On Spline Wavelets

Dedicated to Professor Charles Chui on the occasion of his 65th birthday

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Abstract. Spline wavelet is an important aspect of the constructive theory of wavelets. This paper consists of three parts. The first part surveys the joint work of Charles Chui and me on the construction of spline wavelets based on the duality principle. It also includes a discussion of the computational and algorithmic aspects of spline wavelets. The second part reviews our study of asymptotically time-frequency localization of spline wavelets. The third part introduces the application of Shannon wavelet packet in the sub-band decomposition in multiple-channel synchronized transmission of signals.

§1. Introduction

In this paper, I give a brief survey of my joint work with Charles Chui on spline wavelet. In the theory of wavelet analysis, an important aspect is to construct wavelet bases of a function space, say of the Hilbert space $L^2(\mathbb{R})$ ([32], [64], [65]). Recall that a (standard) wavelet basis of $L^2(\mathbb{R})$ is the basis generated by the dilatations and translates of a “wavelet” function $\psi$. More precisely, if the set of functions

$$\{\psi_{j,k}\}_{j,k \in \mathbb{Z}}, \quad \psi_{j,k} = 2^{j/2}\psi(2^j x - k), \quad (1)$$

forms an unconditional basis (also called a Riesz basis) of $L^2(\mathbb{R})$, then we call $\psi$ a wavelet and call the system (1) a wavelet basis. Three of most basic properties of a wavelet are its regularity, its decay speed, and the order of its vanishing moments. Therefore, a wavelets is usually chosen to be compactly supported (or exponentially decay) and to have certain degree of regularity so that they are “local” in both $x$-domain (time domain) and $\omega$-domain (frequency domain). Then each element in the
wavelet basis (1) has a finite time-frequency window. Thus, wavelet bases are useful tools for “local” analysis of functions. Many books and papers already explain the importance of wavelet bases in harmonic analysis and in various applications. (See [6], [7], [32], [49], [64], [65].)

In the earlier time in the history of wavelet analysis, people tried to construct a wavelet basis by finding the wavelet function \( \psi \) directly. It is Mayer and Mallat who create Multiresolution Analysis (MRA) [64], [65], which provides a powerful framework for the construction of wavelet bases.

**Definition 1.** A multiresolution analysis of \( L^2 \) is a nest of subspaces of \( L^2 \):

\[
\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots
\]

that satisfies the following conditions.

1. \( \bigcap_{j \in \mathbb{Z}} V_j = \{0\} \),
2. \( \bigcup_{j \in \mathbb{Z}} V_j = L^2 \),
3. \( f(\cdot) \in V_j \) if and only if \( f(2\cdot) \in V_{j+1} \), and
4. there exists a function \( \phi \in V_0 \) such that \( \{ \phi(x-n) \}_{n \in \mathbb{Z}} \) is an unconditional basis of \( V_0 \), i.e., \( \{ \phi(x-n) \}_{n \in \mathbb{Z}} \) is a basis of \( V_0 \), and there exist two constants \( A, B > 0 \) such that, for all \( (c_n) \in l^2 \), the following inequality holds:

\[
A \sum_{n \in \mathbb{Z}} |c_n|^2 \leq \left\| \sum_{n \in \mathbb{Z}} c_n \phi(\cdot - n) \right\|^2 \leq B \sum_{n \in \mathbb{Z}} |c_n|^2.
\] (2)

The function \( \phi \) described in Definition 1 is called a **MRA generator**. Furthermore, if \( \{ \phi(x-n) \}_{n \in \mathbb{Z}} \) is an orthonormal basis of \( V_0 \), then \( \phi \) is called an **orthonormal MRA generator**. Since \( \phi \in V_0 \) is also in \( V_1 \), we can establish a **two-scale equation** for \( \phi \):

\[
\phi(x) = 2 \sum_{m \in \mathbb{Z}} h_m \phi(2x - m), \quad (h_m)_{m \in \mathbb{Z}} \in l^2,
\] (3)

where \( h = (h_m)_{m \in \mathbb{Z}} \) is called the **mask** of \( \phi \).

Taking the Fourier transform of (3), we obtain

\[
\hat{\phi}(\omega) = H(e^{-i\omega/2}) \hat{\phi}(\omega/2), \quad H(z) = \sum_{m \in \mathbb{Z}} h_m z^m,
\] (4)

which represents the two-scale equation of \( \phi \) in the frequency domain. Here, \( H(z) \) is the **symbol** of \( \phi \).

The MRA approach to wavelet basis can be described as follows. Let \( W_j \) be a complement of \( V_j \) with respect to \( V_{j+1} \):

\[
W_j \oplus V_j = V_{j+1}, \quad j \in \mathbb{Z}.
\]
Then we have

\[ L^2(\mathbb{R}) = \bigoplus_{j \in \mathbb{Z}} W_j, \quad W_j \cap W_k = \{0\}, \quad j \neq k, \]
\[ g \in W_j \iff g(2^j) \in W_{j+1}, \quad j \in \mathbb{Z}. \]

Let \( \psi \in W_0 \) be such a function that \( \{\psi(\cdot - n)\}_{n \in \mathbb{Z}} \) forms a Riesz basis of \( W_0 \). Then \( \{\tilde{\psi}_{j,k}\}_{j,k \in \mathbb{Z}} \) forms a wavelet basis of \( L^2(\mathbb{R}) \). Since \( W_0 \subset V_1 \), there is a sequence \( g \in l^2 \) such that the function \( \psi \) satisfies

\[ \psi(t) = 2 \sum_{k \in \mathbb{Z}} g_k \phi(2^j - k), \quad g \in l^2, \quad (5) \]

where \( g = (g_k) \) is the mask of \( \psi \). The Fourier transform of \( \psi \) is

\[ \hat{\phi}(\omega) = G(e^{-i\omega/2}) \hat{\phi}(\omega/2), \quad G(z) = \sum_{m \in \mathbb{Z}} g_m z^m, \]

where \( G(z) \) is the symbol of \( \psi \). According to the analysis above, once we have an MRA generator \( \phi \) (and its mask \( h \)) to construct a wavelet basis becomes to find the mask \( g \) of \( \psi \).

Let us consider the decomposition of a function \( f \in L^2 \) into the wavelet series:

\[ f = \sum_{j,k \in \mathbb{Z}} d_{j,k} \psi_{j,k} \quad (6) \]

or

\[ f = \sum_{j \in \mathbb{Z}} c_{j,k} \phi_{j,k} + \sum_{j \geq J \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} d_{j,k} \psi_{j,k}. \quad (7) \]

To find the coefficients \( (d_{j,k}) \) and \( (c_{j,k}) \) in the decompositions (6) and (7), we need the dual bases of \( \{\phi_{j,k}\} \) and \( \{\psi_{j,k}\} \), say \( \tilde{\phi}_{j,k} \) and \( \tilde{\psi}_{j,k} \), respectively. The theory of the construction of wavelet bases studies the methods to find \( \{\psi_{j,k}\}, \{\tilde{\phi}_{j,k}\}, \) and \( \{\tilde{\psi}_{j,k}\} \) from \( \{\phi_{j,k}\} \).

Among various wavelet bases, spline wavelet bases play an important role due to their beautiful structure and powerful ability in computation. Spline wavelet is an important aspect of the constructive theory of wavelets. The paper consists of three parts. The first part surveys the joint work of Charles Chui and me on the construction of spline wavelets based on the duality principle. It also includes a discussion of the computational and algorithmic aspects of spline wavelets. The second part review our study of asymptotically time-frequency localization of spline wavelets. The third part introduces the applications of Shannon wavelet packet in the sub-band decomposition in multiple-channel synchronized transmission of signals. As a survey paper, the proofs of all results will not be included. Instead, we only refer the original papers which show the proofs in details.
§2. Duality Principle and Construction of Spline Wavelets

As we mentioned in the introduction, to construct a wavelet basis based on an MRA generator \( \phi \) and to compute the coefficients of the wavelet series, we need to find the relation among \( \phi, \psi, \tilde{\phi}, \) and \( \tilde{\psi}. \) Their relations can be formulated to Duality principle. Charles and I established the duality principle in [16], [17], and [18]. Its comprehensive description was included in Charles’ book [6]. The book and the papers were published more than ten years ago. Since then, a lot of generalizations of the duality principle have been developed for multivariate wavelet bases (see [44], [45], [46], [47], [48], [53], [54], [56], [57], [58], [66], [67], [68]), multiwavelet bases (see [14], [36], [37]), and wavelet frames (see [8], [9], [10], [12], [34], [70], [71], [72]). The references listed here are far from complete ones. The readers can refer the references in the papers mentioned above. However, today giving a review of the original idea of the duality principle still makes sense. I now introduce the principle of its original formulation in a concise way.

**Definition 2.** Assume a scaling function

\[
\phi(t) = 2 \sum_{k \in \mathbb{Z}} h(k) \phi(2t - k)
\]

(8)

generates an MRA \( \{ V_j \}_{j \in \mathbb{Z}}. \) If a scaling function

\[
\tilde{\phi}(t) = 2 \sum_{k \in \mathbb{Z}} \tilde{h}(k) \tilde{\phi}(2t - k)
\]

(9)
satisfies

\[
< \phi_{0,n}, \tilde{\phi}_{0,m} >= \delta_{nm},
\]

(10)

then \( \phi \) is called a dual scaling function of \( \tilde{\phi}. \)

The dual scaling function \( \tilde{\phi} \) also generates an MRA \( \{ \tilde{V}_j \}_{j \in \mathbb{Z}}, \) called a dual MRA of \( \{ V_j \}_{j \in \mathbb{Z}}. \) It is not hard to see that there is a complement of \( \tilde{V}_j \) with respect to \( \tilde{V}_{j+1}, \) say \( \tilde{W}_j, \) which satisfies

\[
L^2(\mathbb{R}) = \bigoplus_{j \in \mathbb{Z}} \tilde{W}_j, \quad \tilde{W}_j \cap \tilde{W}_k = \{0\}, \quad j \neq k,
\]

\[
\tilde{g} \in \tilde{W}_j \iff \tilde{g} (2^j) \in \tilde{W}_{j+1}, \quad j \in \mathbb{Z}.
\]

and there is a wavelet \( \tilde{\psi} \in \tilde{W}_0 \) such that \( \{ \tilde{\psi}_{jk} \}_{k \in \mathbb{Z}} \) forms a Riesz basis of \( \tilde{W}_j \) and

\[
\langle \tilde{\psi}_{i,k}, \tilde{\psi}_{j,l} \rangle = \delta_{ij} \delta_{kl}, \text{ for all } i,j,k,l \in \mathbb{Z}.
\]

(11)

We call \( \tilde{\psi} \) a dual wavelet of \( \psi \) and call \( \{ \psi_{jk} \}_{j,k \in \mathbb{Z}} \) and \( \{ \tilde{\psi}_{jk} \}_{j,k \in \mathbb{Z}} \) dual bases of \( L^2. \) There is a sequence \( \tilde{g} \in l^2 \) such that the function \( \tilde{\psi} \) satisfies

\[
\tilde{\psi}(t) = 2 \sum_{k \in \mathbb{Z}} \tilde{g}(k) \tilde{\phi}(2t - k).
\]

(12)
Let orthonormal scaling functions and wavelets.

Semi-orthogonal scaling functions and wavelets.

Now see how to apply the duality principle to some special cases.

Satisfy the condition such that it generates a stable scaling function \( \tilde{\phi} \) by

\[ \sum_k h(k)\tilde{h}(k - 2l) = \delta_{0l}, \]

\[ \sum_k g(k)\tilde{g}(k - 2l) = \delta_{0l}, \]

\[ \sum_k h(k)\tilde{g}(k - 2l) = 0, \]

\[ \sum_k \tilde{h}(k)g(k - 2l) = 0, \]

which is equivalent to

\[
\begin{align*}
\overline{H}(z)\tilde{H}(z) + \overline{H}(-z)\tilde{H}(-z) &= 1, \\
G(z)\tilde{G}(z) + G(-z)\tilde{G}(-z) &= 1, \\
\overline{G}(z)\tilde{H}(z) + G(-z)\tilde{H}(-z) &= 0, \\
G(z)\tilde{G}(z) + \overline{G}(-z)\tilde{H}(-z) &= 0.
\end{align*}
\]

where \( H(z) = \sum h(k)z^k, \tilde{H}(z) = \sum \tilde{h}(k)z^k, G(z) = \sum g(k)z^k, \) and \( \tilde{G}(z) = \sum \tilde{g}(k)z^k. \)

By duality principle, finding \( \tilde{\phi}, \psi, \) and \( \tilde{\psi} \) from \( \phi, \) can be completed by finding \( \tilde{H}(z), G(z), \) and \( \tilde{G}(z) \) from \( H(z). \) Here the symbol \( \tilde{H}(z) \) must satisfy the condition such that it generates a stable scaling function \( \phi. \) We now see how to apply the duality principle to some special cases.

1. **Orthonormal scaling functions and wavelets** (see [63], [64]). In this case, \( \phi = \tilde{\phi} \) and \( \psi = \tilde{\psi}. \) Hence, \( \tilde{H} = H \) and \( \tilde{G} = G. \) The equation (13) is reduced to

\[
\begin{align*}
|H(z)|^2 + |H(-z)|^2 &= 1, \\
G(z) &= z^{2l-1}H(-z), \\
\tilde{H}(z) &= H(z), \\
\tilde{G}(z) &= G(z),
\end{align*}
\]

where \( \Gamma = \{z \in \mathbb{C}; \ |z| = 1\}. \)

2. **Semi-orthogonal scaling functions and wavelets** (see [17], [18]). In this case, \( V_j = \tilde{V}_j \) and \( W_j = \tilde{W}_j. \) Let \( \sum_{k \in \mathbb{Z}} |\phi(\omega + 2k\pi)|^2 = \Pi(e^{-i\omega}). \) Then we have

\[
|H(z)|^2 \Pi(z) + |H(-z)|^2 \Pi(-z) = \Pi(z^2).
\]

Applying the equation (13), we have

\[
\begin{align*}
\tilde{H}(z) &= \frac{H(z)\Pi(z)}{H(z)\Pi(z)}, \\
\tilde{G}(z) &= z^{2l-1}H(-z)\Pi(-z)K(z^2), \\
G(z) &= \frac{z^{2l-1}H(z)\Pi(z)K(z^2)}{\Pi(z^2)},
\end{align*}
\]

where \( \Gamma = \{z \in \mathbb{C}; \ |z| = 1\}. \)
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where $K$ is in the Wiener class $W$ and $K(z) \neq 0$ on $\Gamma$.

3. **Compactly supported, biorthogonal scaling functions and wavelets** (see [25]). In this case, $H$ and $\tilde{H}$ both are finite Laurent polynomials. The equation (13) now is reduced to

\[
\overline{H(z)} \tilde{H}(z) + \overline{\tilde{H}(-z)} H(-z) = 1, \\
G(z) = z^{2l-1} \overline{\tilde{H}(-z)}, \\
\tilde{G}(z) = z^{2l-1} \overline{H(-z)},
\]

$z \in \Gamma$. (15)

If $\phi$ (i.e., $H(z)$) is given, then $\tilde{H}$ can be solved from the first equation in (15). The solution $\tilde{H}$ is not unique. We choose those symbols $\tilde{H}$ such that they generate stable scaling functions $\phi$ in $L^2$. Then using the remain two equations of (15) we generate $\psi$ and $\tilde{\psi}$ respectively.

We now apply the duality principle to spline wavelets. Let $m$ be a positive integer and let $N_m$ denote the $m^{th}$ order B-spline with integer knots (simply, the $m^{th}$ order cardinal B-spline), which is defined recursively by

\[
N_m = N_{m-1} * N_1, \quad N_1 = \chi_{[0,1)}.
\]

The Fourier transform of $N_m$ is

\[
\hat{N}_m(\omega) = \left(\frac{1 - e^{-i\omega}}{i\omega}\right)^m.
\]

For any $k, j \in \mathbb{Z}$, we set

\[
N_{m,k,j}(x) = N_m(2^k x - j)
\]

and abbreviate it to $N_{k,j}(x)$ if $m$ is fixed. We also write $N_k(x) = N_{k,0}(x)$. The following important properties of $N_m$ make (cardinal) B-splines important generators of wavelet bases.

**Theorem 2.** ([30], [73]) The cardinal B-spline of order $m$ satisfies the following:

1. $\text{supp } N_m = [0, m]$, and $\{N_{m,k}\}_{k \in \mathbb{Z}}$ is locally linearly independent on any open interval $(a, b) \subset \mathbb{R}$,
2. $N_m \in C^{m-2}(\mathbb{R})$, and $N_m$ is a polynomial of exact degree $m - 1$ on each interval $[k, k + 1], 0 \leq k \leq m - 1$, and the $m^{th}$ order splines achieve approximation order $m$,
3. $N_m(x) > 0$, for all $x \in (0, m)$. It satisfies the scaling equation $N_m = \frac{1}{m} \sum_{k=0}^{m-1} \binom{m}{k} N_{m,k,1}$, and therefore its symbol is $(1 + z^2)^m$,
4. $N_m(x)$ is symmetric with respect to $x = m/2$:

\[
N_m(x) = N_m(m - x). \quad (16)
\]
By Theorem 2, the cardinal B-spline of order \( m \) generates an MRA
\[
\cdots \subset V_{m,-1} \subset V_{m,0} \subset V_{m,1} \cdots
\]
where \( V_{m,0} = \text{span}\{N_{m,k}\}_{k \in \mathbb{Z}} \).

Let \( \{W_{m,k}\}_{k \in \mathbb{Z}} \) be the corresponding wavelet subspace sequence such that \( V_{m,k} \oplus W_{m,k} = V_{m,k+1} \), and \( V_{m,k} \perp W_{m,k} \). A wavelet in \( W_{m,0} \), which generates the wavelet basis, is called a spline-wavelet.

The Euler-Frobenius polynomial (of order \( m-1 \)):
\[
E_{m-1}(z) = m - 2 \sum_{j=0}^{\infty} N_m(j+1)z^j, \quad m \geq 2
\]
plays an important role in the construction of spline-wavelets. When \( m = 2n \), the Euler-Frobenius polynomial has an even degree and we rewrite it in a symmetric form:
\[
\Pi_n(z) = z^{-n+1}E_{2n-1}(z), \quad n \geq 1.
\]

**Theorem 3.** [73] The Laurent polynomial \( \Pi_n(z) \) has the following properties.

1. \( \Pi_n(z) > 0, \forall z \in \Gamma, \) and all of its \( 2n-2 \) roots are positive. Assume \( \{r_j\}_{j=1}^{2n-2} \) are all zeros of \( \Pi_n(z) \) arranged increasingly:
\[
0 < r_{n,1} < \cdots < r_{n,n-1} < 1 < r_{n,n} < \cdots < r_{n,2n-2}.
\]

Then
\[
(17) \quad r_{n,j}r_{n,2n-1-j} = 1.
\]

2. It satisfies the following identity.
\[
P_n(z)\Pi_n(z) + P_n(-z)\Pi_n(-z) = \Pi_n(z^2). \tag{18}
\]

where \( P_n(z) = (1+z^2)^n \left( \frac{1+z^{-1}}{2} \right)^n \).

Applying the duality principle and (18), we can construct various spline wavelets.

1. **Orthonormal spline-wavelets of the order** \( n \). [Lemarié (1988), Mallat (1989)] Let the orthonormal scaling spline be denoted by \( N_n^\perp \) and the corresponding wavelet denoted by \( \Psi_n \). Then the symbol of \( N_n^\perp \) is
\[
H_n^\perp(z) = \left( \frac{1+z}{2} \right)^n \sqrt{\Pi_n(z) \sqrt{\Pi_n(z^2)}},
\]
and the Fourier transform of the orthonormal wavelet \( \Psi \) is
\[
\hat{\Psi}_n(\omega) = -e^{-i\omega/2}H_n^\perp(-e^{-i\omega/2})\hat{N}_n^\perp(\omega/2).
\]
Note that $N_n^\perp$ is a linear combination of $\{N_{n,k}\}_{k \in \mathbb{Z}}$. Let
\[
\frac{1}{\sqrt{\Pi_n(e^{-i\omega})}} = \sum_{k \in \mathbb{Z}} c_k e^{-ik\omega}.
\]
Then we have $N_n^\perp = \sum_{k \in \mathbb{Z}} c_k N_{n,k}$.

2. **Compactly supported, semi-orthogonal spline-wavelet of order $n$.** [Chui and Wang (1992)] Let $N_n$ be the scaling function. In (14), choosing $K(z) = 1, l = n$, we have
\[
\hat{\psi}_n(\omega) = G_n(e^{-i\omega/2})\hat{N}_n\left(\frac{\omega}{2}\right), \quad G_n(z) = z^{2n-1}\left(1 - \frac{z^{-1}}{2}\right)^n \Pi_n(-z).
\]
The support of $\psi_n$ is $[0, 2n - 1]$. Hence, $\psi_n$ is a compactly supported, semi-orthogonal spline-wavelet corresponding to $N_n$. The symbol of the dual of $N_n$ is
\[
\hat{H}_n(z) = \left(\frac{1+2z}{2}\right)^n \Pi(z) / \Pi(z^2),
\]
and the symbol of the dual of $\psi_n$ is
\[
\hat{G}_n(z) = z^{2n-1}\left(1 - \frac{z^{-1}}{2}\right)^n \Pi(z^2).
\]

3. **Interpolating spline-wavelets of order $n$.** [Chui and Wang 1991, Chui and Li 1993] An interpolating spline (also called a cardinal spline) of order $2n$, denoted by $L_{2n} \in V_{2n,0}$, is defined by $L_{2n}(k) = \delta_{0,k}, \forall k \in \mathbb{Z}$. By the identity $\sum_{k \in \mathbb{Z}} \hat{L}_{2n}(\omega + 2k\pi) = 1$, the interpolating spline $L_{2n}$ is represented by the B-spline $N_{2n}$ in the following way.
\[
\hat{L}_{2n}(\omega) = \frac{e^{i\omega} \hat{N}_{2n}(\omega)}{\Pi_n(e^{-i\omega})}.
\]
There is an interesting relation between $L_{2n}$ and a semi-orthogonal spline wavelet of order $n$. From (19), we have
\[
e^{-i\omega/2} \hat{L}_{2n}^{(n)}(\omega/2) = (-1)^n e^{-i\omega/2}(1 - e^{i\omega/2})^n \Pi_n(e^{-i\omega/2}) \hat{N}_n(\omega/2)
\]
\[
= G(e^{-i\omega/2})\hat{N}_n(\omega/2)
\]
where
\[
G(z) = z \left(\frac{1 - z^{-1}}{2}\right)^n \Pi_n(-z) K(z^2), \quad K(z^2) = \frac{(-2)^n}{\Pi_n(z)\Pi_n(-z)}.
\]
Hence, by (14), $\psi^I_n := L_{2n}^{(n)}(2x - 1 - n) \in W_{n,0}$ is a semi-orthogonal spline-wavelet of order $n$, which is called an interpolating spline-wavelet. Applying (14), we obtain the symbols of the duals of $N_n$ and $\psi_n^I$, respectively.
§3. Computational of Spline-Wavelets

Wavelet bases are enable us to develop fast algorithms for decomposing a function into a wavelet series and recovering a function from its wavelet series, which are called Fast Wavelet Transform (FWT) algorithms and Fast Inverse Wavelet Transform (FIWT) algorithms respectively. Let \( \psi \) and \( \tilde{\psi} \) be a pair of dual wavelets. Then a function \( f \in L^2 \) can be expanded as a wavelet series:

\[
\begin{align*}
f &= \sum_{j,k \in \mathbb{Z}} \tilde{b}_{j,k} \psi_{j,k} \\
&\quad \text{(or } f = \sum_{j,k \in \mathbb{Z}} b_{j,k} \tilde{\psi}_{j,k} \text{)}
\end{align*}
\]

where

\[
\begin{align*}
\tilde{b}_{j,k} &= \int_{-\infty}^{\infty} f(t) \tilde{\psi}_{j,k}(t) \, dt \\
b_{j,k} &= \int_{-\infty}^{\infty} f(t) \psi_{j,k}(t) \, dt
\end{align*}
\]

However, this expansion is not effective in computation. In scientific computation, we need to digitalize functions. Let \( \phi \) and \( \tilde{\phi} \) be the dual scaling functions corresponding to \( \psi \) and \( \tilde{\psi} \) respectively, and they generate the dual MRA \( \{V_n\} \) and \( \{\tilde{V}_n\} \). Assume \( \phi, \tilde{\phi}, \psi, \) and \( \tilde{\psi} \) are defined by (8), (9), (5) and (12) respectively. To discretize a function \( f \in L^2 \), we choose a sufficient large \( n \) such that \( ||f - f_n|| \) is smaller than a tolerance \( \epsilon \), where \( f_n \in V_n \) is a projection of \( f \) on \( V_n \). Let \( c_n = (c_{nm}) \) be the coefficient sequence of \( f_n \) : \( f_n = \sum_{m \in \mathbb{Z}} c_{nm} \phi_{nm} \). The sequence \( c_n \), as the discrete representation of \( f \), is the initial data for the fast wavelet transform. Assume we decompose \( f_n \) into

\[
f_n = f_0 + g_0 + \cdots + g_{n-1}, \tag{20}
\]

where

\[
f_j = \sum c_{j,k} \phi_{j,k} \in V_j
\]

and

\[
g_j = \sum d_{j,k} \psi_{j,k} \in W_j.
\]

Write \( c_j = (c_{j,k}) \) and \( d_j = (d_{j,k}) \). Then FWT derives \( c_j \) and \( d_j \) from \( c_{j+1} \).

**FWT algorithm**:

\[
\begin{align*}
c_{j,k} &= \sqrt{2} \sum \hat{h}(l-2k) c_{j+1,l} \\
d_{j,k} &= \sqrt{2} \sum \tilde{g}(l-2k) c_{j+1,l}
\end{align*}
\]

and FIWT recovers \( c_{j+1} \) from \( c_j \) and \( d_j \),

**FIWT algorithm**:

\[
\begin{align*}
c_{j+1,l} &= \sqrt{2} \sum h(l-2k) c_{j,k} + \sqrt{2} \sum h(l-2k) d_{j,k}.
\end{align*}
\]
Let $H, G, \tilde{H}, \tilde{G}$ be operators on $l^2$ defined by

\[
H a(n) = \sqrt{2} \sum_{k \in \mathbb{Z}} a_k h(n - 2k),
\]
\[
G a(n) = \sqrt{2} \sum_{k \in \mathbb{Z}} a_k g(n - 2k),
\]
\[
\tilde{H} a(n) = \sqrt{2} \sum_{k \in \mathbb{Z}} a_k \tilde{h}(k - 2n),
\]
\[
\tilde{G} a(n) = \sqrt{2} \sum_{k \in \mathbb{Z}} a_k \tilde{g}(k - 2n),
\]
respectively. Then, the FWT and FIWT algorithms can be represented as

\[
c_{j-1} = \tilde{H} c_j, \quad d_{j-1} = \tilde{G} d_j
\]
and

\[
c_j = H^* c_{j-1} + G^* d_{j-1}
\]
respectively. Iterating FWT and FIWT algorithms, we complete multilevel decomposition and recovering using the following decomposition pyramid algorithm

\[
\begin{array}{cccccc}
\vdots & \tilde{H} & \vdots & \tilde{H} & \vdots & \tilde{H} \\
\tilde{G} & \vdots & \tilde{G} & \vdots & \tilde{G} & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
d_n & \vdots & d_{n-2} & \vdots & d_2 & \vdots \\
\end{array}
\]

(23)

and recovering pyramid algorithm

\[
\begin{array}{cccccc}
\vdots & H & \vdots & H & \vdots & H \\
G & \vdots & G & \vdots & G & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_0 & \vdots & c_2 & \vdots & c_1 & \vdots \\
\end{array}
\]

(24)

Let us now analyze the FWT algorithm based on compactly supported, semi-orthogonal spline-wavelet of order $m$, in which the filters $\tilde{H}_m$ and $\tilde{G}_m$ are infinite impulse responses (IIR). Hence, to apply the FWT algorithm, we have to truncate $\tilde{H}_m$ to $\tilde{H}_{m,n} = \{\tilde{h}_k\}_{k=-n}^n$ and $\tilde{G}_m$ to $\tilde{G}_{m,n} = \{\tilde{g}_k\}_{k=-n}^n$. As an example, we analyze the truncate error of the filter $\tilde{H}_{m,n}$.

We estimate of the truncated error of the filter $\tilde{H}_{m,n}$ by

\[
E_n(H_m) = \|H_m - \tilde{H}_{m,n}\|_{l^2}.
\]

For any $c \in l^2$, we have

\[
\|\tilde{H}_m * c - \tilde{H}_{m,n} * c\|_{l^2} \leq E_n(H_m) \|c\|_{l^2},
\]

(25)

and the error estimate (25) is sharp. Charles and I (1992) obtain the following estimate [23].
Theorem 4. Let \( \tilde{H}_m \) be the symbol of the semi-orthogonal spline dual to \( N_m \). Let \( \{r_{m,k}\}_{k=1}^{2m-2} \) be the root set of \( \Pi_m(z) \) defined in (17). For any positive integer \( m \), there is an integer \( n_m \) such that, for all \( n \geq n_m \),

\[
\|\tilde{H}_m - \tilde{H}_{m,n}\|_2 \leq \left(2^{m-1} \sum_{j=1}^{m-1} \frac{r_{m,n}}{\pi_m(r_{m,k})}\right).
\]  

(26)

Particularly, the estimate (26) is true for all \( n \geq 0 \) when \( m = 2, 3, 4 \).

In application, two most useful splines are linear spline \((m = 2)\) and cubic spline \((m = 4)\). For these two splines, we have the following.

Corollary 1. We have

\[
\|\tilde{H}_2 - \tilde{H}_{2,n}\|_2 \leq 0.73205 \times (0.26795)^n, \quad \forall n \geq 0
\]

and

\[
\|\tilde{H}_4 - \tilde{H}_{4,n}\|_2 \leq 4.1952 \times (0.5352805)^n, \quad \forall n \geq 0.
\]

§4. Asymptotic characteristic of time-frequency localization

The wavelet transform gives localized time-frequency information of signals (or functions), which is measured by the time-frequency windows of scaling functions and wavelets [33]. Since a scaling function provides a low-pass filter while a wavelet provides a band-pass filter, their time-frequency measurements should be formulated in different way. Let \( \phi \in L^2 \) and \( \hat{\phi} \) be its Fourier transform defined by

\[
\hat{\phi}(\omega) = \int_{-\infty}^{\infty} \phi(x)e^{-i\omega x} dx.
\]

In the time-frequency analysis, the time-frequency window of \( \phi \) is formulated as follows. Its time center is defined by

\[
t_\phi = \lim_{N \to \infty} \int_{-N}^{N} x |\phi(x)|^2 dx / \int_{-\infty}^{\infty} |\phi(x)|^2 dx
\]

and its frequency center is defined by

\[
w_\phi = \lim_{N \to \infty} \int_{-N}^{N} \omega |\hat{\phi}(\omega)|^2 d\omega / \int_{-\infty}^{\infty} |\hat{\phi}(\omega)|^2 d\omega.
\]

The quantities

\[
\Delta_\phi = \left(\int_{-\infty}^{\infty} (x - t_\phi)^2 |\phi(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |\phi(x)|^2 dx\right)^{\frac{1}{2}}
\]
and 

\[ \Delta \hat{\phi} = \left( \frac{\int_{-\infty}^{\infty} (\omega - \omega_0)^2 |\hat{\phi}(\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |\hat{\phi}(\omega)|^2 d\omega} \right)^{\frac{1}{2}} \]

are called the time and frequency localization radii of \( \phi \). If both \( \Delta_\phi \) and \( \Delta \hat{\phi} \) are finite, we say that \( \phi \) is a window function and denote the measure of its time-frequency window by

\[ M(\phi) = \Delta_\phi \Delta \hat{\phi}. \]

The following uncertainty principle is well known.

**Theorem 5.** If \( \phi \) is a window function, then

\[ M(\phi) \geq \frac{1}{2} \]

and equality holds if and only if

\[ \phi = kG_\sigma(x - \mu), \quad k \neq 0, \quad \sigma > 0, \quad \mu \in \mathbb{R}, \]

where

\[ G_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma^2}} \]

is a Gaussian function.

Note that the Gaussian function \( G_\sigma \) is not a scaling function.

Because a wavelet \( \psi \) satisfies \( \hat{\psi}(0) = 0 \), from the point of view of signal processing, it is a bandpass filter. As a bandpass filter, \( \hat{\psi} \) treats positive and negative frequency bands separately. Therefore, the notion of the frequency window of a bandpass filter \( \psi \) has to be modified. More precisely, we have to consider two frequency centers for a bandpass filter \( \psi \). The positive frequency center is defined by

\[ \omega^+_{\psi} = \frac{\int_{0}^{\infty} \omega |\hat{\psi}(\omega)|^2 d\omega}{\int_{0}^{\infty} |\hat{\psi}(\omega)|^2 d\omega} \]

and the negative frequency center is defined by

\[ \omega^-_{\psi} = \frac{\int_{-\infty}^{0} \omega |\hat{\psi}(\omega)|^2 d\omega}{\int_{-\infty}^{0} |\hat{\psi}(\omega)|^2 d\omega}. \]
Then the positive and negative frequency localization radii of $\psi$ are defined respectively by

$$
\Delta^+_{\psi} = \left( \int_0^\infty (\omega - \omega^+_{\hat{\psi}})^2 |\hat{\psi}(\omega)|^2 \, d\omega \right)^{\frac{1}{2}} /
\left( \int_0^\infty |\hat{\psi}(\omega)|^2 \, d\omega \right)^{\frac{1}{2}}
$$

and

$$
\Delta^-_{\psi} = \left( \int_{-\infty}^0 (\omega - \omega^-_{\hat{\psi}})^2 |\hat{\psi}(\omega)|^2 \, d\omega \right)^{\frac{1}{2}} /
\left( \int_{-\infty}^0 |\hat{\psi}(\omega)|^2 \, d\omega \right)^{\frac{1}{2}}.
$$

When $\psi$ is a real-valued function, $|\hat{\psi}|$ is an even function, so that $\omega^-_{\hat{\psi}} = -\omega^+_{\hat{\psi}}$ and $\Delta^+_{\psi} = \Delta^-_{\psi}$. In wavelet analysis, we usually only consider real-valued wavelets, and can therefore ignore $\omega^-_{\hat{\psi}}$ and $\Delta^-_{\psi}$.

A bandpass (real-valued) window function $\psi$ with center $(t_{\psi}, \omega^+_{\hat{\psi}})$ and radii $\Delta_{\psi}$ and $\Delta^+_{\psi}$ has time-frequency localization measurement

$$
M^+_{\psi}(\psi) := \Delta_{\psi} \Delta^+_{\psi}.
$$

Charles and I prove the following uncertain principle for bandpass window functions [22].

**Theorem 6.** If $\psi \in L^2 \cap L^1$ is a real-valued symmetric or anti-symmetric function that satisfies $t\psi(t) \in L^2$, $\psi' \in L^2$, and $\hat{\psi}(0) = 0$, then

$$
M^+_{\psi}(\psi) > \frac{1}{2}.
$$

Furthermore, the lower bound $\frac{1}{2}$ cannot be improved and cannot be attained.

According to Theorem 5, no scaling function can achieve the optimal lower bound of the window measure either. This motivates our study of scaling functions and wavelets that asymptotically achieve the optimal bound of $\frac{1}{2}$.

Recall that a real-valued sequence $a$ is called a Pólya frequency sequence if all the minors of the bi-infinite matrix $A$ with $(i,j)$th entries given by

$$
A_{ij} = a_{j-i}, \quad i, j \in \mathbb{Z},
$$

are non-negative.
\[ J.Z. \text{ Wang} \]

(where \( a = \{a_j\} \) and \( a_n := 0 \) if \( n \) is not in the index set of \( a \)) are non-negative, that is,

\[
A \left( i_1, \cdots, i_p \right) := \det_{k, \ell=1, \cdots, p} A_{k,j_\ell} \geq 0,
\]

for all integers \( p \geq 1 \) and \( i_1 < \cdots < i_p \), \( j_1 < \cdots < j_p \). (See [60] for the properties of Pólya frequency sequences.) If \( a \) is a finite sequence, its length will be denoted by \(|a|\).

Now let \( \phi \in L^1 \) be a scaling function with the mask \( a \), and define

\[
B_\phi \left( e^{-i\omega} \right) = \sum_{n \in \mathbb{Z}} \left| \hat{\phi} \left( \omega + 2n\pi \right) \right|^2.
\]

Assume its corresponding semi orthogonal wavelet \( \psi \) is defined by

\[
\hat{\psi}(\omega) = C_\psi e^{-i\frac{\omega}{2}} a \left( -e^{-i\frac{\omega}{2}} \right) B_\phi \left( -e^{-i\frac{\omega}{2}} \right) \hat{\phi} \left( \frac{\omega}{2} \right),
\]

where \( C_\psi \) is a positive constant so chosen that \( \|\hat{\psi}\|_\infty = 1 \).

If \( a \) is a symmetric finite Pólya frequency sequence, we will call \( \phi \) a stoplet and \( \psi \) a cowlet, respectively. We denote the standard deviation of \( \phi \) by

\[
\sigma = \left( \int_{-\infty}^{\infty} \phi(x) (x - t_\phi)^2 \, dx \right)^{\frac{1}{2}}.
\]

Charles and I reveal that the time-frequency localization of semi-orthogonal spline-wavelets is asymptotically optimal. On the contract, the size of time-frequency window of orthonormal two-scaling functions and wavelets grows to infinity as the smoothness increases. For a scaling function of spline-type, we have the following [21].

**Theorem 7.** For each \( n \), let \( a^n = \{a^n_j\}_{j=0}^{k_n} \) be a finite symmetric Pólya frequency sequence with symbol

\[
a_n(z) = \left( \frac{1+z}{2} \right)^{k_n} p_n(z),
\]

for some polynomial \( p_n(z) \) that satisfies \( p_n(1) = 1, p_n(-1) \neq 0 \) and \( \deg p_n \leq Cn \), where \( C \) is a positive constant independent of \( n \). Let \( \phi_n \) be the stoplet determined by \( a^n \) and \( \sigma_n \) be its standard variation. Then

1. \( \lim_{n \to \infty} \sigma_n = +\infty \);
2. the following limits hold:

\[
\lim_{n \to \infty} \left\| \hat{\phi}_n \left( \frac{\omega}{\sigma_n} \right) e^{i\frac{\omega}{2\sigma_n} \tau} - e^{-i\frac{\omega}{2} \tau} \right\|_{L^p} = 0, \quad 1 \leq p < \infty,
\]
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and

\[
\lim_{n \to \infty} \left\| \sigma_n \phi_n \left( \sigma_n x + \frac{k_n}{2} \right) - \frac{1}{2\pi} e^{-\frac{x^2}{2}} \right\|_{L^q} = 0, \quad 2 \leq q < \infty; \tag{32}
\]

(3) furthermore,

\[
\lim_{n \to \infty} \frac{1}{\sigma_n} \Delta \phi_n = \lim_{n \to \infty} \sigma_n \Delta \phi_n = \frac{1}{\sqrt{2}} \tag{33}
\]

so that

\[
\lim_{n \to \infty} M(\phi_n) = \frac{1}{2}. \tag{34}
\]

For a semi-orthogonal wavelet of spline-type, we have the following [22].

**Theorem 8.** Let \( \psi_n \) be the cowlets corresponding to the stoplets \( \phi_n \) as in Theorem 7. Then

1. for each \( n \), there is a unique \( \omega_n \) in \((0, \infty)\), at which the function \( |\hat{\psi}_n(\omega)| \) attains its absolute maximum value;

2. \( \pi \leq \omega_n \leq 2\pi \), and \( \tau_n := \sqrt{|\hat{\psi}_n''(\omega_n)|} \to \infty \) as \( n \to \infty \);

3. the following limits hold:

\[
\lim_{n \to \infty} \left\| e^{\frac{i\omega_n}{\tau_n}} \hat{\psi}_n \left( \frac{\omega}{\tau_n} \right) - e^{-\frac{(\omega - \omega_n\tau_n)^2}{2}} \right\|_{L^p(0, +\infty)} = 0, \quad 1 \leq p < \infty, \tag{35}
\]

so that for even \( n \),

\[
\lim_{n \to \infty} \left\| \tau_n \psi_n \left( \tau_n x + \frac{1}{2} \right) - \frac{1}{2\pi} \cos (\tau_n \omega_n x) e^{-\frac{x^2}{2}} \right\|_{L^q} = 0, \quad 2 \leq q < +\infty, \tag{36}
\]

and for odd \( n \),

\[
\lim_{n \to \infty} \left\| \tau_n \psi_n \left( \tau_n x + \frac{1}{2} \right) - \frac{1}{2\pi} \sin (\tau_n \omega_n x) e^{-\frac{x^2}{2}} \right\|_{L^q} = 0, \quad 2 \leq q < +\infty; \tag{37}
\]

4. furthermore,

\[
\lim_{n \to \infty} \frac{1}{\tau_n} \Delta \psi_n = \lim_{n \to \infty} \tau_n \Delta \psi_n = 1 \tag{38}
\]

so that

\[
\lim_{n \to \infty} M^+ (\psi_n) = \frac{1}{2}. \tag{39}
\]

As an application of Theorems 7 and 8, we have the following corollaries for B-spline and for the compactly supported, semi-orthogonal spline-wavelet.
Corollary 2. We have
\[
\lim_{n \to \infty} \left\| \sqrt{\frac{n}{12}} N_n \left( \sqrt{\frac{n}{12}} \omega \sin \omega \right) \right\|_{L^p} = 0, \quad 1 \leq p < \infty, \tag{40}
\]
and
\[
\lim_{n \to \infty} \left\| \sqrt{\frac{n}{12}} N_n \left( \sqrt{\frac{n}{12}} \frac{x}{2} \right) - \frac{1}{2\pi} e^{-\frac{x^2}{4}} \right\|_{L^q} = 0, \quad 2 \leq q < \infty. \tag{41}
\]
Furthermore,
\[
\lim_{n \to \infty} \sqrt{\frac{n}{12}} \Delta N_n = \lim_{n \to \infty} \sqrt{\frac{n}{12}} \Delta \hat{N}_n = \frac{1}{\sqrt{2}}, \tag{42}
\]
so that
\[
\lim_{n \to \infty} M(N_n) = \frac{1}{2}. \tag{43}
\]

Corollary 3. Let \( \psi_n \) be the compactly supported, semi-orthogonal spline wavelet corresponding to \( N_n \). Let \( \omega_0 \) (\( \approx 1.6367\pi \)) be the unique value in \((0, \infty)\), at which the function \( C(\omega) := \frac{8(1 - \cos \omega)}{\omega(2\pi - \omega)^2} \) attains its absolute maximum value. Set \( \alpha = \sqrt{\frac{1}{8} C'(\omega_0)}(\approx 0.3745) \). Then for \( 1 \leq p < \infty \),
\[
\lim_{n \to \infty} \left\| e^{i \omega \sqrt{n} \psi N_n} \left( \frac{\omega}{\alpha \sqrt{n}} \right) - e^{-\frac{(\omega - \alpha \omega_0 \sqrt{n})^2}{4}} \right\|_{L^p(0, +\infty)} = 0, \tag{44}
\]
and therefore, for even \( n \) and \( 2 \leq q < +\infty \),
\[
\lim_{n \to \infty} \left\| \alpha \sqrt{n} \psi N_n \left( \alpha \sqrt{n} x + \frac{1}{2} \right) - \frac{1}{2\pi} \cos \left( \alpha \omega_0 \sqrt{n} x \right) e^{-\frac{x^2}{4}} \right\|_{L^q} = 0, \tag{45}
\]
and for odd \( n \) and \( 2 \leq q < +\infty \),
\[
\lim_{n \to \infty} \left\| \alpha \sqrt{n} \psi N_n \left( \alpha \sqrt{n} x + \frac{1}{2} \right) - \frac{1}{2\pi} \sin \left( \alpha \omega_0 \sqrt{n} x \right) e^{-\frac{x^2}{4}} \right\|_{L^q} = 0. \tag{46}
\]
Moreover,
\[
\lim_{n \to \infty} \frac{1}{\alpha \sqrt{n}} \Delta \psi N_n = \lim_{n \to \infty} \alpha \sqrt{n} \Delta^+ \psi N_n = \frac{1}{\sqrt{2}}, \tag{47}
\]
so that
\[
\lim_{n \to \infty} M^+(\psi N_n) = \frac{1}{2}. \tag{48}
\]

We remark that (40), (41) and (44)–(46) were already established by the authors of [74] in a different way.

For orthonormal scaling function and wavelets, we have the following remarkable results [20].
Theorem 9. Let \( \phi_n \) be the orthonormal scaling function with the symbol
\[
P(z) = \left( \frac{1 + z}{2} \right)^n S_n(z),
\]
where \(|S_n(e^{-i\omega})| \leq C 2^n |\sin^{\frac{n}{2}}(\frac{\omega}{2})|\), \(\frac{\pi}{2} \leq |\omega| \leq \pi\). Let \( \psi_n \) be the corresponding orthonormal wavelet. Then
\[
\lim_{n \to \infty} \| \hat{\phi}_n \|_{L^2} = 0,
\]
\[
\lim_{n \to \infty} \Delta \hat{\phi}_n = \frac{\pi}{\sqrt{3}},
\]
\[
\lim_{n \to \infty} \Delta \phi_n = \infty.
\]
and
\[
\lim_{n \to \infty} \| \hat{\psi}_n \|_{L^2} = 0,
\]
\[
\lim_{n \to \infty} \Delta^+ \hat{\psi}_n = \frac{\pi}{2\sqrt{3}},
\]
\[
\lim_{n \to \infty} \Delta \psi_n = \infty.
\]
Consequently, the sizes of the time-frequency windows of both \( \phi_n \) and \( \psi_n \) tend to infinity:
\[
\lim_{n \to \infty} M(\phi_n) = \infty, \quad \lim_{n \to \infty} M^+(\psi_n) = \infty.
\]

Recently, the uncertainty principle for scaling functions and wavelets are discussed in various angles by Goh, Goodman, Lee, and other authors in the papers [5], [38], [39], [40], [42], and their references.

§5. Sub-band code and general sample theory

In signal transmission, the sub-band coding is a main method for multiple-channel synchronized transmission. In multi-channel transmission, a signal is decomposed into several sub-signal by its frequency distribution, where each sub-signal has a certain frequency band. The corresponding coding method is called sub-band coding. The sub-band coding must be consistent of the sampling method. On the other hand, an important purpose of sub-band coding is to achieve the lower bite rate. Wavelet theory provides a useful tool for the study of sub-band coding.

In signal processing, all continuous-time signals \( f(t) \) are often considered to be real-valued and band-limited. A signal \( f \in L^2 \) is said to be band-limited if \( \text{supp} \hat{f} \subset [-B, B] \), where \( B > 0 \). Assume \( f \) is a real-valued function. Then \( \text{supp} \hat{f} \) is a symmetric set (with respect to the origin) on
\( \mathbb{R} \), i.e., \( x \in \text{supp} \hat{f} \iff -x \in \text{supp} \hat{f} \). Therefore, in the study we can use the bounded set
\[
\text{supp}^+ \hat{f} := (\text{supp} \hat{f}) \cap [0, \infty)
\]
to substitute \( \text{supp} \hat{f} \). The well-known sampling theorem allows us to recover the continuous-time band-limited signal \( f(t) \) from a certain sample set \( f(kT), t > 0 \), by using the sampling function
\[
\phi(t) = \text{sinc} t := \frac{\sin \pi t}{\pi t}.
\]
A precise statement of this theorem is the following [59].

**Theorem 10 (Shannon Sampling Theorem)** A continuous-time and real-valued band-limited signal \( f(t) \) with
\[
\text{supp}^+ \hat{f} \subset [0, 2\pi \sigma]
\]
has the infinite series representation
\[
f(t) = \sum_{k=-\infty}^{\infty} f(kT) \text{sinc} (2\sigma(t - kT))
\]
where \( T = \frac{1}{2\sigma} \).

In signal processing, we always assume \( 2\pi \sigma \) in (49) is the least upper bound (lub) of \( \text{supp}^+ \hat{f} \). Thus, we call \( \sigma \) the highest band of \( f \). Since the set
\[
\{\text{sinc} (2\sigma(t - kT))\}_{k \in \mathbb{Z}}
\]
is linearly independent, the sampling theorem asserts that to completely recover a signal \( f(t) \) from its sample set \( \{f(k/\mu)\} \), the sampling rate \( \mu \) (also called the sampling frequency) must satisfy \( \mu \geq 2\sigma \), where \( 2\sigma \) is the smallest sampling rate for the completely recovering of \( f \), called the Nyquist frequency or Nyquist rate of \( f(t) \).

For example, to sample a speech signal with highest band 4 kHz, the sampling rate must be at least 8 kHz to avoid distortion; and the sampling rate of high-quality music signals (with highest band 22.05 kHz) is at least 44.1 kHz. Then 8 kHz and 44.1 kHz are their Nyquist rates.

However, most of speech signals do not cover all bands in \([0, \sigma]\), but only a collection of sub-intervals in \([0, \sigma]\). Assume a signal \( f \) has a positive lowest band \( \mu \) and the highest band \( \nu \), i.e.,
\[
\text{supp}^+ \hat{f} \subset [2\pi \mu, 2\pi \nu].
\]
Then \( \sigma := \nu - \mu \) is called the bandwidth of \( f(t) \). For such a bandpass signal, if it is sampled by using the sample method in Theorem 10, then
its Nyquist rate is $2\sigma_2$. However, if $\sigma_1/\sigma$ is an integer, the sampling rate
is reduced from the (standard) Nyquist frequency $2\nu$ to $2\sigma$ (see [59]).
As we will show later, the rate $2\sigma$ is the smallest rate for the completely
recovering. For this reason, $2\sigma$ is also called the Nyquist frequency for
bandpass signals (when $\mu/\sigma$ is an integer). When $\mu/\sigma$ is not an integer,
it was shown in [61] that the smallest sampling rate for complete recovery
of $f(t)$ is given by $2\sigma_m$,

$$\sigma_m = \frac{1 + \mu/\sigma}{1 + [\mu/\sigma]}$$

where the notation $[x]$ stands for the integer part of $x$. We call $\sigma_m/\nu (\leq 1)$
the bit rate of the coding method for $f$, which samples $f$ using its band-
pass property (50).

The sampling theorem for bandpass signals has been applied in the
study of sub-band coding (see [28], [29], [50], [75]), and is often considered a
fundamental result for multiple-channel synchronized transmission. Since
the signals to be transmission are usually sampled by $\{f(k/2\nu)\}$, where $\nu$
is the highest band of $f$. In multiple-channel synchronized transmission,
we need to do the following. (1) To determine the nearly lowest sampling
rate of a signal $f$. (2) To extract a sub-sampling data of $\{f(k/2\nu)\}$ that
achieves the rate. (3) To develop a fast algorithm that recovers $f$ from
the sub-sampling data of the signal. To explain the tasks more clearly, we
give the following.

Definition 3. A sub-band decomposition of a bandlimited signal $f(t)$ is
the following.

$$f(t) = \sum_{k=1}^{n} f_k(t),$$

where $\text{supp}^+ \hat{f}_k \subset [2\pi \mu_k, 2\pi \nu_k]$,

$$0 \leq \mu_1 < \nu_1 \leq \mu_2 < \nu_2 \leq \cdots \leq \mu_n < \nu_n,$$

and $\mu_k$ and $\nu_k$ satisfy the sub-band decomposition conditions:
(i) $\mu_k/\sigma_k, k = 1, \ldots, n$, are integers;
(ii) $\sigma_k/\sigma_\ell, k, \ell = 1, \ldots, n$, are rationales.

If $\mu_k$ and $\nu_k$ in (52) are the lowest and highest bands of $f_k$, then
Condition (i) ensures that the smallest sampling rate of each sub-band is
equal to its Nyquist frequency, and Condition (ii) ensures the existence
of some positive integer $N$ and some $\sigma > \sigma_k, k = 1, \ldots, n$, such that
$N\sigma_k/\sigma, k = 1, \ldots, n$, are integers.

Definition 4. Let $S = \{rk\}_{k \in \mathbb{Z}, r \in (0, \infty)}$, $V = \{\lambda k + \tau\}_{k \in \mathbb{Z}, \lambda \in (0, \infty), \tau \in \mathbb{R}}$, and $V \subset S$. Then we say the set $V$ has a compression
rate $\lambda/r$ with respect to $S$. 
We have the following.

**Theorem 11.** [24] If \( f \) has a sub-band decomposition (51) and \( f \) is sampled by \( S := \{ f(k/2^n) \}_{k \in \mathbb{Z}} \), then (1) each \( f_k \) can be sampled by a subset \( S_k \subset S \) with the compression rate \( \nu_n/\sigma_k \) (or the bit rate \( \sigma_k/\nu_k \)). (2) \( S_k \cap S_j = \emptyset \), \( k \neq j \).

This property allows the feasibility of bit allocation for each sub-band signal (see [51]).

To complete recover a band-limited signal \( f(t) \) with sub-band decomposition given in (51), we only need to recover each \( f_k(t) \), which has the sampling rate \( \leq 2\sigma_k = 2(\nu_k - \mu_k) \). Then \( f \) can be recovered by a sampling (or coding) rate

\[
2\sigma_s := 2 \sum_{k=1}^{n} \sigma_k.
\]

We call \( 2\sigma_s \) the sub-band sampling rate (or coding rate) corresponding to the sub-band decomposition (51). It is also known that if \( \hat{f}_k \) has the exact (positive) support \( [\mu_k, \nu_k] \), i.e., \( \text{supp}^+ \hat{f}_k = [2\pi \mu_k, 2\pi \nu_k] \), then its Nyquist rate is equal to \( 2\sigma_k \). Therefore, we give the following.

**Definition 5.** Let

\[
\sigma_f = \frac{\text{mes}(\text{supp}^+ f)}{\pi},
\]

where the notation “mes” stands for the Lebesgue measure. Then \( 2\sigma_f \) is called the theoretical Nyquist frequency of a band-limited signal \( f(t) \).

It is obvious that for each sub-band decomposition of \( f \), its sub-band coding rate is no less than its theoretical Nyquist frequency \( 2\sigma_f \). From the point of view of signal transmission, a good sub-band sampling should achieve a sampling rate as close to \( 2\sigma_f \) as possible. Charles and I in [24] apply Shannon wavelet packets in the study of subband-coding.

According to [49], the function \( \phi(t) = \text{sinc} \) is called the Shannon scaling function and the function \( \psi(t) = 2\text{sinc}(2t) - \text{sinc} \) is called the Shannon wavelet. Let

\[
\cdots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots,
\]

be the MRA generated by \( \phi(t) : V_n = \{ f \in L^2 : \text{supp} \hat{f} \subset [-2^n \pi, 2^n \pi] \} \). Let \( \{ W_n \}_{n \in \mathbb{Z}} \) be the corresponding wavelet subspaces generated by \( \psi \). Then \( W_n \perp V_n, \ W_n + V_n = V_{n+1} \). We have

\[
\hat{\phi}(\omega) = \chi_{[-\pi, \pi]}(\omega)
\]

\[
\hat{\psi}(\omega) = \chi_{[-2\pi, -\pi] \cup [\pi, 2\pi]}(\omega).
\]
On Spline Wavelets

Let $p_0(\omega)$ be the $2\pi$-periodic function:

$$p_0(\omega) = \begin{cases} 1, & \omega \in [-\frac{\pi}{2}, \frac{\pi}{2}) \\ 0, & \omega \in (-\pi, -\frac{\pi}{2}) \cup [\frac{\pi}{2}, \pi) \end{cases}$$

and write $p_1(\omega) = p_0(\omega + \pi)$.

Then, we have

$$\hat{\phi}(\omega) = p_0(\omega/2)\hat{\phi}(\omega/2),$$
$$\hat{\psi}(\omega) = p_1(\omega/2)\hat{\phi}(\omega/2).$$

The Shannon wavelet packets can be constructed as follows. Write

$$\{ \mu_0(t) = \phi(t), \mu_1(t) = \psi(t) \}$$

Then, we have

$$\begin{cases} \hat{\mu}_0(\omega) = p_0(e^{-i\omega/2})\hat{\mu}_0(\omega/2), \\ \hat{\mu}_1(\omega) = p_1(e^{-i\omega/2})\hat{\mu}_0(\omega/2). \end{cases}$$

For even $n$, we set

$$\begin{cases} \hat{\mu}_{2n}(\omega) = p_0(e^{-i\omega/2})\hat{\mu}_n(\omega/2), \\ \hat{\mu}_{2n+1}(\omega) = p_1(e^{-i\omega/2})\hat{\mu}_n(\omega/2), \end{cases}$$

and for odd $n$, set

$$\begin{cases} \hat{\mu}_{2n}(\omega) = p_1(e^{-i\omega/2})\hat{\mu}_n(\omega/2), \\ \hat{\mu}_{2n+1}(\omega) = p_0(e^{-i\omega/2})\hat{\mu}_n(\omega/2), \end{cases}$$

Then the collection $\{\mu_l\}_{l=0}^\infty$ is a family of Shannon wavelet packets.

It can be easily verified that

$$\mu_l(t) = (l + 1)sinc((l + 1)t) - lsinc(lt),$$

or, equivalently,

$$\hat{\mu}_l = \chi_{[-(l+1)\pi, -(l-1)\pi) \cup [l\pi, (l+1)\pi)}, \quad l = 0, 1, 2, \ldots$$

Write

$$\mu_{l,j,k}(t) = 2^{j/2} \mu_l(2^j t - k).$$

We have

$$\hat{\mu}_{l,j,k}(\omega) = e^{i2^{-j}k\omega} \chi_{[-2^j(l+1)\pi, -2^j(l-1)\pi) \cup [2^j(l-1)\pi, 2^j(l+1)\pi)}. $$
Define
\[ U^j_l = \text{clos}_{L^2} \text{ span } \{ 2^{j/2} \mu_l (2^j t - k) : k \in \mathbb{Z} \}, \quad j \in \mathbb{Z}, \quad l \in \mathbb{Z}^+. \]
The each function in \( U^j_l \) is a bandpass signal with lowest band \( 2^{j-1} l \), highest band \( 2^{j-1} (l+1) \), and bandwidth \( 2^{j-1} \). In addition, it also satisfies the sub-band coding condition (i). For any \( n = 0, 1, 2, \cdots \), we have
\[ U^n_{j+1} = U^{2n}_j \oplus U^{2n+1}_j, \quad U^{2n}_j \perp U^{2n+1}_j, \quad j \in \mathbb{Z}. \]
Therefore, for \( j \geq 1 \) and \( k \geq 0 \), we have
\[ W_j = U^{2k}_j \oplus U^{2k+1}_j \oplus \cdots \oplus U^{2k+1-1}_j. \]
Let \( I_{j, l} = [2^j l \pi, 2^j (l+1) \pi] \). Let \( \Lambda \) and \( \Gamma \) be two subsets of the integer set. Then the set \{ \( I_{j, l} : j \in \Lambda, l \in \Gamma \) \} forms a dyadic partition of \( \mathbb{R}^+ := [0, \infty) \) if \( \cup_{j \in \Lambda, l \in \Gamma} I_{j, l} = \mathbb{R}^+ \) and \( \text{mes}(I_{j, l} \cap I_{j', l'}) = 0, (j, l) \neq (j', l') \). It can be proved that if \( \{ I_{j, l} : j \in \Lambda, l \in \Gamma \} \) is a dyadic partition of \( \mathbb{R}^+ \), then the set of \( \{ \psi_{j, l, k} : j \in \Lambda, l \in \Gamma, k \in \mathbb{Z} \} \) is an orthonormal basis of \( L^2 \) and \( L^2 = \bigoplus_{j \in \Lambda, l \in \Gamma} U^j_l \). Therefore, if \( f \in U^n_0 \), that is, \( \text{supp}^+ \hat{f} \subset I_{0, n} := [n \pi, (n+1) \pi] \), then
\[ f(t) = \sum_{k \in \mathbb{Z}} f(k) \mu_{n, 0, k}(t). \]
By the nice properties of Shannon wavelet packet in the frequency domain, we can prove the following [24].

**Theorem 12.** Let \( f \) be a bandlimited signal with theoretical Nyquist frequency \( 2\sigma_f \). Then for any \( \lambda_f > \sigma_f \), there is a sub-band decomposition (51) of \( f \), with sub-band coding rate no greater than \( 2\lambda_f \). Furthermore, the sub-band coding rate of any sub-band decomposition of \( f \) is at least \( 2\sigma_f \).

**Theorem 13.** Let \( f \) be a bandlimited signal with highest band \( \sigma \) and theoretical Nyquist frequency \( \sigma_f \). Then for any \( \hat{\sigma} > \sigma_f \),
\begin{enumerate}
  \item there exists a sub-band decomposition of \( f \) that achieves bit-rate compression ratio larger than \( \sigma/\hat{\sigma} \),
  \item the sub-band coding can be realized by a Shannon wavelet packet.
\end{enumerate}

The Shannon wavelet packet introduced above has a primary bandwidth \( 1/2 \) (i.e., both \( \phi \) and \( \psi \) have bandwidth \( 1/2 \)). Therefore, each function in the subspace \( U^j_l \) has the dyadic bandwidth \( 2^{j-1} \). In application we need to construct the Shannon wavelet packets with primary band different from \( 2^j \). To do this, for a positive number \( \nu \notin 2^j \), we define
\[ \phi^\nu(t) = (2\nu)^{1/2} \phi(2\nu t), \quad \psi^\nu(t) = (2\nu)^{1/2} \psi(2\nu t). \]
Then both $\phi_\nu(t)$ and $\psi_\nu(t)$ have bandwidth $\nu$. Let

\[
\phi_{j,k}^\nu(t) = 2^{j/2} \phi(2^j t - \frac{k}{2^\nu}) = \left(2^{(j+1)/2} \nu^{1/2} \phi(2^{j+1} \nu x - t)\right),
\]

\[
\psi_{j,k}^\nu(t) = 2^{j/2} \psi(2^j t - \frac{k}{2^\nu}) = \left(2^{(j+1)/2} \nu^{1/2} \psi(2^{j+1} \nu t - k)\right).
\]

Then $\{\psi_{j,k}^\nu : j, k \in \mathbb{Z}\}$ creates the Shannon wavelet packet has primary band $\nu$. The Shannon wavelet library therefore can be constructed as follows. Denote the Shannon wavelet packets with the primary band $\nu$ by $\mathcal{P}^\nu$. Two positive numbers $\nu$ and $\mu$ are said to be binarily similar if there exists an integer $j$ such that $\nu = 2^j \mu$. Let $B \subset \mathbb{R}$ be the set of all numbers that are not binarily similar to each other. Then

\[
\{\mathcal{P}^\nu : \nu \in B\}
\]

constitutes a Shannon wavelet library.

We now have two ways to do the sub-band decomposition for a signal $f$.

1. We use a single Shannon wavelet packet, say $\mathcal{P}^\nu$, in the library for sub-band decomposition. In this case, the decomposition always satisfies the sub-band coding conditions (i) and (ii). In addition, all sub-band functions of a sub-band coding obtained in this way are synchronic, and therefore, no additional code is needed for synchronized transmission.

2. We use the whole Shannon wavelet library for sub-band decomposition. We may decompose a signal $f(t)$ into sub-band signals using several packets:

\[
f(t) = \sum_{l=1}^{m} \sum_{k=1}^{n} f_{lk}(t),
\]

where $f_{lk}(t)$ and $f_{l'k'}(t)$ have different primary bands if and only if $l \neq l'$. Thus, the signal $f$ is decomposed by using $m$ different wavelet packets $\mathcal{P}^\nu$, $1 \leq l \leq m$. The decomposition is a sub-band decomposition if all the ratios $\nu_k/\nu_{k'}$, $1 \leq k, k' \leq m$, are rational numbers. In this case, additional code may be needed for synchronized transmission.

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