Approximation of Weighted Local Mean Operators

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Abstract
Recently, weighted local mean operators are widely used in image processing, compressive sensing, and other areas. A weighted local mean operator changes its characteristics depending on a function content within a local area in order to preserve the function features. The directional diffusion filter and Yaroslavsky neighborhood filter (also called the sigma filter) are discrete versions of such operators. Although these operators are not convolution ones, due to their sparsity, the corresponding numerical algorithms have simple structure and fast performance. In this paper, we study the approximate properties of the weighted local mean operators, particularly focus on their asymptotic expansions, which are related to nonlinear diffusion equations.

Keywords: Weighted local mean, anisotropic diffusion, approximation, asymptotic expansion, neighborhood filters.

1 Introduction

Using local integral mean to approximate functions is well-known in the classic approximation. Let \( f \) be a function defined on the \( d \)-dimensional \((d-D)\) Euclidean space \( \mathbb{R}^d \) and write \( B_h = \{ x \in \mathbb{R}^d, \|x\| < h \}, h > 0 \), where \( \|x\| \) denotes the norm of the \( d-D \) vector \( x \). The local integral mean of \( f \) is defined by

\[
A_h(f)(x) = \frac{1}{\text{Vol}(B_h)} \int_{B_h} f(x + z) \, dz, \quad \forall x \in \mathbb{R}^d,
\]

(1)

where \( \text{Vol}(B_h) = \frac{h^d \pi^{d/2}}{\Gamma(d/2 + 1)} \) is the volume of \( B_h \). The local mean operator in (1) is a convolution one, which produces an isotropic mean for \( f \) in its domain. Hence, the operator does not preserve the function features that are characterized by the gradient of \( f \). In many applications, like image enhancement, for example, edge is an important feature that should be preserved. However, the local mean in (1) blurs edge. In order to preserve such features of functions, recently, the weighted local mean operators are introduced. These operators change their characteristics depending on the contents of functions such that small weights are assigned to feature areas while large weights are assigned to harmonic areas, so that a slower diffusion is created across region boundaries while a faster diffusion acts on harmonic areas. Suppose \( f \) is a differentiable function whose gradient is denoted by \( \nabla f \). The first type of weighted local mean operators is defined as follows:

\[
W_h(f)(x) = \frac{1}{S(x)} \int_{B_h} \exp \left( -\frac{< \nabla f(x), z >}{h^2} \right) f(x + z) \, dz, \quad \forall x \in \mathbb{R}^d,
\]

(2)

where \( < \cdot , \cdot > \) denotes the inner product in \( \mathbb{R}^d \), \( h(> 0) \) is a time-like parameter, and

\[
S(x) = \int_{B_h} \exp \left( -\frac{< \nabla f(x), z >}{h^2} \right) \, dz
\]

(3)

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is the normalization factor. The operator $W_h$ is nonlinear and has the smallest weight in the gradient direction of $f$ and the largest weight in the level-space directions of $f$ so that $W_h$ preserves the region boundaries of $f$. In this paper, we call $W_h$ the nonlinear directionally local mean (DLM) operator. A similar but slightly modified version of $W_h$ for $d = 2$ was introduced for removing image noise in [6] and also for the study of diffusion maps in [3].

The second type of weighted local mean operators is defined as follows:

$$Y_h(f)(x) = \frac{1}{D(x)} \int_{B_h} \exp \left( -\frac{|f(x+z) - f(x)|^2}{h^2} \right) f(x+z) \, dz, \quad \forall x \in \mathbb{R}^d, \quad (4)$$

where $h > 0$ is a parameter and

$$D(x) = \int_{B_h} \exp \left( -\frac{|f(x+z) - f(x)|^2}{h^2} \right) \, dz \quad (5)$$

is the normalization factor. The operator $Y_h$ is also nonlinear and has the similar behavior as $W_h$. When $d = 2$, $Y_h$ was known as Yaroslavsky neighborhood filter in [7] and the sigma filter in [5]. Therefore, we will call $Y_h$ the nonlinear Yaroslavsky local mean (YLM) operator.

The approximate properties of $W_h$ and $Y_h$ describe their important characters. In this paper, we study the approximation of $W_h$ and $Y_h$, particularly put emphasis on their asymptotic expansions as $h \to 0$. In Section 2, we introduce the notations and preliminaries. In Section 3, we study the approximation of $W_h$. In Section 4, we study the approximation of $Y_h$.

## 2 Preliminaries and notations

In this section, we assemble the main notions and notations, which will be used throughout the paper. Assume that $X \subset \mathbb{R}^d$ is a locally compact set. Let $C(X)$ denote the linear space of all real-valued continuous functions defined on $X$. The sup-norm of $f \in C(X)$ is defined by

$$\|f\|_X = \sup_{x \in X} |f(x)|. \quad (6)$$

If the set $X$ need not to be stressed, we simply write the norm as $\|f\|$.

We by $C^k(X)$ denote the space of the functions that have all $k^{th}$ order continuous partial derivatives. The space $C^k(\mathbb{R}^d)$ is often abbreviated to $C^k$.

Let $X - x = \{ z \in \mathbb{R}^d, \ z + x \in X \}$ and $X_x = \{ z \in \mathbb{R}^d, \ \pm z + x \in X \}$. The (point-wise) first modulus and the (point-wise) second modulus of continuity of $f \in C(X)$ at $x \in X(\subset \mathbb{R}^d)$ are defined by

$$\omega(f, x, \delta) = \sup_{z \in B_h \cap (X - x)} |f(x+z) - f(x)|, \quad \delta > 0,$$

and

$$\omega^s(f, x, \delta) = \sup_{z \in B_h \cap X_x} |f(x+z) + f(x-z) - 2f(x)|, \quad \delta > 0,$$

respectively.

The first (global) modulus and the second (global) modulus of continuity on $X$ are defined by $\omega(f, X, \delta) = \sup_{x \in X} \omega(f, x, \delta)$ and $\omega^s(f, X, \delta) = \sup_{x \in X} \omega^s(f, x, \delta)$ respectively.

As usual, Gamma function is defined by

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} \, dt, \quad s > 0,$$

and the (real-valued) modified Bessel function of the first kind is defined by

$$I_n(s) = \frac{1}{\pi} \int_0^\pi e^{-s \cos t} \cos(nt) \, dt, \quad n > -1/2, \ s \in \mathbb{R},$$

2
which can also be represented by
\[
I_n(s) = \frac{s^n}{2^n \Gamma\left(n + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_0^\pi e^{-s \cos t} \sin^{2n} t \, dt.
\]

The Kummer’s function is defined by
\[
M(a; b; s) = \frac{\Gamma(b)}{\Gamma(b - a) \Gamma(a)} \int_0^1 t^{a-1}(1 - t)^{b-a-1} \, dt, \quad s \in \mathbb{R},
\]
which is also called the confluent hypergeometric function of the first kind with the notation \( {}_1 F_1 (a; b; s) \). More details of Bessel functions and the Kummer’s functions can be found in [1] and [4].

For \( n \in \mathbb{Z}^d \) and \( k \in \mathbb{Z}^d_+ \), we write \( n! = n_1!n_2\cdots n_d! \) and \( |n| = n_1 + n_2 + \cdots + n_d \). For \( s \in (0, \infty)^d \), we write \( \Gamma(s) = \Gamma(s_1) \cdots \Gamma(s_d) \). For \( x \in \mathbb{R}^d \), we define the \( k \)-power of \( x \) by \( x^k = x_1^{k_1}x_2^{k_2} \cdots x_d^{k_d} \) and say that \( x^k \) is an odd power if there is \( j \) such that \( k_j \) is odd. The \( k \)-moment of a function \( f \in C(X) \) is defined by
\[
m_X(f, x^k) = \int_X x^k f(x) \, dx.
\]

When \( X = B_h \), we denote it by \( m_h(f, x^k) \).

3 Approximation of the directionally local mean operators

To study the approximation of the nonlinear DLM operator \( W_h \), we first generalize it to the following linear DLM operator (associated with \( g \in C^1 \)):
\[
W_{g,h}(f)(x) = \frac{1}{S_{g,h}(x)} \int_{B_h} \exp \left( -\frac{\langle \nabla g(x), z \rangle}{h^2} \right) f(x + z) \, dz, \quad x \in \mathbb{R}^d,
\]
where
\[
S_{g,h}(x) = \int_{B_h} \exp \left( -\frac{\langle \nabla g(x), z \rangle}{h^2} \right) \, dz, \quad x \in \mathbb{R}^d.
\]

It is obvious that \( W_{f,h}(f)(x) = W_h(f)(x) \). We define the non-normalized kernel of \( W_{g,h} \) by
\[
W_{g,h}(x, z) = \exp \left( -\frac{\langle \nabla g(x), z \rangle}{h^2} \right)
\]
and the normalized kernel by
\[
w_{g,h}(x, z) = \frac{1}{S_{g,h}(x)} W_{g,h}(x, z).
\]

Later, we will simply denote them by \( S_g(x), w_g(x, z), \) and \( W_g(x, z) \) if the parameter \( h \) is not stressed.

It is easy to verify that the kernel \( w_g(x, z) \) satisfies (1) \( w_g(x, z) \geq 0 \); (2) \( w_g(x, z) = w_g(x, -z) \); and (3) \( \int_{B_h} w_g(x, z) \, dz = 1 \). By these properties, we immediately get the following:

**Theorem 1** Suppose \( g \in C^1 \). Then
\[
|W_{g,h}(f)(x) - f(x)| \leq \omega^*(f, x, h), \quad f \in C,
\]
and, for each compact set \( X \subset \mathbb{R}^d \),
\[
|W_{g,h}(f)(x) - f(x)| \leq \omega^*(f, X, h), \quad x \in X.
\]
Furthermore, if \( f \in C^2 \), then
\[
|W_{g,h}(f)(x) - f(x)| = O(h^2).
\]
The theorem is proved.

Theorem 2

The DLM operator

Definition 1

Let \( F \) be an operator on \( C \). We say that \( F \) has the linearly-reproducing property if it reproduces all linear functions, that is, \( F(f) = f \) for each linear function \( f \in C \).

In many applications, the linearly-reproducing property is important. For instance, in image denoising, the linearly-reproducing filters can reduce the staircase effect \([2, 6]\). We now prove the following:

Theorem 2

The DLM operator \( W_{g,h} \) has the linearly-reproducing property.

Proof. For a linear function \( L \) on \( \mathbb{R}^d \), there is \( a \in \mathbb{R}^d \) and \( c \in \mathbb{R} \) such that, for each \( y \in \mathbb{R}^d \), \( L(y) = c + \sum_{j=1}^{d} a_j y_j \). Hence, \( L(x + z) - L(x) = \sum_{j=1}^{d} a_j z_j \). By \( w_g(x, z) = w_g(x, -z) \), we have

\[
\int_{B_h} w_g(x, z) z_j dz = 0, \quad 1 \leq j \leq d,
\]

which yields

\[
W_g(L)(x) = \int_{B_h} w_g(x, z)L(x + z) dz = \int_{B_h} w_g(x, z)L(x) dz = L(x).
\]

The theorem is proved. \( \blacksquare \)

The problem of associating integral operators with PDEs is a very natural and fruitful research area. Note that the corresponding PDEs can be derived from the asymptotic expansions of these operators. For example, the Gauss-Weierstrass integral

\[
G_t(v)(x) = \frac{1}{(\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{t}} v(y) dy, \quad \forall v \in C^2,
\]

has the asymptotic expansion

\[
G_t(v)(x) = v(x) + t \Delta v(x) + o(t), \quad t \to 0^+,
\]

where \( \Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} \) is the spatial Laplacian operator. It is known that \( \{G_t\}_{t>0} \) is a semigroup and

\[
\begin{align*}
\frac{\partial}{\partial t} u(x, t) &= \Delta u(x, t), \quad t > 0, \\
u(x, 0) &= v(x), \quad x \in \mathbb{R}^d.
\end{align*}
\]

If we set \( t = h^2 \) and consider \( t \) as a time variable, then the operator \( W_{g,h} \) also produces a diffusion process. Although the family \( \{W_{g,h}\}_{h>0} \) is no longer a semigroup, its asymptotic expansion describes the diffusion process in terms of PDEs. We now give the asymptotic expansion of \( W_{g,h} \) in the following:
Theorem 3 Let \( f, g \in C^2 \) and \( B = \{ b_1, b_2, \ldots, b_d \} \) be a local o.n. basis at \( x \) associated with \( g \). Write \( p = \| \nabla g(x) \| \) and define
\[
L_{g,x} = \begin{cases} 
\frac{1}{2(d+2)} \left( \frac{\partial^2}{\partial b_i^2} A(p^2) \right) + B(p^2) \sum_{j=2}^d \frac{\partial^2}{\partial b_j^2}, & \text{if } p \neq 0, \\
\frac{1}{2(d+2)} \Delta, & \text{if } p = 0,
\end{cases}
\]
(13)
where
\[
A(u) = \frac{M \left( \frac{1}{2}; 1 + d \frac{1}{2}, -u \right)}{M \left( \frac{1}{2}; 1 + d \frac{1}{2}, -u \right)}, \quad B(u) = \frac{M \left( \frac{1}{2}; 2 + d \frac{1}{2}, -u \right)}{M \left( \frac{1}{2}; 1 + d \frac{1}{2}, -u \right)}.
\]
(14)
Then, \( W_{g,h} \) has the following asymptotic expansion.
\[
W_{g,h,f}(x) = f(x) + h^2 (L_{g,x} f)(x) + O(h^2), \quad h \to 0.
\]
(15)
Furthermore, if \( f \in C^4 \), then
\[
W_{g,h,f}(x) = f(x) + h^2 (L_{g,x} f)(x) + O(h^4), \quad h \to 0.
\]
(16)

In applications, the case of \( d = 2 \) is particularly useful. We present the special form of (13) for \( d = 2 \) in the following:

Corollary 4 Let \( f, g \in C^2 \) and \( B = \{ b_1, b_2 \} \) be the local o.n. basis at \( x \) associated with \( g \). Write \( p = \| \nabla g(x) \| \). Then the operator \( L_{g,x} \) in Theorem 3 has the following form.
\[
L_{g,x} = \begin{cases} 
\frac{1}{2} \left( \frac{1}{2} \right)^2 \frac{t_1 \left( \frac{1}{2} \right)}{t_0 \left( \frac{1}{2} \right)} \frac{\partial^2}{\partial b_1^2} + \frac{1}{2} \frac{t_2 \left( \frac{1}{2} \right) + (p^2 - 1) t_1 \left( \frac{1}{2} \right)}{t_0 \left( \frac{1}{2} \right) + t_1 \left( \frac{1}{2} \right)} \frac{\partial^2}{\partial b_2^2}, & \text{if } p \neq 0, \\
\frac{1}{8} \Delta, & \text{if } p = 0.
\end{cases}
\]

By the relation \( W_{f,h,f}(x) = W_{h,f}(x) \), applying Theorem 3, we immediately obtain the following:

Theorem 5 Let \( f \in C^2(\mathbb{R}^d) \) and \( B = \{ b_1, b_2, \ldots, b_d \} \) be a local o.n. basis at \( x \) associated with \( f \). Then, the nonlinear DLM operator \( W_h \) has the following asymptotic expansion.
\[
W_h f(x) = f(x) + h^2 (L_{f,x} f)(x) + O(h^2), \quad h \to 0.
\]
(17)
Furthermore, if \( f \in C^4(\mathbb{R}^d) \), then
\[
W_h f(x) = f(x) + h^2 (L_{f,x} f)(x) + O(h^4), \quad h \to 0.
\]
(18)

To prove Theorem 3 and Corollary 4, we establish two lemmas.

Lemma 1 Assume \( p = \| \nabla g(x) \| \neq 0 \) and \( d \geq 2 \). Let \( B = \{ b_1, b_2, \ldots, b_d \} \) be a local o.n. basis at \( x \) associated with \( g \). Then we have
\[
m_h(W_{g,h}, z^k) = 0, \quad z^k \text{ is odd,}
\]
(19)
\[
S_{g,h}(x) = \frac{h^{d-d_2}}{\Gamma(1 + \frac{d}{2})} M \left( \frac{1}{2}; 1 + \frac{d}{2}, -p^2 \right),
\]
(20)
\[
m_h(W_{g,h}, z^3) = \frac{h^{2+d_2}}{2 \Gamma(2 + \frac{d}{2})} M \left( \frac{3}{2}; 2 + \frac{d}{2}, -p^2 \right),
\]
(21)
\[
m_h(W_{g,h}, z^j) = \frac{h^{2+d_2}}{2 \Gamma(2 + \frac{d}{2})} M \left( \frac{1}{2}; 2 + \frac{d}{2}, -p^2 \right), \quad 2 \leq j \leq d.
\]
(22)
We have, for $g$, then (23), (24), and (25) are derived from Lemma 1. Assume

$$m_h(W_{g,h}, 1) = \int_{B_h} \exp \left( -\frac{p^2 z^2}{h^2} \right) \, dz$$

$$= \frac{(d - 1)\pi^{d/2-1/2}}{\Gamma \left( \frac{d+1}{2} \right)} \int_0^h \int_0^\pi \exp \left( -\frac{p^2 r^2 \cos^2(\theta)}{h^2} \right) r^{d-1} \sin^{d-2}(\theta) dr d\theta$$

$$= \frac{h^d \pi^\frac{d}{2}}{\Gamma(1 + \frac{d}{2})} M \left( \frac{1}{2}; 1 + \frac{d}{2}; -p^2 \right).$$

Equation (20) is obtained. Similarly,

$$m_h(W_{g,h}, z_j^2) = \int_{B_h} z_j^2 \exp \left( -\frac{p^2 z^2}{h^2} \right) \, dz$$

$$= \frac{(d - 1)\pi^{d/2-1/2}}{\Gamma \left( \frac{d+1}{2} \right)} \int_0^h \int_0^\pi \exp \left( -\frac{p^2 r^2 \cos^2(\theta)}{h^2} \right) r^{d+1} \sin^2(\theta) \sin^{d-2}(\theta) d\theta dr$$

$$= \frac{h^{2+d} \pi^\frac{d}{2}}{2\Gamma(2 + \frac{d}{2})} M \left( \frac{1}{2}; 2 + \frac{d}{2}; -p^2 \right), \quad 2 \leq j \leq d,$$

which yields (21). Finally, for $2 \leq j \leq d,$

$$m_h(W_{g,h}, z_j^2) = \int_{B_h} z_j^2 \exp \left( -\frac{p^2 z^2}{h^2} \right) \, dz$$

$$= \frac{\pi^{d/2-1/2}}{\Gamma \left( \frac{d+1}{2} \right)} \int_0^h \int_0^\pi \exp \left( -\frac{p^2 r^2 \cos^2(\theta)}{h^2} \right) r^{d+1} \sin^2(\theta) d\theta dr$$

$$= \frac{h^{2+d} \pi^\frac{d}{2}}{2\Gamma(2 + \frac{d}{2})} M \left( \frac{1}{2}; 2 + \frac{d}{2}; -p^2 \right), \quad 2 \leq j \leq d.$$
Theorem 6

When \( p \neq 0 \), by Lemma 2 and (27), we have

\[
W_{g,h}f(x) = f(x) + \frac{h^2}{2(d+2)} \Delta f(x) + o(h^2).
\]

When \( p = 0 \), by the direct computation, we have

\[
W_{g,h}f(x) = f(x) + \frac{h^2}{2(d+2)} \left( A(p^2) \frac{\partial^2 f(x)}{\partial b_i^2} + B(p^2) \sum_{j=2}^{d} \frac{\partial^2 f(x)}{\partial b_j^2} \right) + o(h^2),
\]

where \( A(u) \) and \( B(u) \) are given by (13). The proof of (15) is completed. If \( g \in f^4(\mathbb{R}^4) \), then in (27) the term \( o(|z|^2|) \) can be replaced by \( O(|z|^4) \), which yields (16). 

**Proof of Corollary 4.** When \( d = 2 \), \( \Gamma \left(1 + \frac{d}{2}\right) = \Gamma(2) = 1 \) and

\[
M \left( \frac{1}{2}; 1 + \frac{d}{2}; -p^2 \right) = M \left( \frac{1}{2}; 2; -p^2 \right) = e^{-\frac{x^2}{p^2}} \left( I_0 \left( \frac{p^2}{2} \right) + I_1 \left( \frac{p^2}{2} \right) \right).
\]

By the identities

\[
I_1(x) = \frac{x}{2} e^{-x} M \left( \frac{3}{2}; 3; 2x \right), \\
I_1(-x) = -I_1(x),
\]

we have

\[
M \left( \frac{3}{2}; 3; -p^2 \right) = \frac{-4e^{-\frac{x^2}{p^2}}}{p^2} I_1 \left( -\frac{p^2}{2} \right) = \frac{4e^{-\frac{x^2}{p^2}}}{p^2} I_1 \left( \frac{p^2}{2} \right).
\]

Applying the recurrent formula

\[
(b - 1)M(a; b - 1; x) - aM(a + 1; b; x) - (b - a - 1)M(a; b; x) = 0,
\]

we have

\[
M \left( \frac{1}{2}; 3; -p^2 \right) = \frac{4}{3} M \left( \frac{1}{2}; 2; -p^2 \right) - \frac{1}{3} M \left( \frac{3}{2}; 3; -p^2 \right)
\]

\[
= \frac{4e^{-\frac{x^2}{p^2}}}{3} \left( I_0 \left( \frac{p^2}{2} \right) + I_1 \left( \frac{p^2}{2} \right) \right) - \frac{4e^{-\frac{x^2}{p^2}}}{3p^2} I_1 \left( \frac{p^2}{2} \right)
\]

\[
= \frac{4e^{-\frac{x^2}{p^2}}}{3p^2} \left( p^2 I_0 \left( \frac{p^2}{2} \right) + (p^2 - 1) I_1 \left( \frac{p^2}{2} \right) \right).
\]  

The identities (30), (31), and (33) lead us to the corollary. 

To reveal the anisotropic diffusion behavior of the directionally local mean operator, we give the properties of the functions \( A(u) \) and \( B(u) \) in the following:

**Theorem 6** The function \( A(u) \) is decreasing on \((0, \infty)\) and the function \( B(u) \) is increasing on \((0, \infty)\). Furthermore,

\[
A(0) = B(0) = 1,
\]

\[
\lim_{u \to \infty} A(u) = 0,
\]

\[
\lim_{u \to \infty} B(u) = \frac{d + 2}{d + 1}.
\]
**Proof.** We first prove that $A(u)$ is decreasing on $(0, \infty)$. By (13), we have

$$\frac{d}{du} A(u) = \frac{M' \left( \frac{3}{2}; 2 + \frac{d}{2}; -u \right) M \left( \frac{1}{2}; 1 + \frac{d}{2}; -u \right) - M \left( \frac{3}{2}; 2 + \frac{d}{2}; -u \right) M' \left( \frac{1}{2}; 1 + \frac{d}{2}; -u \right)}{M^2 \left( \frac{1}{2}; 1 + \frac{d}{2}; -u \right)}.$$ 

The identity

$$M'(a; b; -u) = -\frac{a}{b} M(a + 1, b + 1, -u)$$

yields

$$M' \left( \frac{3}{2}; 2 + \frac{d}{2}; -u \right) M \left( \frac{1}{2}; 1 + \frac{d}{2}; -u \right) - M \left( \frac{3}{2}; 2 + \frac{d}{2}; -u \right) M' \left( \frac{1}{2}; 1 + \frac{d}{2}; -u \right)$$

$$= -\frac{3}{4 + d} M \left( \frac{5}{2}; 3 + \frac{d}{2}; -u \right) M \left( \frac{1}{2}; 1 + \frac{d}{2}; -u \right) + \frac{1}{2 + d} M^2 \left( \frac{3}{2}; 2 + \frac{d}{2}; -u \right)$$

$$= -C \left( \int_0^1 e^{-u t^2} (1 - t)^{\frac{d+1}{2}} dt \int_0^1 e^{-u t^2} (1 - t)^{\frac{d+1}{2}} dt - \left( \int_0^1 e^{-u t^2} (1 - t)^{\frac{d+1}{2}} dt \right)^2 \right)$$

$$\leq 0, \quad \forall u \in (0, \infty),$$

where $C = \frac{\Gamma^2 \left( 2 + \frac{d}{2} \right)}{d + 2 \Gamma^2 \left( \frac{d}{2} + 1 \right)}$. Hence $A(u)$ is decreasing. By the recurrent formula (32), we re-write $B(u)$ as the following:

$$B(u) = \frac{(d + 2)M \left( \frac{1}{2}; \frac{d}{2} + 1; -u \right) - M \left( \frac{3}{2}; \frac{d}{2} + 2; -u \right)}{(d + 1)M \left( \frac{1}{2}; \frac{d}{2} + 1; -u \right)}$$

$$= \frac{d + 2}{d + 1} - \frac{1}{d + 1} A(u),$$

which yields the increase of $B(u)$. Note that the equation (34) is a trivial consequence of $M(a; b; 0) = 1$. To prove (35), applying Kummer’s transform $M(a; b; -x) = e^{-x} M(b - a; b; x)$, we have

$$M \left( \frac{3}{2}; 2 + \frac{d}{2}; -u \right) = e^{-u} M \left( \frac{1}{2}; \frac{d}{2} + 1; u \right),$$

$$M \left( \frac{1}{2}; 1 + \frac{d}{2}; -u \right) = e^{-u} M \left( \frac{1}{2}; 1 + \frac{d}{2}; u \right).$$

Applying the Poincaré-Type expansions of $M \left( \frac{d+1}{2}; 2 + \frac{d}{2}; u \right)$ and $M \left( \frac{d+1}{2}; 1 + \frac{d}{2}; u \right)$, we have

$$M \left( \frac{3}{2}; 2 + \frac{d}{2}; u \right) = \frac{\Gamma \left( 2 + \frac{d}{2}, \frac{d}{2} \right)}{\Gamma \left( \frac{d}{2} \right)} u^{-\frac{d}{2}} \left( \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma \left( \frac{3}{2} + k \right)}{k! \Gamma \left( \frac{d+1}{2} - k \right)} u^{-k} + O \left( u^{-n-1} \right) \right),$$

$$M \left( \frac{1}{2}; 1 + \frac{d}{2}; u \right) = \frac{\Gamma \left( 1 + \frac{d}{2}, \frac{d}{2} \right)}{\Gamma \left( \frac{d}{2} \right)} u^{-\frac{d}{2}} \left( \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma \left( \frac{1}{2} + k \right)}{k! \Gamma \left( \frac{d+1}{2} - k \right)} u^{-k} + O \left( u^{-n-1} \right) \right),$$

which yield

$$A(u) = \frac{d + 2}{2} u^{-1} (1 + o(1)),$$

so that (35) holds. Finally, by (38) and (35), we obtain (36). \qed

Let us set $t = h^2$ as a time variable. Then, the operator $\mathbf{W}_{g,h}$ describes a diffusion processing. Theorem 6 illustrates that, as the magnitude of the gradient of $g$ enlarges, the diffusion speed in the gradient direction reduces and tends to 0, while the diffusion speed along the tangent space of $g$ increases up to the ceiling $\frac{d+2}{d+1}$. Thus, in the area where $\|\nabla g(x)\|$ is small, the operator produces a
nearly isotropic diffusion, while in the area where $|\nabla g(x)|$ is large, it almost stops the diffusion in the gradient direction of $g$.

In Figures 1 and 2, the graphs of the functions $A(p^2)$ and $B(p^2)$ are presented for $d = 2$ and $d = 4$ respectively. They show that $A(p^2)$ is decreasing and approximates 0, while $B(p^2)$ is increasing and approximates $\frac{d+2}{d+1}$.

Figure 1: The graphs of the functions $A(p^2)$ (left) and $B(p^2)$ (right) for $d = 2$.

Figure 2: The graphs of the functions $A(p^2)$ (left) and $B(p^2)$ (right) for $d = 4$.

4 Approximation of Yaroslovsky local mean operators

We now discuss YLM operator $Y_h$ defined in (4). Suppose $g \in C$. Again, we first generalize $Y_h$ to the following:

$$Y_{g,h}(f)(x) = \frac{1}{D_{g,h}(x)} \int_{B_h} \exp \left( -\frac{|g(x + z) - g(x)|^2}{h^2} \right) f(x + z) \, dz, \quad \forall x \in \mathbb{R}^d, \quad (42)$$

where $D_{g,h}(x)$ is the normalization factor defined by

$$D_{g,h}(f)(x) = \int_{B_h} \exp \left( -\frac{|g(x + z) - g(x)|^2}{h^2} \right) \, dz, \quad \forall x \in \mathbb{R}^d. \quad (43)$$

We denote the non-normalized kernel of $Y_{g,h}$ by

$$K_{g,h}(x, z) = \exp \left( -\frac{|g(x + z) - g(x)|^2}{h^2} \right) \quad (44)$$
and denote the normalized kernel by

\[ k_{g,h}(x, z) = \frac{1}{D_{g,h}(x)} K_{g,h}(x, z). \]  

(45)

We will simply denote them by \( D_g(x), K_g(x, z), \) and \( k_g(x, z) \) if the parameter \( h \) is not stressed.

It is easy to verify that the kernel \( k_g(x, z) \) is nonnegative and satisfies the normalization condition \( \int_{B_h} k_g(x, z) \, dz = 1 \). Therefore, we immediately get the following:

**Theorem 7** Let \( f \in C \). Then

\[ |Y_{g,h}(f)(x) - f(x)| \leq \omega(f, x, h). \]

Therefore, for each compact set \( X \subset \mathbb{R}^d \),

\[ |Y_{g,h}(f)(x) - f(x)| \leq \omega(X, f, h), \quad x \in X. \]

Unlike the kernel of the DLM operator, the kernel \( k_{g,h}(x, z) \) is not symmetric with respect to \( z \). Therefore, it has no linearly-reproducing property. We now present the asymptotic expansion of \( Y_{g,h} \) in the following:

**Theorem 8** Let \( f, g \in C^2(\mathbb{R}^d) \) and \( B = \{ b_1, b_2, \ldots, b_d \} \) be a local o.n. basis at \( x \) associated with \( g \). Let \( L_{g,x} \) be the operator in (13). Define

\[
Q_{g,x} = \begin{cases} 
3E(p^2)g_{11}(x) + F(p^2) \sum_{j=2}^{d} g_{jj}(x) \frac{\partial}{\partial B_1} + 2F(p^2) \sum_{j=2}^{d} g_{jj}(x) \frac{\partial}{\partial B_j}, & p \neq 0, \\
0, & p = 0,
\end{cases}
\]

(46)

where

\[
E(u) = \frac{M\left(\frac{3}{2}; 3 + \frac{d}{2}; -u\right)}{M\left(\frac{1}{2}; 1 + \frac{d}{2}; -u\right)}, \quad F(u) = \frac{M\left(\frac{1}{2}; 1 + \frac{d}{2}; -u\right)}{M\left(\frac{3}{2}; 3 + \frac{d}{2}; -u\right)}.
\]

(47)

Then, \( Y_{g,h} \) has the following asymptotic expansion:

\[ Y_{g,h}(f)(x) = f(x) + h^2 \left( L_{g,x} - \frac{p}{(d+2)(d+4)} Q_{g,x} \right) f(x) + o(h^2), \quad h \to 0. \]

(48)

The following corollary presents the special form of \( Q_{g,x} \) for \( d = 2 \).

**Corollary 9** When \( d = 2 \), the functions \( E(u) \) and \( F(u) \) in (47) have the following forms.

\[
E(u) = \frac{4\left( -p^2I_0 \left( \frac{2}{2} \right) + (p^2 + 4)I_1 \left( \frac{2}{2} \right) \right)}{p^2(I_0 \left( \frac{2}{2} \right) + I_1 \left( \frac{2}{2} \right))}, \quad F(u) = \frac{4\left( p^2I_0 \left( \frac{2}{2} \right) + (p^2 - 4)I_1 \left( \frac{2}{2} \right) \right)}{p^2(I_0 \left( \frac{2}{2} \right) + I_1 \left( \frac{2}{2} \right))}.
\]

(49)

Since in application we are more interested in the nonlinear YLM operator \( Y_h \), we give the asymptotic expansion of \( Y_h \) in the following:

**Theorem 10** Let \( f \in C^2 \) and \( B = \{ b_1, b_2, \ldots, b_d \} \) be a local o.n. basis at \( x \) associated with \( f \). Write \( p = \| \nabla f(x) \| \) and let \( A(u) \) and \( B(u) \) be the functions in (13). Then, the nonlinear YLM operator \( Y_h \) has the following asymptotic expansion:

\[ Y_h f(x) = f(x) + \frac{h^2}{2(d+2)} \left( G(p^2) \sum_{j=2}^{d} \frac{\partial^2 f(x)}{\partial b_j^2} + A(p^2) \sum_{j=2}^{d} \frac{\partial^2 f(x)}{\partial b_j^2} \right) + o(h^2), \quad h \to 0. \]

(50)
Lemma 4 Assume \( g \in C^1 \), \( d \geq 2 \), and \( p = \| \nabla g(x) \| \neq 0 \). Let \( \mathcal{B} = \{ \mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_d \} \) be a local o.n. basis at \( x \) associated with \( g \). Then we have

\[
m_h(W_{g,h}, z^1) = \frac{3h^{4+d} \pi^{d/2}}{4\Gamma(3 + \frac{d}{2})} M \left( \frac{5}{2}; 3 + \frac{d}{2}; -p^2 \right).
\]

and, for \( 2 \leq j \leq d \),

\[
m_h(W_{g,h}, z^2_1 z^2_j) = \frac{h^{4+d} \pi^{d/2}}{4\Gamma(3 + \frac{d}{2})} M \left( \frac{3}{2}; 3 + \frac{d}{2}; -p^2 \right).
\]

Proof. We have

\[
m_h(W_{g,h}, z^1) = \int_{B_h} z^1 \exp \left( -\frac{p^2 z^2}{h^2} \right) dz
= \frac{(d-1) \pi^{d/2-1/2}}{\Gamma \left( \frac{d+1}{2} \right)} \int_0^h \int_0^{2\pi} \exp \left( -\frac{p^2 r^2 \cos^2(\theta)}{h^2} \right) \sin^{d-2}(\theta) d\theta dr
= \frac{3h^{4+d} \pi^{d/2}}{4\Gamma(3 + \frac{d}{2})} M \left( \frac{5}{2}; 3 + \frac{d}{2}; -p^2 \right)
\]

and, for \( 2 \leq j \leq d \),

\[
m_h(W_{g,h}, z^2_1 z^2_j) = \int_{B_h} z^1 z^2_j \exp \left( -\frac{p^2 z^2}{h^2} \right) dz
= \frac{\pi^{d/2-1/2}}{\Gamma \left( \frac{d+1}{2} \right)} \int_0^h \int_0^{2\pi} \exp \left( -\frac{p^2 r^2 \cos^2(\theta)}{h^2} \right) \sin^{d-2}(\theta) d\theta dr
= \frac{h^{4+d} \pi^{d/2}}{4\Gamma \left( 3 + \frac{d}{2} \right)} M \left( \frac{3}{2}; 3 + \frac{d}{2}; -p^2 \right).
\]

The lemma is proved. \( \blacksquare \)

Lemma 4 Assume \( g \in C^2 \) and \( p = \| \nabla g(x) \| \neq 0 \). Then

\[
D_{g,h}(x) = S_{g,h}(x) + o(h^3), \quad h \to 0.
\]

(55)
Proof. Let $B$ be a local o.n. basis at $x$ associated with $g$. We have

$$K_{g,h}(x,z) = \exp \left( -\frac{1}{\hbar^2} \left( \frac{p z_1 + \sum_{i=1}^{d} \sum_{j=1}^{d} g_{ij}(x) z_i z_j}{2} + o(|z^2|) \right)^2 \right)$$

$$= e^{-\frac{p z_1^2}{\hbar^2}} \left( 1 - \frac{p z_1}{\hbar^2} \sum_{i=1}^{d} \sum_{j=1}^{d} g_{ij}(x) z_i z_j + o(|z^3|) \right),$$

which yields

$$D_{g,h}(x) = m_h(K_{g,h}, 1) = m_h(W_{g,h}, 1) - \frac{p}{\hbar^2} \sum_{i=1}^{d} \sum_{j=1}^{d} g_{ij}(x)m_h(W_{g,h}, z_1 z_i z_j) + o(h^3).$$

By (19), $m_h(W_{g,h}, z_1 z_i z_j) = 0$, which yields (55). □

Proof of Theorem 8. When $p = 0$, by simple computation, we obtain

$$Y_{g,h} f(x) = f(x) + \frac{\hbar^2}{2(d+2)} \Delta f(x) + o(h^2).$$

When $p \neq 0$, by (56) and (26), we have

$$K_{g,h}(x)(f(x + z) - f(x))$$

$$= W_{g,h}(x,z) \left( \sum_{j=1}^{d} f_j(x) z_j + \sum_{j=1}^{d} \sum_{j=1}^{d} f_{ij}(x) z_i z_j \frac{1}{2} - \frac{2p z_1}{\hbar^2} \left( \sum_{j=1}^{d} f_j(x) z_j \right) + \sum_{j=1}^{d} \sum_{j=1}^{d} g_{ij}(x) z_i z_j + o(|z^2|) \right),$$

which yields

$$Y_{g,h}(f)(x) = f(x) + \frac{1}{2} \sum_{j=1}^{d} \frac{m_h(W_{g,h}, z_j^2)}{D_{g,h}(x)} f_{jj}(x) - \frac{2p z_1}{\hbar^2} \sum_{j=1}^{d} \frac{m_h(W_{g,h}, z_j^2)}{D_{g,h}(x)} g_{jj}(x) f_j(x)$$

$$- \frac{p}{\hbar^2} \left( m_h(W_{g,h}, z_1^4) g_{11}(x) + \sum_{j=2}^{d} \frac{m_h(W_{g,h}, z_1^2 z_j^2)}{D_{g,h}(x)} g_{jj}(x) \right) f_1(x) + o(h^2).$$

By Lemmas 1, 3, and 4, we have

$$\frac{1}{2} \sum_{j=1}^{d} \frac{m_h(W_{g,h}, z_j^2)}{D_{g,h}(x)} f_{jj}(x) = L_{g,h} f(x) + o(h^3),$$

$$\frac{m_h(W_{g,h}, z_1^4)}{D_{g,h}(x)} = \frac{3h^4}{(d+4)(d+2)} M \left( \frac{3}{2}; 3 + \frac{d}{2}; -p^2 \right) + o(h^3),$$

$$\frac{m_h(W_{g,h}, z_1^2 z_j^2)}{D_{g,h}(x)} = \frac{h^4}{(d+4)(d+2)} M \left( \frac{3}{2}; 3 + \frac{d}{2}; -p^2 \right) + o(h^3).$$

Hence, (48) holds. The proof is completed. □

Proof of Corollary 9. By the identities (31), (33), and

$$b M(a; b; z) - z M(a; b + 1; z) - b M(a - 1; b; z) = 0,$$
we have

\[ M\left(\frac{3}{2}; 2; -p^2\right) = \frac{3}{p^2} \left( M\left(\frac{1}{2}; 3; -p^2\right) - M\left(\frac{3}{2}; 3; -p^2\right) \right) \]

\[ = 4e^{-\frac{\pi^2}{p^2}} \left( I_0 \left( \frac{p^2}{2} \right) + \left( 1 - \frac{1}{p^2} \right) I_1 \left( \frac{p^2}{2} \right) - \frac{3}{p^2} I_1 \left( \frac{p^2}{2} \right) \right) \]

\[ = 4e^{-\frac{\pi^2}{p^2}} \left( p^2 I_0 \left( \frac{p^2}{2} \right) + (p^2 - 4) I_1 \left( \frac{p^2}{2} \right) \right). \]  

(62)

Then, by (32) and (62), we have

\[ M\left(\frac{5}{2}; 4; -p^2\right) = 2M\left(\frac{3}{2}; 3; -p^2\right) - M\left(\frac{3}{2}; 4; -p^2\right) \]

\[ = 4e^{-\frac{\pi^2}{p^2}} \left( -p^2 I_0 \left( \frac{p^2}{2} \right) + (p^2 + 4) I_1 \left( \frac{p^2}{2} \right) \right). \]  

(63)

Finally, by (63), (62), and (30), we obtain (49). \( \square \)

**Proof of Theorem 10.** By \( Y_h f(x) = Y_{f,h} f(x) \), we have \( \frac{\partial f}{\partial x} = \| \nabla f(x) \| = p \) and \( \frac{\partial f}{\partial x} = 0, 2 \leq j \leq d \). Then, the operator \( Q_{f,x} \) is reduced to

\[ Q_{f,x} = 3pE(p^2)D_{11}f(x) + pF(p^2) \sum_{j=2}^{d} D_{jj}f(x). \]

By (48), we obtain

\[ Y_h f(x) = f(x) + \frac{h^2}{2(d+2)} \left( A(p^2) - 6p^2 \frac{d}{d+4} E(p^2) \right) D_{11}f(x) \]

\[ + \frac{h^2}{2(d+2)} \left( B(p^2) - 2p^2 \frac{d}{d+4} F(p^2) \right) \sum_{j=2}^{d} D_{jj}f(x) + o(h^2), \quad h \to 0. \]  

(64)

By (61), we have

\[ M\left(\frac{3}{2}; 3 + \frac{d}{2}; -p^2\right) = \frac{4 + d}{2p^2} \left( M\left(\frac{1}{2}; 2 + \frac{d}{2}; -p^2\right) - M\left(\frac{3}{2}; 2 + \frac{d}{2}; -p^2\right) \right), \]  

(65)

and, by (32) and (65), we have

\[ M\left(\frac{5}{2}; 3 + \frac{d}{2}; -p^2\right) = \frac{4 + d}{3} \left( M\left(\frac{3}{2}; 2 + \frac{d}{2}; -p^2\right) - \frac{d + 1}{3} M\left(\frac{3}{2}; 3 + \frac{d}{2}; -p^2\right) \right) \]

\[ = \frac{4 + d}{6p^2} \left( (2p^2 + d + 1)M\left(\frac{3}{2}; 2 + \frac{d}{2}; -p^2\right) - (d + 1)M\left(\frac{1}{2}; 2 + \frac{d}{2}; -p^2\right) \right). \]  

(66)

Thus, (50) is yielded from (65) and (66). Finally, (52) is derived from (49) and (51). \( \square \)

The anisotropic diffusion behavior of the nonlinear YLM operator is characterized by \( G(p^2) \) and \( A(p^2) \), where the properties of \( A(p^2) \) is already given in Theorem 6. We present the properties of \( G(p^2) \) in the following:

**Theorem 11** Let the function \( G(u) \) be given by (51). Then \( G(0) = 1, \lim_{u \to \infty} G(u) = 0 \) and there is a \( u_0 \in (0, \infty) \) such that \( G(u) < 0 \) for \( u > u_0 \). Hence, \( G(u) \) has a zero in \( (0, \infty) \).
Proof. By \( A(0) = 1 \) and \( B(0) = 1 \), we get \( G(0) = 1 \). The limit \( \lim_{u \to \infty} G(u) = 0 \) is derived from (41) and (36). To prove that there is \( u_0 > 0 \) such that \( G(u) < 0 \) for \( u > u_0 \), we write

\[
G(u) = (d + 1)B(u) - (2u + d)A(u) = (d + 2) - (2u + d + 1)A(u)
\]

By \( \frac{\Gamma \left( \frac{3}{2} \right)}{\Gamma \left( 2 + \frac{d}{2} \right)} \left( (d + 2)M \left( \frac{1}{2}; 1 + \frac{d}{2}; -u \right) - (2u + d + 1)M \left( \frac{3}{2}; 2 + \frac{d}{2}; -u \right) \right) \)

By (39) and (40),

\[
\frac{\Gamma \left( \frac{3}{2} \right)}{\Gamma \left( 2 + \frac{d}{2} \right)} \left( (d + 2)M \left( \frac{1}{2}; 1 + \frac{d}{2}; -u \right) - (2u + d + 1)M \left( \frac{3}{2}; 2 + \frac{d}{2}; -u \right) \right)
\]

Hence, when \( u \) is large enough, \( G(u) < 0 \). The theorem is proved.

It can be seen that the diffusion behavior of \( Y_h \) is different from \( W_h \) when they act on a same function \( f \). In the gradient direction of \( f \), when the magnitude of the gradient is large, \( Y_h \) causes a backward diffusion, while \( W_h \) still produces a forward one although it becomes very weak. In the directions on the level space of \( f \), \( Y_h \) produces the diffusion that tapers off when the magnitude of the gradient becomes large, while \( W_h \) produces a nearly isotropic one. We show the graphs of \( G(p^2) \) and \( A(p^2) \) in Figure 3 for \( d = 2 \).

![Graphs](image.png)

Figure 3: The graphs of the functions \( G(p^2) \) (left) and \( A(p^2) \) (right) for \( d = 2 \).

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References


