The State Polytope.

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A \textit{polyhedron} is a finite intersection of closed half-spaces in $\mathbb{R}^n$. Thus a polyhedron $P$ can be written as $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ where $A$ is a matrix with $n$ columns.

If $b = 0$, then there exist vectors $u_1, \ldots, u_m \in \mathbb{R}^n$ such that

$$P = \text{pos}(u_1, \ldots, u_m) := \{\lambda_1 u_1 + \cdots + \lambda_m u_m \mid \lambda_1, \ldots, \lambda_m \in \mathbb{R}_+\}$$

A polyhedron of this form is called a \textit{polyhedral cone}. 

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The **polar** of a cone $C$ is defined as

$$C^* = \{ w \in \mathbb{R}^n \mid w \cdot c \leq 0 \ \forall \ c \in C \}.$$ 

A polyhedron $Q$ which is bounded is called a **polytope**. Every polytope $Q$ can be written as the convex hull of a finite set of points $P$.

$$Q = \text{conv}(v_1, \ldots, v_m)$$

$$:= \{ \sum_{i=1}^{m} \lambda_i v_i \mid \forall \lambda_i \in \mathbb{R}_+, \sum_{i=1}^{m} \lambda_i = 1 \}. $$
Let $P$ be a polyhedron in $\mathbb{R}^n$ and $w \in \mathbb{R}^n$, viewed as a linear functional. We define

$$\text{face}_w(P) := \{ u \in P \mid w \cdot u \geq w \cdot v \quad \forall v \in P \}.$$ 

The relation “is a face of” among polyhedra is transitive:

$$\text{face}_{w'}(\text{face}_w(P)) = \text{face}_{w+\epsilon w'}(P)$$
for $\epsilon > 0$ sufficiently small.
We define the \textit{Minkowski sum} of polyhedra $P_1$ and $P_2$ as

$$P_1 + P_2 := \{p_1 + p_2 \mid p_1 \in P_1, p_2 \in P_2\}.$$

A basic fact about the Minkowski sum is the additivity of faces:

$$\text{face}_w(P_1 + P_2) = \text{face}_w(P_1) + \text{face}_w(P_2).$$
Proposition 1. Every polyhedron $P$ can be written as the sum $P = Q + C$ of a polytope $Q$ and a cone $C$. The cone $C$ is unique and is called the recession cone of $P$.

A (polyhedral) complex $\Delta$ is a finite collection of polyhedra in $\mathbb{R}^n$ such that

(i) If $P \in \Delta$ and $F$ is a face of $P$, then $F \in \Delta$;

(ii) If $P_1, P_2 \in \Delta$, then $P_1 \cap P_2$ is a face of $P_1$ and of $P_2$.

The support of a complex $\Delta$ is $|\Delta| := \bigcup_{P \in \Delta} P$. A complex $\Delta$ which consists of cones is called a fan. A fan $\Delta$ is complete if $|\Delta| = \mathbb{R}^n$. 

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If $P \subset \mathbb{R}^n$ is a polyhedron and $F$ is a face of $P$, then the normal cone of $F$ at $P$ is

$$\mathcal{N}_P(F) = \{ w \in \mathbb{R}^n \mid \text{face}_w(P) = F \}.$$ 

Note that $\dim(\mathcal{N}_P(F)) = n - \dim(F)$. If $F$ and $F'$ are faces of $P$, then $F'$ is a face of $F$ if and only if $\mathcal{N}_P(F)$ is a face of $\mathcal{N}_P(F')$.

Hence the collection of normal cones $\mathcal{N}_P(F)$, where $F$ ranges over the faces of $P$, is a fan. This fan is denoted $\mathcal{N}(P)$ and called the normal fan of $P$. 
The support of $\mathcal{N}(P)$ equals the polar $C^*$ of the recession cone $C$.

If $Q$ is a polytope, then its recession cone is $\{0\}$, and its normal fan $\mathcal{N}(Q)$ is a complete fan.
Let \( f = \sum_{i=1}^{m} c_i x^{a_i} \). The Newton polytope of \( f \) is defined as \( \text{New}(f) := \text{conv}\{a_i \mid i = 1, \ldots, m\} \) in \( \mathbb{R}^n \).

**Lemma 2.** \( \text{New}(f \cdot g) = \text{New}(f) + \text{New}(g) \).

- It suffices to show that both polytopes have the same vertices.
- \( \text{face}_w(\text{New}(f)) = \text{New}(\text{in}_w(f)) \)
- The following relation holds for all \( w \) which are sufficiently generic.

\[
\begin{align*}
\text{face}_w(\text{New}(f \cdot g)) &= \text{New}(\text{in}_w(f \cdot g)) \\
&= \text{New}(\text{in}_w(f) \cdot \text{in}_w(g)) \\
&= \text{New}(\text{in}_w(f)) + \text{New}(\text{in}_w(g)) \\
&= \text{face}_w(\text{New}(f)) \\
&\quad + \text{face}_w(\text{New}(g)) \\
&= \text{face}_w(\text{New}(f) + \text{New}(g)).
\end{align*}
\]
Fix $I \subset k[x]$. Two vectors $w, w'$ are equivalent w.r.t. $I \iff \text{in}_w(I) = \text{in}_{w'}(I)$.

**Proposition 3.** Each equivalence class of weight vectors is a relatively open convex polyhedral cone.

**Proof.** Let $C[w]$ denote the equivalence class of $w$. Fix a term order $\prec$. Let $G$ be the reduced Gröbner basis of $I$ w.r.t. $\prec_w$.

$$C[w] = \{w' \in \mathbb{R}^n \mid \text{in}_{w'}(g) = \text{in}_w(g) \quad \forall g \in G\}.$$  

This formula expresses $C[w]$ as an intersection of hyperplanes and open half-spaces: $w' \cdot a = w' \cdot b$ and $w' \cdot a > w' \cdot c$, where $x^a$ and $x^b$ run over the terms of $\text{in}_w(g)$ and $x^c$ runs over the terms of $g$ which do not appear in $\text{in}_w(g)$.
This formula has a geometric reformulation

\[ C[w] = \mathcal{N}_Q(\text{face}_w(Q)), \]
\[ Q := \text{New}( \prod_{g \in G} g ) = \sum_{g \in G} \text{New}(g). \]

Let \( I = (x^2 - y + 2, y^2 - x - 3) \subset k[x, y] \). Let \( w = (3, 3) \), then \( C[w] \) is the following open convex cone:

We define the \textit{Gröbner fan} \( \text{GF}(I) \) to be the set of closed cones \( \overline{C[w]} \) for all \( w \in \mathbb{R}^n \).
From now on we shall assume that $I$ is homogeneous w.r.t. some positive grading $\deg(x_i) = d_i > 0$.

**Theorem 4.** There exists a polytope $\text{State}(I)$ whose normal fan $\mathcal{N}(\text{State}(I))$ coincides with the Gröbner fan $\text{GF}(I)$.

The polytope $\text{State}(I)$ will be called the *state polytope* of $I$. Its construction goes as follows: Denote by $I_d$ the vector space of homogeneous polynomials of degree $d$ in $I$. If $M$ is any monomial ideal, then $\sum M_d$ denotes the sum of all vectors $a \in \mathbb{N}^n$ such that $x^a$ has degree $d$ and lies in $M$. 
\[
\text{State}_d(I) := \text{conv}\{\sum \text{in}_<(I)_d | < \text{ any term order}\}.
\]

Let \( D \) be the largest degree of any element in a minimal universal Gröbner basis of \( I \).

\[
\text{State}(I) := \sum_{d=1}^{D} \text{State}_d(I).
\]

We say that a polytope \( Q \subset \mathbb{R}^n \) is a state polytope for \( I \) if it is strongly isomorphic to \( \text{State}(I) \). In other words, a polytope \( Q \) is a state polytope for \( I \) if its normal fan \( \mathcal{N}(Q) \) equals the Gröbner fan \( \text{GF}(I) \).
Proposition 5. Let $I = \langle f \rangle$, for some homogeneous polynomial $f$. Then $\text{New}(f)$ is a state polytope for $I$.

Proof. $\{f\}$ equals the reduced Gröbner basis w.r.t. any term order. Hence $C[w] = \mathcal{N}_{\text{New}(f)}(\text{face}_w(\text{New}(f)))$. Thus the equivalence classes of term orders are the normal cones of the Newton polytope $\text{New}(f)$. □

Corollary 6. Let $G$ be a universal Gröbner basis of $I$ which is a reduced Gröbner basis of $I$ w.r.t. every term order. Then $\sum_{g \in G} \text{New}(g)$ is a state polytope for $I$. 
Some spectacular applications of Newton polytopes to classical algebraic problems have been found by A. Kouchnirenko, Bernstein, Khovanskij, Gelfand, Krapanov, Zelevinsky, Sturmfels, and many others. The structure of New$(f)$ is deeply related to the geometry of the hypersurface $\{f = 0\}$. Denote by $\log : (\mathbb{C}^*)^n \rightarrow \mathbb{R}^n$ the map

$$(x_1, \ldots, x_n) \longmapsto (\log |x_1|, \ldots, \log |x_n|).$$

For a polynomial $f \in k[x]$, denote by $Z_f$ the hypersurface in $(\mathbb{C}^*)^n$, defined by the equation $f = 0$. The amoeba of $f$ is the subset $\log(Z_f) \subset \mathbb{R}^n$. 
**Theorem 7.** The vertices of $\text{New}(f)$ are in bijection with those connected components of the complement $\mathbb{R}^n \setminus \log(Z_f)$ which contain a convex cone with non-empty interior.

The normal cone $\mathcal{N}(Q)$ is complete, so the amoeba is situated in thin spaces between walls of the translated normal cones. It follows that the combinatorial structure of the Newton polytope $\text{New}(f)$ can be read from the geometry of the hypersurface $Z_f$. 
Suppose we have \(k\) polynomials \(f_1, \ldots, f_k\) in \(k\) variables. These polynomials define functions on the algebraic torus \((\mathbb{C}^*)^k\). We want to find the number of their common roots in this torus.

**Theorem 8.** Let \(A_1, \ldots, A_k \subset \mathbb{Z}^k\) be finite sets such that \(\bigcup_{i=1}^k A_i\) generates \(\mathbb{Z}^k\) as an affine lattice. Let \(Q_i = \text{conv}(A_i)\), and let \(\mathbb{C}^{A_i}\) be the space of polynomials in \(x_1, \ldots, x_k\) with monomials from \(A_i\). Then there exists a dense Zariski open subset \(U \subset \prod \mathbb{C}^{A_i}\) with the following property: for any \((f_1, \ldots, f_k) \in U\), the number of solutions of the system of equations \(f_1(x) = \cdots = f_k(x) = 0\) in \((\mathbb{C}^*)^k\) equals the mixed volume \(\text{vol}_{\mathbb{Z}^k}(Q_1, \ldots, Q_k)\).

Observe that each \(Q_i\) is the Newton polytope of a generic \(f \in \mathbb{C}^{A_i}\).