(X) + \sum_{j=2n}^{r_i} a_{j,1}X^j$$

for $1 \leq i \leq k$ where each $a_{i,j} \in K$. By assumption, for each $0 \leq v \leq n - 1$,

$$X^n + v = C_{1,v}(X)f_1(X) + \cdots + C_{k,v}(X)f_k(X)$$

where each $C_{j,v}(X) \in K[X; S]$ if $c_{j,v}$ is the constant term for each $C_{j,v}(X)$, then a simple degree argument yields

$$X^n + v = c_{1,v}f_1'(X) + \cdots + c_{k,v}f_k'(X)$$

for each $0 \leq v \leq n - 1$. Thus the $K$-vector space generated by $f_1'(X), \ldots, f_k'(X)$ contains $V$, which contradicts that $\dim V = n$. 

ACKNOWLEDGMENTS. The author would like to thank Susan Colley, Vadim Ponomarenko, and two unknown referees for comments that greatly improved the content and exposition of this note.

REFERENCES


Department of Mathematics and Statistics, Sam Houston State University, Box 2206, Huntsville, TX 77341
scott.chapman@shsu.edu