These notes were developed by professor Ken W. Smith for MATH 1410 sections at Sam Houston State University, Huntsville, TX. This material was covered in six 80-minute class lectures at Sam Houston in Summer 2013. (Sections 2.0 and 2.1 were combined in one lecture since 2.0 is a brief review.)

In addition to these class notes, there is a slide presentation version of these notes and a set of Worksheets. All of these are available on Blackboard.

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2 Polynomials

2.0 A review of linear functions

In this chapter we look at polynomial functions, functions of the form

\[ f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0. \]

The first, and easiest example of a polynomial function, is a function of the form,

\[ f(x) = ax + b, \]

those of degree 1.

Since the graphs of these functions are straight lines, these are called linear functions.

2.0.1 A quick review of linear functions

A line joining two points \( P(x_1, y_1) \) and \( Q(x_2, y_2) \) has slope

\[ m := \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \] (1)

(assuming that \( x_1 \) is different from \( x_2 \).) If a line joins two distinct points \( P(x_0, y_1) \) and \( Q(x_0, y_2) \) in which the \( x \)-coordinates are the same, then the line is the vertical line \( x = x_0 \) and we say the slope is “infinite” or “undefined.”

Since, in Euclidean geometry (the geometry of the Cartesian plane) any two points determine a unique line, then the slope of a line is a natural property to identify.

Given two points \( P(x_0, y_1) \) and \( Q(x_0, y_2) \), the difference of \( y \)-values, written \( \Delta y := y_2 - y_1 \) is sometimes called the rise of the line connecting the two points. The difference of \( x \)-values, written \( \Delta x := x_2 - x_1 \) is called the run. Thus the slope of a line is the ratio of the rise to the run.

The slope of a line, as a ratio, has a natural geometric meaning. It identifies how quickly a line rises or falls; it describes the number of units one rises as one moves one unit to the right. For example, if the line has slope 4 and goes through the point (10, 100) then we know that

- when \( x = 11, y = 100 + 4 = 104 \)
- when \( x = 12, y = 104 + 4 = 108 \)
- when \( x = 13, y = 112 \)
- when \( x = 14, y = 116 \)

\[ \vdots \]

etc.

This geometric meaning is often more important than mere equations about slope!

2.0.2 Equations for lines

Suppose we have a point \( P(x_1, y_1) \) on a line of slope \( m \). Then given any other point \( Q(x, y) \) on the line,

\[ m = \frac{y - y_1}{x - x_1}. \]

If we solve for \( y \) (as is our custom), then

\[ y - y_1 = m(x - x_1) \]

(2)
and so

\[ y = m(x - x_1) + y_1 \]  

(3)

Either of these equations (equation 2 or equation 3) will be called the “point-slope” form for a line, since it is created out of a single point and the slope.

Another equation for a line has form \( Ax + By = C \) where \( A, B \) and \( C \) are constants associated with the line. This form is called \textbf{symmetric} because, unlike the other forms, it does not attempt to single out a special variable \( (y) \) and express the equation in terms of that special variable.

A favorite form for the equation of a line is the \textbf{slope-intercept} form. We began by discussing linear equations \( y = a_1x + a_0 \). It is easy (isn’t it?) to see that \( a_1 \) represents the slope of this line. (Increase \( x \) by 1. What happens to \( y \)?)

The value \( a_0 \) represents the \( y \)-intercept since it is the value of \( y \) when \( x \) is zero.

The constants \( a_1 \) and \( a_0 \) are more popularly replaced by \( m \) and \( b \) and we speak of the equation

\[ y = mx + b \]  

(4)
as the \textbf{slope-intercept} form for a line. (It is not clear why we use the letter “\( m \)” for slope, or \( b \) for the \( y \)-intercept.)

\textbf{Worked exercises.}

1. Find the equation for the line of slope 4 passing through the point (6, 20).
   
   (a) Put your answer in \textbf{slope-intercept} form.
   
   (b) Put your answer in \textbf{point-slope} form.
   
   (c) Put your answer in \textbf{symmetric} form.

\textbf{Solutions.}

(a) If \( y = 4x + b \) is a line of slope 4 and (6, 20) is on the line then \( 20 = 4(6) + b \implies b = -4 \).

Answer: \( y = 4x - 4 \).

(b) In point-slope form we first write the formula for the slope using an arbitrary point \( (x, y) \) and the point (6, 20).

So our answer begin with \( 4 = \frac{y - 20}{x - 6} \). Clear denominators to get \( 4(x - 6) = y - 20 \).

(c) If we wish a “symmetric” form for our line, we just start with one of the previous equations and use simple algebra to move expressions involving \( x \) and \( y \) to one side and all constants to the other. For example, in the previous part, we found

\[ 4(x - 6) = y - 20. \]

Multiply out the left side and then move the \( y \) to the left side by subtracting \( y \) from both sides:

\[ 4x - 24 = y - 20. \]

\[ 4x - y - 24 = -20. \]

Now add 24 to both sides so that the lefthand side has only the variables and the righthand side is just a constant.

\[ 4x - y = 4 \]

This is the symmetric form.
2. Find the equation for the line passing through the points (3, 8) and (6, 20). Put your answer in point-slope form.

**Solution.** First we find the slope of the line passing through (3, 8) and (6, 20). It is 
\[
\frac{20 - 8}{6 - 3} = \frac{12}{3} = 4.
\]
Since the line goes through the point (6, 20) and has slope 4, this is the same line as in problem 2. So our answer is
\[
4(x - 6) = y - 20.
\]

3. Find the \(x\)-intercept of the line passing through the points (3, 8) and (6, 20).

**Solution.** Set \(y = 0\) in the solution from problem 3:
\[
4(x - 6) = 0 - 20
\]
\[
(x - 6) = \frac{-20}{4}
\]
\[
x - 6 = -5
\]
\[
x = 1.
\]
The answer is \((1, 0)\).

4. The line \(y = f(x)\) has slope 4 and passes through the point \((12400, 999900)\). Find \(f(12402)\).

**Solution.** A line of slope 4 has the property that every step to the right creates a rise of 4. So 2 steps to the right (from \(x = 12400\) to \(x = 12402\)) creates a rise of 8.

Since we began at height 999900 we must end at \(999900 + 8 = 999908\).

5. The graph of \(y = f(x)\) is a straight line of slope 12. If \(f(gazillion) = google\) then what is \(f(gazillion + 3)\)?

**Solution.** \(f(gazillion + 3) = google + (3)(12) = google + 36\)

2.0.3 The average rate of change of a function

Suppose two points \(P\) and \(Q\) are on the graph \(y = f(x)\) of a function \(f\). Since \(f\) is a function, then by the vertical line test, these two points cannot have the same \(x\)-value. Let’s suppose that we concentrate on the point \(P\) and write its coordinates as \((x, f(x))\). The other point, \(Q\) has \(x\)-coordinate \(x + h\) for some value of \(h\). (So \(h\) is the “run” between the points \(Q\) and \(P\).) Then the coordinates of \(Q\) are \(Q(x + h, f(x + h))\).

The slope of the line joining \(P\) to \(Q\) is
\[
m := \frac{\Delta y}{\Delta x} = \frac{f(x + h) - f(x)}{(x + h) - x} = \frac{f(x + h) - f(x)}{h}.
\]
This expression, \(\frac{f(x + h) - f(x)}{h}\), sometimes called the difference quotient, is the slope of the line connecting the points \(P(x, f(x))\) and \(Q(x + h, f(x + h))\). It is the average rate of change (ARC) of the function \(f(x)\) between the points \(P\) and \(Q\).

The ARC is a critical concept in calculus. Let’s do an example or two.
Worked Examples.

1. Consider the quadratic function \( f(x) = x^2 \). Find the difference quotient for this function.

   **Solution.** We compute
   \[
   \frac{f(x + h) - f(x)}{h} = \frac{(x + h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{2xh + h^2}{h} = 2x + h.
   \]

2. Find the average rate of change of \( f(x) = x^2 \) as \( x \) varies from \( x = 2 \) to \( x = 5 \).

   **Solution.** The slope of the line through \((2, 4)\) and \((5, 25)\) is \( \frac{25 - 4}{5 - 2} = \frac{21}{3} = 7 \).

2.0.4 Other resources for linear functions

In the free textbook, *Precalculus*, by Stitz and Zeager (version 3, July 2011, available at [stitz-zeager.com](http://www.stitz-zeager.com)) this material is covered in section 2.1.

In the free textbook, *Precalculus, An Investigation of Functions*, by Lippman and Rasmussen (Edition 1.3, available at [www.opentextbookstore.com](http://www.opentextbookstore.com)) this material is covered in section 1.3 and sections 2.1 and 2.2.


There are lots of online resources for studying lines and their properties. Here are some I recommend.

1. **Notes on graphing lines** from Dr. Paul’s webpage at Lamar University.
2. **Videos on graphing lines** from Khan Academy,
3. **How to graph lines, from ThatTutorGuy** at Stanford University.

Homework.

As class homework, please complete **Worksheet 2.0, Linear Functions**, available through the class webpage.
2.1 Quadratic functions and parabolas

2.1.1 Quadratic functions

We first looked at polynomials of simple form, of degree 1: \( f(x) = mx + b \). Now we move on to a more interesting case, polynomials of degree 2, the quadratics.

Quadratic functions have form \( f(x) = a_2x^2 + a_1x + a_0 \) or, to use other notation, \( f(x) = ax^2 + bx + c \).

The graph of a quadratic polynomial is a parabola. All of the graphs of quadratic functions can be created by transforming the parabola \( y = x^2 \) in some way. Just as we have standard forms for the equations for lines (point-slope, slope-intercept, symmetric), we also have a standard form for a quadratic function. Every quadratic function can be put in the following standard form, where \( a, h \) and \( k \) are real numbers.

\[
f(x) = a(x - h)^2 + k \tag{6}
\]

We can see, from our understanding of transformation, that if we begin with the equation for the simple parabola \( y = x^2 \) and shift that parabola to the right \( h \) units, stretch it vertically by a factor of \( a \) and then shift the parabola up \( k \) units, we will have the graph for \( y = a(x - h)^2 + k \).

The point \((0, 0)\) on the parabola \( y = x^2 \) is called the vertex of the parabola; it is moved by these transformations to the new point \((h, k)\).

In general, if we are given the equation \( f(x) = ax^2 + bx + c \) for a quadratic function, we can change the function into the above form by first factoring out the term \( a \) so that \( f(x) = ax^2 + bx + c = a(x^2 + \frac{b}{a}x + \frac{c}{a}) \).

We then complete the square on the expression \( x^2 + \frac{b}{a}x + \frac{c}{a} \).

2.1.2 Completing the square

A major tool for solving quadratic equations is to turn a quadratic into an expression involving a sum of a square and a constant term. This technique is called “completing the square.”

Recall that if we square the linear term \( x + A \) we get

\[
(x + A)^2 = x^2 + 2Ax + A^2.
\]

Most of us are not surprised to see the \( x^2 \) or \( A^2 \) come up in the expression but the expression \( 2Ax \), the “cross-term” is also a critical part of our answer. In the expansion of \((x + A)(x + A)\) we sum \( Ax \) twice.

A test of our equation.

Here is a test of our understanding of the simple equation \((x + A)^2 = x^2 + 2Ax + x^2\). Fill in the blanks:

1. \((x + \underline{\_})^2 = x^2 + 4x + \underline{\_}\)
2. \((x + \underline{\_})^2 = x^2 + 10x + \underline{\_}\)
3. \((x + \underline{\_})^2 = x^2 + 2x + \underline{\_}\)
4. \((x + \underline{\_})^2 = x^2 - 2x + \underline{\_}\)
5. \((x + \underline{\_})^2 = x^2 - 6x + \underline{\_}\)
6. \((x + \underline{\_})^2 = x^2 + 7x + \underline{\_}\)

Solutions. Our answers are:

1. \((x + 2)^2 = x^2 + 4x + 4\)
2. \((x + 5)^2 = x^2 + 10x + 25\)
3. \((x + 1)^2 = x^2 + 2x + 1\)
4. \((x-1)^2 = x^2 - 2x + 1\)
5. \((x-3)^2 = x^2 - 6x + 9\)
6. \((x+\frac{7}{2})^2 = x^2 + 7x + \frac{49}{4}\)

**Applying this idea.**

If one understands this simple idea, that we can predict the square by looking at the coefficient of \(x\), then we can rewrite any quadratic polynomial into an expression involving a perfect square. For example, since

\[x^2 + 4x + 4 = (x + 2)^2\]

then a polynomial that begins \(x^2 + 4x\) must involve, somehow, \((x + 2)^2\). For example:

\[x^2 + 4x = (x + 2)^2 - 4\]
\[x^2 + 4x + 7 = (x + 2)^2 + 3\]
\[x^2 + 4x - 5 = (x + 2)^2 - 9\]

Since \(x^2 - 6x\) is the beginning of

\[x^2 - 6x + 9 = (x - 3)^2\]

then

\[x^2 - 6x = (x - 3)^2 - 9.\]

and so

\[x^2 - 6x + 2 = (x - 3)^2 - 7,\]
\[x^2 - 6x + 10 = (x - 3)^2 + 1,\]
\[x^2 - 6x - 3 = (x - 3)^2 - 12,\]

(etc.)

Since \(x^2 - 3x\) is the beginning of

\[x^2 - 3x + \frac{9}{4} = (x - \frac{3}{2})^2\]

then

\[x^2 - 3x = (x - \frac{3}{2})^2 - \frac{9}{4}.\]

This is useful if we are trying to put an equation \(y = f(x)\) of a quadratic into the standard form \(y = a(x - h)^2 + k\). For example, if

\[y = x^2 + 4x + 7\]

is the equation of a parabola then we can complete the square, writing \(x^2 + 4x = (x + 2)^2 - 4\) and so \(x^2 + 4x + 7 = (x + 2)^2 - 4 + 7 = (x + 2)^2 + 3\). Our equation is now

\[y = (x + 2)^2 + 3.\]

We see that the vertex of the parabola is \((-2, 3)\).
An extra step occurs if the coefficient of \( x^2 \) is not 1.

What if the coefficient of \( x^2 \) is not one? How do we complete the square on something like \( 2x^2 + 8x + 15 \)? Once again we focus on the first two terms, \( 2x^2 + 8x \). Factor out the coefficient of \( x^2 \) so that

\[
2x^2 + 8x = 2(x^2 + 4x).
\]

Now complete the square inside the parenthesis so that \( x^2 + 4x = (x^2 + 4x + 4) - 4 = (x + 2)^2 - 4 \).

Therefore

\[
2x^2 + 8x = 2((x + 2)^2 - 4) = 2(x+2)^2 - 8.
\]

If \( y = 2x^2 + 8x + 15 \) then completing the square gives

\[
y = 2x^2 + 8x + 15 = 2(x+2)^2 - 8 + 15 = 2(x+2)^2 + 7.
\]

Some worked problems.

1. What is the vertex of the parabola with equation \( y = 2x^2 + 8x + 15 \)?

**Solution.** Write

\[
y = 2x^2 + 8x + 15 = 2(x+2)^2 + 7.
\]

The parabola with equation \( y = 2(x+2)^2 + 7 \) has vertex \((-2, 7)\).

2. Find the standard form for a parabola with vertex \((2, 1)\) passing through \((4, 5)\).

**Solution.** A parabola with vertex \((2, 1)\) passing through \((4, 5)\) has standard form \( y = a(x-h)^2 + k \) where \((h, k)\) is the coordinate of the vertex. Here \( h = 2 \) and \( k = 1 \). By plugging in the point \((4, 5)\) we see that \( a = 1 \). So our answer is

\[
y = 1 \quad (x - 2)^2 + 1
\]

3. A parabola with vertex \((2, 1)\) passing through \((4, 9)\) has what standard form?

**Solution.** Here the point \((4, 9)\) tells us that \( a = 2 \). So the answer is

\[
y = 2 \quad (x - 2)^2 + 1
\]

4. Find the vertex of the parabola with graph given by the equation \( y = 2x^2 - 12x + 4 \).

**Solution.** Completing the square, we see that

\[
2x^2 - 12x + 4 = 2(x^2 - 6x) + 4 = 2((x-3)^2 - 9) + 4 = 2(x-3)^2 - 18 + 4 = 2(x-3)^2 - 14.
\]

So the vertex is \((3, -14)\).

5. Describe, precisely, the transformations necessary to move the graph of \( y = x^2 \) into the graph of \( y = 2x^2 - 12x + 4 \), given above.

**Solution.** Since the standard form for the parabola is \( y = 2(x-3)^2 - 14 \) then we must do the following transformations, in this order:

(a) Shift \( y = x^2 \) right by 3.

(b) Stretch the graph by 2 in the vertical direction.

(c) Shift the graph down by 14.
2.1.3 The quadratic formula

We obtain a nice general formula for solutions to a quadratic equation if we complete the square on a general, arbitrary quadratic function. Let’s see how this works.

Consider the general quadratic function

\[ f(x) = ax^2 + bx + c \]

where \(a, b\) and \(c\) are unknown constants. We can put this function into standard form by completing the square.

First we factor out \(a\) from the first two terms,

\[ f(x) = a(x^2 + \frac{b}{a} x) + c. \]

Then we complete the square on \(x^2 + \frac{b}{a} x\). Since \(x^2 + \frac{b}{a} x\) is the beginning of the expression for \((x + \frac{b}{2a})^2\) we compute

\[ (x + \frac{b}{2a})^2 = x^2 + \frac{b}{a} x + \frac{b^2}{4a^2}. \]

Subtracting \(\frac{b^2}{4a^2}\) from both sides gives us

\[ (x + \frac{b}{2a})^2 - \frac{b^2}{4a^2} = x^2 + \frac{b}{a} x. \]

So

\[ f(x) = a(x^2 + \frac{b}{a} x) + c = a((x + \frac{b}{2a})^2 - \frac{b^2}{4a^2}) + c. \]

If we distribute the \(a\) into the term \(\frac{b^2}{4a^2}\) we have

\[ f(x) = a(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a} + c. \]

Now get a common denominator for the last expression:

\[ f(x) = a(x + \frac{b}{2a})^2 - \frac{b^2}{4a} + c. \]

This is the “standard form” for an arbitrary quadratic. (Gosh, I hope no one tries to memorize this answer! It is much easier, when given a general quadratic, to quickly complete the square to get the standard form. If there is anything one might try to memorize, it is the general solution that this result gives us, below.)

Now that we have completed the square on a general quadratic, we could ask where that quadratic is zero. If we wish to solve the equation

\[ ax^2 + bx + c = 0 \]

we might instead use the standard form and solve

\[ a(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a} = 0. \]

This is easy to do: Add \(\frac{b^2 - 4ac}{4a}\) to both sides:

\[ a(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a} \]
and divide both sides by \( a \)
\[
(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2},
\]
and then take the square root of both sides, keeping in mind that we want both the positive and negative square roots:
\[
x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}.
\]
The denominator on right side can be simplified
\[
x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{2a}}.
\]
Finally solve for \( x \) by subtracting \( \frac{b}{2a} \) from both sides
\[
x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}
\]
and combine the right hand side using the common denominator:
\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\]
This is the quadratic formula and this expression is worth memorizing.

**The Quadratic Formula.**

The general solution to the quadratic equation
\[
ax^2 + bx + c = 0
\]
is
\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{7}
\]

**Definition.** The expression \( b^2 - 4ac \) under the radical sign in equation \(7 \) is called the discriminating of the quadratic equation and is sometimes abbreviated by a Greek letter, capitalized delta: \( \Delta \). With this notation, the quadratic formula says that the general solution to the quadratic equation \( ax^2 + bx + c = 0 \) is
\[
x = \frac{-b \pm \sqrt{\Delta}}{2a}.
\]

### 2.1.4 The equation of a circle

We digress for a moment from our study of polynomials to consider another important quadratic equation which appears often in calculus. This is an equation in which both \( x \) and \( y \) are squared, the equation for a circle.

Suppose we have a circle centered at the point \((a, b)\) with radius \( r \). Let \((x, y)\) be a point on the circle. We can draw a right triangle (see the figure on the next page) with short sides of lengths \((x - a)\) and \((y - b)\) and hypotenuse of length \( r \).
By the Pythagorean Theorem,
\[(x - a)^2 + (y - b)^2 = r^2.\] (8)

This is the general equation for a circle.

If we are given a quadratic equation in which both \(x^2\) and \(y^2\) both occur with coefficient 1 then we can recover the equation for a circle by completing the square.

For example, suppose we are given the equation
\[x^2 + 4x + y^2 + 2y = 6.\]

We can complete the square on \(x^2 + 4x\), rewriting that as \((x^2 + 4x + 4) - 4 = (x + 2)^2 - 4\) and also complete the square on \(y^2 + 2y\) writing that as \(y^2 + 2y = (y + 1)^2 - 1\). So
\[x^2 + 4x + y^2 + 2y = 6\]
becomes
\[(x + 2)^2 - 4 + (y + 1)^2 - 1 = 6.\]

or
\[(x + 2)^2 + (y + 1)^2 = 11.\]

This is an equation for a circle with center \((-2, -1)\) and radius \(\sqrt{11}\).

2.1.5 Other resources for quadratic functions

In the free textbook, *Precalculus*, by Stitz and Zeager (version 3, July 2011, available at [stitz-zeager.com](http://stitz-zeager.com)) this material is covered in section 2.3.

In the free textbook, *Precalculus, An Investigation of Functions*, by Lippman and Rasmussen (Edition 1.3, available at [www.opentextbookstore.com](http://www.opentextbookstore.com)) this material is covered in sections 3.1 and 3.2.


There are lots of online resources for studying quadratic equations. Here are two sets of resources, in addition to the class notes and class presentations.

1. [Notes on graphing parabolas](http://www.mathsisfun.com) from Dr. Paul’s webpage at Lamar University.
2. [Video quadratic equations and completing the square](http://www.khanacademy.org) from Khan Academy,
3. Some good stuff at the [mathisfun website](http://www.mathsisfun.com).
Homework.

As class homework, please complete **Worksheet 2.1, Quadratic Functions**, available through the class webpage.
2.2 Polynomial functions and their graphs

2.2.1 Definition of a polynomial

A polynomial of degree \( n \) is a function of the form

\[
f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0
\]

where \( n \) is a nonnegative integer (so all powers of \( x \) are nonnegative integers) and the elements \( a_n, a_{n-1}, \ldots, a_2, a_1, a_0 \) are real numbers.

The integer \( n \) (the highest exponent on \( x \)) is the degree of the polynomial. The real number \( a_n \), the coefficient of \( x^n \), is called the leading coefficient of the polynomial. The constant term is \( a_0 \); it corresponds to the \( y \)-intercept of \( f(x) \).

For example, the polynomial

\[
f(x) = 2x^5 + \pi x^3 + \sqrt{3} x^2 - \frac{13}{7} x + 23
\]

has degree five, with leading coefficient 2 and constant term 23.

The simplest polynomials are the constant functions

\[
f(x) = a_0
\]

(whose graphs are straight lines) and the linear functions

\[
f(x) = a_1 x + a_0.
\]

We have already looked at these, along with the functions of degree two, the quadratics

\[
f(x) = a_2 x^2 + a_1 x + a_0.
\]

Since \( f(x) \) does not involve square roots of the variable \( x \), nor does it have denominators involving the variable \( x \), then the domain of a polynomial is \( (-\infty, \infty) \).

2.2.2 Polynomials are continuous and smooth

Graphs of polynomials are particularly nice. They are continuous, without holes or gaps.

For example, the graphs below are not the graphs of polynomials.

![Figure 2. Graphs with a discontinuity at \( x = 1 \).](image)

Because the graph of a polynomial is continuous, it obeys the [Intermediate Value Theorem](#). This means that if the function takes on a particular \( y \)-value in one place and a different \( y \)-value in another, then the function takes on all possible \( y \)-values between the two.
More explicitly, suppose \( a \) and \( b \) are two real numbers with \( f(a) < f(b) \). Then given any real number \( u \) between \( f(a) \) and \( f(b) \), there is an \( x \)-value \( c \) between \( a \) and \( b \) such that \( f(c) = u \).

A continuous functions such as polynomials cover all \( y \)-values between \( f(a) \) and \( f(b) \) (“intermediate” to \( f(a) \) and \( f(b) \).) Here is a picture from Wikipedia, displaying this relationship.

![Figure 3. The Intermediate Value Theorem](image)

Graphs of polynomials are also “smooth”, they have no sharp corners or cusps. In the picture below, the graph on the left has a sharp corner at \((1, 1)\). The graph on the right has a cusp at the origin. Neither of these graphs could be the graph of a polynomial.

![Figure 4. Graph with a corner; graph with a cusp](image)

**Summary.**

The following properties of a polynomial \( f(x) \) should be visible in the graph of \( y = f(x) \):

1. The domain is all real numbers, \((-\infty, \infty)\).
2. The function is continuous.
3. The function is “smooth.”
2.2.3 End-behavior of the graphs of polynomials

Consider the simplest polynomials, the so-called power functions like $f(x) = x, f(x) = x^2, f(x) = x^3, f(x) = x^4, \ldots$ All of these functions have a form such as that of $f(x) = x^3$ or $f(x) = x^4$.

If the exponent on a power function is even, then the $y$-values go to $+\infty$ whether $x$ is going to $-\infty$ or $\infty$.

![Figure 5. Graphs of $y = x^2$ and $y = x^4$](image)

We say the behavior at infinity (or the “end behavior”) of the polynomial is $\nwarrow \nearrow$, mimicking the action of the graph away from the $x$-axis.

If the degree of the polynomial is even but the leading coefficient is negative, then the end-behavior mimics that of a power function reflected across the $x$-axis. The end behavior should look like the reflection, that is, it will be $\nwarrow \searrow$.

On the other hand, if the graph is $y = x^3$ or a similar graph where the exponent on $x$ is to an odd power, such as

![Figure 6. Graphs of $y = x^3$ and $y = x^5$](image)

then the end behavior is $\nwarrow \nearrow$, mimicking the action of the graph far away from the $y$-axis.
For a general polynomial, the leading term, the part of highest degree, will begin to dominate the graph as \( x \) grows in absolute value. So, ultimately, the graph of \( y = 2x^5 + 23x^4 - 77x^3 + 2x^2 - 100x + 40 \) will begin to look like the graph of \( y = x^5 \) as \( x \) gets larger in absolute value than the coefficients of the other terms. The end behavior of the fifth degree polynomial \( f(x) = 2x^5 + 23x^4 - 77x^3 + 2x^2 - 100x + 40 \) is \( ↗ \downarrow \).

But if the leading coefficient is negative then the end behavior of a polynomial of odd degree looks like \( \searrow \nearrow \). For example, the graph of \( y = -2x^5 + 23x^4 - 77x^3 + 2x^2 - 100x + 40 \) will rise off to the left of the \( y \)-axis and will drop off to the right of the \( y \)-axis. When \( x \) is large in absolute value, the polynomial \( f(x) = -2x^5 + 23x^4 - 77x^3 + 2x^2 - 100x + 40 \) begins to look a lot like \( -2x^5 \).

### 2.2.4 Turning points of a polynomial of degree \( n \)

A polynomial of even degree and positive leading coefficient (such as \( f(x) = 3x^6 + 2x - 7 \)) has a graph with end behavior \( \nearrow \searrow \). Since it drops from the left as we get close to the \( y \)-axis and then rises far off to the right, it must turn around an odd number of times. The local maximums and minimums, where the graph changes direction, are called turning points and the number of turning points gives us some clue to the degree of the polynomial. In particular, the number of turning points is always less than the degree.

For example, the graph of \( f(x) = x^4 - 3x^2 + x + 1 \) (below) crosses the \( x \)-axis four times (near \( x = -1.7, x = -0.4, x = 1, x = 1.25 \)) and has three obvious turning points, around \( x = -1.3, x = 0.2, \) and \( x = 1.1 \).

![Graph of \( y = x^4 - 3x^2 + x + 1 \)](image)

**Figure 7.** Graph of \( y = x^4 - 3x^2 + x + 1 \)

Sometimes a pair of turning points can merge and disappear. If we take the coefficient of \( x^2 \) in the previous example and change \(-3\) to \(-2\) or \(-1\) a pair of turning points eventually disappear.

Here (next page) are the graphs of \( f(x) = x^4 - 2x^2 + x + 1 \) and \( f(x) = x^4 - x^2 + x + 1 \). In the final graph, there is only one turning point, around \( x = -0.9 \).
2.2.5 A first look at the Fundamental Theorem of Algebra

The $x$-intercepts of the graph of a polynomial $f(x)$ are called the “zeroes” (or “roots”) of the polynomial. They are the $x$-values for which $f(x) = 0$.

It is easy to create a polynomial with prescribed zeroes. Suppose we wanted a polynomial with zeroes at $x = -2, x = 1, x = 3$ and $x = 4$. Then we could just multiply:

$$f(x) = (x + 2)(x - 1)(x - 3)(x - 4)$$

If we evaluate this function at $x = -2$ then the first term is zero and so $f(-2)$ is zero. If we evaluate this function at $x = 1$ then the second term is zero and so, in a similar way $f(1) = 0$. And so on.

Here is the graph of $y = (x + 2)(x - 1)(x - 3)(x - 4)$.

Figure 9. Graph of $y = (x + 2)(x - 1)(x - 3)(x - 4)$

Note that this function has four zeroes (or roots or $x$-intercepts), occurring at $x = -2, 1, 3$ and 4.

One might observe that if we wanted four different zeroes (such as $x = -2, 1, 3$ and 4 in this case) then the polynomial should have degree 4. There is a vague sense in which this (the number of zeroes) is the meaning of degree.

We might hope that a polynomial of degree $n$ has $n$ zeroes. This is almost true. We will elaborate on this more in a later lesson. But here is a first draft of the Fundamental Theorem of Algebra.
Fundamental Theorem of Algebra (first version):

A polynomial of degree \( n \) has \textit{at most} \( n \) zeroes.

We can often find \( n \) zeroes if we are willing to count some zeroes as occurring more than once.

For example the polynomial \( x^4 - 4x^2 + 3 = (x^2 - 1)(x^2 - 3) \) has four zeroes, occurring at \( x = -1, x = 1, x = -\sqrt{3} \) and \( x = \sqrt{3} \). But if we alter the polynomial a little, dropping the constant term, we have \( x^4 - 4x^2 = x^2(x^2 - 4) = x^2(x - 2)(x + 2) = (x - 0)(x - 2)(x + 2) \). This has zeroes at \( x = 2, -2 \) and 0. But the zero at \( x = 0 \) occurs because of the factor \( x^2 \); we should count that zero \textit{twice}. So we say that \( f(x) = x^4 - 4x^2 = (x - 0)(x - 0)(x - 2)(x + 2) \) has zeroes \(-2, 0, 0, 2\). If we count the zero \( x = 0 \) twice, we have as many zeroes as the degree.

Here are the graphs of these two different polynomials.

![Graphs of polynomials](image)

Figure 10. Graphs of \( y = x^4 - 4x^2 + 3 \) and \( y = x^4 - 4x^2 \)

Notice that the zero at \( x = 0 \), which occurs twice in the polynomial on the right, is visible as a zero where the curve does not go through the \( x \)-axis (like the other places where a zero occurs) but instead the curve moves close and \textit{kisses} the \( x \)-axis before moving away.

Two worked problems.

1. Give a polynomial of degree 3 with roots (zeroes) \( x = 0, x = 1, x = 3 \).

   \textbf{Solution.} All solutions will have the form \( a(x - 1)(x - 3) \) where \( a \) is some real number.

   (So one answer might simply be \( x(x - 1)(x - 3) \).)

2. Give the polynomial of degree 3 with roots \( x = 0, x = 1, x = 3 \) passing through the point \((-1, 16)\).

   \textbf{Solution} Polynomials with roots \( x = 0, x = 1, x = 3 \) will have the form \( ax(x - 1)(x - 3) \) where \( a \) is some real number. Here we need to find \( a \). Substitute \( x = -1 \) into the expression \( f(x) = ax(x - 1)(x - 3) \) to see that \( f(-1) = -8a \). The polynomial we are after has \( f(-1) = 16 \) so \( a = -2 \).

   \textbf{Answer:} \( -2x(x - 1)(x - 3) \)

2.2.6 The sign diagram for a polynomial

A convenient aid to graphing a polynomial is to locate the zeroes of the polynomial and then draw a “sign diagram.” A sign diagram is a convenient way to keep up with the sign of the polynomial in the regions between the zeroes.
For example, consider the polynomial

\[ g(x) = -2(x - 1)^2(x + 3)(x - 4). \]

This polynomial has zeroes at \( x = 1, x = -3 \) and \( x = 4 \). In order, from smallest to largest, these zeroes are \(-3, 1, 4\).

The intermediate value theorem assures us that the only way the graph of the polynomial \( g(x) \) crosses the \( x \)-axis is at a zero, so in each region between the zeroes, \((-\infty, -3), (-3, 1), (1, 4), \) and \((4, \infty), \) the polynomial has a particular sign; it is either positive or negative. (Visualize the zeroes of the polynomials as fences, separating the regions \((-\infty, -3), (-3, 1), (1, 4), \) and \((4, \infty).\))

To the left of \( x = -3 \) we can find some test value (such as \( x = -4 \)) and determine the sign of \( g(-4) \). We really don’t need to completely compute the value of \( g(-4) \) but merely find its sign: \( g(-4) = -2(-5)^2(-1)(-8) \) must be negative because we are multiplying five negative numbers together. Since minus signs cancel in pairs, the fact that we have multiplied an odd number of negative numbers gives us a negative number.

Between \( x = -3 \) and \( x = 1 \), pick a nice number – \( x = 0 \) is the best! – and compute the sign of \( g(0) \). This tells us that for every \( x \)-value between \( x = -3 \) and \( x = 1 \), \( g(x) \) must be positive.

Between \( x = 1 \) and \( x = 4 \), pick a number like \( x = 2 \) and compute the sign of \( g(2) \). Here \( g(2) = -2(1)^2(5)(-2) \) is negative.

Finally, to the right of \( x = 4 \) pick a number, say \( x = 5 \) and find the sign of \( g(5) \). It is negative in this case, due to the leading coefficient \(-2\).

We collect this all in a chart, the sign diagram, which shows us the signs of \( g(x) \) in the regions created by the zeroes.

\[
\begin{array}{c|c|c|c|c}
\text{(-)} & (+) & (+) & (-) \\
-3 & 1 & 4 \\
\end{array}
\]

**Figure 11.** The sign diagram of \( g(x) = -2(x - 1)^2(x + 3)(x - 4) \)

From this diagram, we know that as \( x \) approaches \(-3\) from the left, the graph of \( g(x) \) rises to the \( x \)-axis and passes through the \( x \)-axis at \( x = -3 \), then stays above the \( x \)-axis until \( x = 1 \) when it drops back to the axis, kisses the \( x \)-axis and bounces back up, staying above the \( x \)-axis until \( x = 4 \) when it passes through the \( x \)-axis and drops below it as \( x \) continues to the right.

Here, below, is the true graph of \( y = g(x) \).

![Graph of \( y = -2(x - 1)^2(x + 3)(x - 4) \)](image)

**Figure 12.** The graph of \( y = -2(x - 1)^2(x + 3)(x - 4) \)
A worked problem.

Let’s finish our analysis of the polynomial \( g(x) = -2(x - 1)^2(x + 3)(x - 4) \), above. Here are some typical questions one might be asked about \( g(x) \).

1. Describe the end behavior of the graph of \( y = g(x) \).
   Solution. This is a fourth degree polynomial with leading coefficient negative. So the end behavior is ↙↘.

2. Find the real zeroes of \( g(x) \).
   Solution. \( x = 1 \) (twice), \( x = -3 \), \( x = 4 \).

3. Find the \( y \)-intercepts of \( g(x) \).
   Solution. \((0, 24)\).

4. Determine the maximal number of turning points of the graph of \( y = g(x) \).
   Solution. Since the polynomial has degree four then it has at most \( \text{three} \) turning points.

5. Draw the sign diagram of \( y = g(x) \) and then sketch the graph.
   Solution. The sign diagram is drawn above in figure 11; the graph is drawn in figure 12.

2.2.7 Other resources for graphs of polynomial functions

In the free textbook, *Precalculus*, by Stitz and Zeager (version 3, July 2011, available at stitz-zeager.com) this material is covered in section 3.1.


There are lots of online resources for studying polynomials. Here are some I recommend.

1. Dr. Paul’s online math notes on polynomials.
2. Videos on polynomials from Khan Academy,

Homework.

As class homework, please complete Worksheet 2.2, Polynomial Functions available through the class webpage.

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2.3 Zeroes of polynomials and long division

2.3.1 Zeroes of polynomial functions

The Fundamental Theorem of Algebra tells us that every polynomial of degree \(n\) has at most \(n\) zeroes. Indeed, if we are willing to count multiplicity of zeroes and also count complex numbers (more on that later) then a polynomial of degree \(n\) has exactly \(n\) zeroes!

A major goal to understanding a polynomial is to write it in terms of its zeroes. Each zero \(c\) corresponds to a factor \(x - c\) so understanding the zeroes of a polynomial is equivalent to completely factoring the polynomial.

Consider the polynomial graphed below.

![Graph of a polynomial](image)

**Figure 13.** A certain polynomial

From the graph, can we see the number of turning points, the degree of the polynomial and the zeroes of the polynomial? Furthermore, from this graph can we in fact write out the polynomial exactly?

**Solution.**

The graph apparently has two turning points and so probably has degree three. It has zeroes \(x = -1, x = 1\) and \(x = 2\) which agrees with our guess that the degree is 3.

Since the graph has zeroes at \(-1, 1\) and \(2\) and presumably has degree 3, then it should have form

\[
f(x) = a(x + 1)(x - 1)(x - 2)
\]

for some unknown \(a\). (The unknown \(a\) is the leading term of this polynomial.)

Can we guess the leading coefficient \(a\) from the graph? Since the graph goes through the point \((0, 2)\) then \(f(0) = 2\). We see by direct computation from the formula above that \(f(0) = 2a\) so \(a = 1\). Therefore the polynomial graphed in figure 13 above must be

\[
f(x) = (x + 1)(x - 1)(x - 2).
\]

**Another Example.**

Find a polynomial \(f(x)\) of degree 3 with zeroes \(x = -1, x = 1\) and \(x = 2\) where the graph of \(y = f(x)\) goes through the point \((3, 16)\).

**Solution.** Because the zeroes are \(-1, 1, 2\) then factors of the polynomial should be \(x + 1, x - 1\) and \(x - 2\). If \(f(x) = a(x + 1)(x - 1)(x - 2)\) then \(16 = f(3) = a(3 + 1)(3 - 1)(3 - 2) = 8a\) so \(a = 2\). So the
answer is \[ f(x) = 2(x + 1)(x - 1)(x - 2). \]

These examples are intended to demonstrate that our understanding of a polynomial is very closely related to our knowledge of its zeroes.

In the next section we concentrate on dividing polynomials by smaller ones, with an eye to eventually factoring the polynomial and finding all its zeroes.

2.3.2 The Division Algorithm

The Division Algorithm for polynomials promises that if we divide a polynomial by another polynomial, then we can do this in such a way that the remainder is a polynomial with degree smaller than that of the divisor.

We will make that precise in a moment, but let us first review the Division Algorithm for integers, and, as we do this, review “long division.”

Suppose that we wish to divide 23 by 5. We notice that 5 goes into 23 at most 4 times and that
\[ 20 = 5 \cdot 4. \]

So we may take 20 away from 23, leaving a remainder of 3. We write that (in the United States) as a long division problem in the following form:

\[
\begin{array}{c|c}
4 & 23 \\
\hline
5 & 20 \\
\hline
3 & \\
\end{array}
\]

We say that dividing 5 into 23 leaves a quotient of 4 and a remainder of 3.

There are equivalent ways to write this. We can write
\[
\frac{23}{5} = 4 + \frac{3}{5}
\]
or
\[
23 = (4)(5) + 3.
\]

Let us do a more complicated example. Suppose we divide 231 by 5. The easiest way to do this is to take advantage of our decimal notation and first divide 23 by 5 (as before) and note that 5 goes into 23 4 times. If 5 goes into 23 4 times then 5 goes into 230 at least 40 times.

Indeed, 5 goes into 231 at least 40 times. If we use 40 as our (temporary) quotient, we have a remainder of 31.

However, this remainder 31 is at least as big as the divisor 5 so we can divide 5 into 31 a few more times (6) and get a remainder of 1. We write this long division as

\[
\begin{array}{c|c}
46 & 231 \\
\hline
5 & 200 \\
\hline
31 & 30 \\
\hline
1 & \\
\end{array}
\]

and say that 231 divided by 5 leaves a quotient of 46 and a remainder of 1. Thus
\[
\frac{231}{5} = 46 + \frac{1}{5}
\]
or
\[
231 = (46)(5) + 1.
\]
We can do the same computations (the “Division Algorithm”) with polynomials.
Let us divide the polynomial $2x^4 - 3x^3 + 5x - 36$ by $x^2 + x + 2$. Let’s write this as a long division problem:

$$
\begin{array}{r}
  x^2 + x + 2
\end{array}
\overline{\begin{array}{r}
  2x^4 - 3x^3 \\
  + 5x - 36
\end{array}}
\]

We keep things simple (as we did when dividing 231 by 5) by focusing on part of the problem. If we just try to divide $2x^4$ by $x^2$ we would get $2x^2$. Let’s use $2x^2$ as the first guess at our quotient.

$$
\begin{array}{r}
  2x^2 \\
\end{array}
\overline{\begin{array}{r}
  2x^4 - 3x^3 \\
  + 5x - 36
\end{array}}
\]

We multiply the divisor $x^2 + x + 2$ by the quotient $2x^2$ to obtain $2x^4 + 2x^3 + 4x^2$ and subtract this from the original polynomial. This leaves a remainder of $-5x^3 - 4x^2 + 5x - 36$.

$$
\begin{array}{r}
  2x^2 \\
\end{array}
\overline{\begin{array}{r}
  2x^4 - 3x^3 \\
  - 2x^4 - 2x^3 - 4x^2 \\
  - 5x^3 - 4x^2 + 5x
\end{array}}
\]

Are we done?

No. Recall that in our long division of 231 by 5, we got a temporary remainder of 31 which was larger than the divisor and so we divided again, dividing 5 into the new remainder. In a similar way, here our remainder is also larger than the divisor — the remainder has larger degree than the divisor — and so we can divide into it again.

Since $-5x^3$ divided by $x^2$ is $-5x$, let us guess that $x^2 + x + 2$ goes into the temporary remainder $-5x^3 - 4x^2 + 5x - 36$ about $-5x$ times. This gives another layer of our long division.

$$
\begin{array}{r}
  2x^2 - 5x \\
\end{array}
\overline{\begin{array}{r}
  2x^4 - 3x^3 \\
  - 2x^4 - 2x^3 - 4x^2 \\
  - 5x^3 - 4x^2 + 5x
\end{array}}
\]

We multiply the divisor $x^2 + x + 2$ by $-5x$ and subtract....

$$
\begin{array}{r}
  2x^2 - 5x \\
\end{array}
\overline{\begin{array}{r}
  2x^4 - 3x^3 \\
  - 2x^4 - 2x^3 - 4x^2 \\
  - 5x^3 - 4x^2 + 5x \\
  5x^3 + 5x^2 + 10x
\end{array}}
\]

Are we done here?

No. Now the degree of the remainder is the same as the degree of the divisor, which means we can go one more step.

$$
\begin{array}{r}
  2x^2 - 5x + 1 \\
\end{array}
\overline{\begin{array}{r}
  2x^4 - 3x^3 \\
  - 2x^4 - 2x^3 - 4x^2 \\
  - 5x^3 - 4x^2 + 5x \\
  5x^3 + 5x^2 + 10x
\end{array}}
\]

$$
\begin{array}{r}
  x^2 + 15x - 36 \\
\end{array}
\overline{\begin{array}{r}
  2x^2 - 5x + 1 \\
\end{array}}
\]

$$
\begin{array}{r}
  - x^2 - x - 2 \\
\end{array}
\overline{\begin{array}{r}
  x^2 + 15x - 36 \\
\end{array}}
\]

$$
\begin{array}{r}
  14x - 38
\end{array}
\]
At this point we have a remainder now of degree smaller than the degree of the divisor and so we are forced to stop.

Our quotient is \(2x^2 - 5x + 1\) and our remainder is \(14x - 38\).

We may write this out as either
\[
\frac{2x^4 - 3x^3 + 5x - 36}{x^2 + x + 2} = 2x^2 - 5x + 1 + \frac{14x - 38}{x^2 + x + 2}
\]
or
\[
2x^4 - 3x^3 + 5x - 36 = (2x^2 - 5x + 1)(x^2 + x + 2) + 14x - 38.
\]

Let us do another example, this time one which is a little bit simpler. Let us divide \(2x^4 - 3x^3 + 5x - 36\) by just \(x - 2\). Here is the long division.

\[
\begin{array}{c|ccccc}
& 2x^3 & + x^2 & + 2x & + 9 \\
\hline
x-2 & 2x^4 & - 3x^3 & + 5x & - 36 \\
 & - 2x^4 & + 4x^3 & \\
\hline
 & - x^3 & + 2x^2 & \\
 & - x^3 & + 2x^2 & + 5x & \\
\hline
 & - 2x^2 & + 9x & \\
 & - 2x^2 & + 4x & \\
\hline
 & 9x & - 36 & \\
 & - 9x & + 18 & \\
\hline
 & - 18 & \\
\end{array}
\]

In this case
\[
\frac{2x^4 - 3x^3 + 5x - 36}{x - 2} = 2x^3 + x^2 + 2x + 9 - \frac{18}{x - 2}
\]
or
\[
2x^4 - 3x^3 + 5x - 36 = (2x^3 + x^2 + 2x + 9)(x - 2) - 18.
\]

We summarize our work in this section by explicitly stating the Division Algorithm as a theorem.

**Theorem.** (The Division Algorithm)
Suppose \(f(x)\) and \(d(x)\) are polynomials with real coefficients. We may divide \(f(x)\) by \(d(x)\) and obtain a quotient \(q(x)\) and a remainder \(r(x)\), so that
\[
f(x) = q(x)d(x) + r(x) \tag{9}
\]
where the degree of \(r(x)\) is strictly less than the degree of the divisor \(d(x)\).

**2.3.3 The Remainder Theorem**
We digress for a moment to discuss the value of the remainder in a long division problem.

Here is an example. Observe that if
\[
f(x) = 2x^4 - 3x^3 + 5x - 36 = (2x^3 + x^2 + 2x + 9)(x - 2) - 18
\]
then
\[
f(2) = (2 \cdot 2^3 + 2^2 + 2 \cdot 2 + 9)(2 - 2) - 18.
\]
Ignore the first expression on the right side involving sums of powers of 2. The critical concept here is that we have $2 - 2 = 0$ in the expression for $f(2)$ and any numbers times zero is zero. Thus this expression simplifies to

$$f(2) = q(2) \cdot (0) - 18 = -18.$$  

More generally: if, by the division algorithm we divide a polynomial $f(x)$ by $d(x)$ and obtain a quotient $q(x)$ and a remainder $r(x)$, so that

$$f(x) = q(x)d(x) + r(x) \quad (10)$$

then if $c$ is a zero of $d(x)$ then $f(c) = q(c) \cdot 0 + r(c) = r(c)$.

This is especially important if the divisor polynomial is nice and linear. Suppose we divide a polynomial $f(x)$ by $x - c$ and obtain a quotient $q(x)$ and a remainder $r$. Then

$$f(x) = q(x)(x - c) = r$$

and so

$$f(c) = q(c)(c - c) + r = q(c) \cdot 0 + r = r.$$  

**The Remainder Theorem.**

If $f(x)$ is a polynomial then $f(c)$ is the remainder obtained by dividing $f(x)$ by $x - c$.

**A Worked Example.**

The polynomial

$$f(x) = x^{100} - 3x^{98} - 2x^{97} + 5x^4 - 7x^2 + 3,$$

when divided by $x - 2$ gives a quotient $q(x)$ and a remainder $r(x) = 55$. (Just take my word for it – I did a computation on a computer algebra system at [Wolfram Alpha](https://www.wolframalpha.com).)

Given this information, what is $f(2)$? (Why?)

**Solution.** We can write $f(x) = (x - 2)q(x) + 55$. So if we evaluate this expression at $x = 2$ we have $f(2) = 0 \cdot q(2) + 55 = 55$. So $f(2) = 55$.

**Another Example.**

A certain polynomial $f(x)$ of degree 999, when divided by $x^2 - 9$ gives a quotient $q(x)$ and a remainder $r(x) = 4x + 11$. What is $f(3)$? (Why?)

**Solution.**

We are given that $f(x) = (x^2 - 9)q(x) + 4x + 11$. Then $f(3) = (3^2 - 9)q(3) + 4(3) + 11 = 0 \cdot q(3) + 12 + 11 = 23$. So $f(3) = 23$.

The Remainder Theorem helps motivate a shorthand notation for dividing a polynomial by a nice linear factor of the form $x - c$. This shorthand notation is called synthetic division.

**2.3.4 Synthetic division**

Let’s review our long division when we divided $2x^4 - 3x^3 + 5x - 36$ by $x - 2$. 


If we really don’t want to write all these $x$’s down, what really matters is the coefficients appear in each row. If we take into account that we are subtracting, we might organize these coefficients as

\[
\begin{array}{c}
-2 & 4 \\
-1 & 2 \\
-2 & 4 \\
-9 & 18
\end{array}
\]

and, just for simplicity, not write down the powers of $x$. We can then condense all this work into a short array, something like this:

\[
\begin{array}{cccc}
2 & -3 & 0 & 5 & -36 \\
4 & 2 & 4 & 18 \\
2 & 1 & 2 & 9 & -18
\end{array}
\]

This short array (a couple of lines) captures all the work we did in our long division problem. This line of steps in long division is sometimes called “synthetic division”. Here is how this works.

If we want to divide $2x^4 - 3x^3 + 5x - 36$ by $x - 2$, we write down the coefficients of the larger polynomial across the first line. Then, since we are dividing by $x - 2$ (which has a zero at $x = 2$) we write 2 on the far left. (Note that we write 2, not $-2$! This 2 is a potential zero of the polynomial; it will be our $c$ in the computation $f(c)$.)

\[
\begin{array}{cccc}
2 & -3 & 0 & 5 & -36 \\
2 & -3 & 0 & 5 & -36
\end{array}
\]

Then we pull down the first coefficient.

\[
\begin{array}{cccc}
2 & -3 & 0 & 5 & -36 \\
2 & -3 & 0 & 5 & -36
\end{array}
\]

We multiply it by the original $c = 2$ to obtain 4 and place that below the next coefficient.

\[
\begin{array}{cccc}
2 & -3 & 0 & 5 & -36 \\
2 & -3 & 0 & 5 & -36
\end{array}
\]
We add the coefficients. In this case \(-3 + 4 = 1\).

\[
\begin{array}{cccccc}
 & 2 & -3 & 0 & 5 & -36 \\
2 & 4 \\
\hline
 & 2 & 1 & 2 \\
\end{array}
\]

We again multiply 1 by \(c = 2\) and place that below the next coefficient and add.

\[
\begin{array}{cccccc}
 & 2 & -3 & 0 & 5 & -36 \\
2 & 4 & 2 \\
\hline
 & 2 & 1 & 2 & 9 \\
\end{array}
\]

We continue this process one coefficient at a time to get

\[
\begin{array}{cccccc}
 & 2 & -3 & 0 & 5 & -36 \\
2 & 4 & 2 & 4 \\
\hline
 & 2 & 1 & 2 & 9 & 18 \\
\end{array}
\]

and finally

\[
\begin{array}{cccccc}
 & 2 & -3 & 0 & 5 & -36 \\
2 & 4 & 2 & 4 & 18 \\
\hline
 & 2 & 1 & 2 & 9 & -18 \\
\end{array}
\]

Now we read off the meaning of the bottom row. At the far right is the remainder \(r = -18\). The rest of the bottom row gives the quotient \(2x^3 + x^2 + 2x + 9\).

Let’s work a few more examples.

**Worked Examples.**

1. Divide \(3x^5 - 8x^3 - 2x + 10\) by \(x - 2\).

   **Solution.**

   Here we use \(c = 2\) in our problem. Don’t forget to write all the coefficients of \(3x^5 - 8x^3 - 2x + 10\); they are 3, 0, -7, 0, -2, 10.

   \[
   \begin{array}{cccccc}
   & 3 & 0 & -8 & 0 & -2 & 10 \\
2 & 6 & 12 & 8 & 16 & 28 \\
\hline
   & 3 & 6 & 4 & 8 & 14 & 38 \\
   \end{array}
   \]

   Answer: \(3x^5 - 8x^3 - 2x + 10 = (3x^4 + 6x^3 + 4x^2 + 8x + 14)(x - 2) + 38\)

2. Divide \(3x^5 - 8x^3 - 2x + 10\) by \(x + 2\).

   **Solution.**

   Here we use \(c = -2\) in our problem.

   \[
   \begin{array}{cccccc}
   & 3 & 0 & -8 & 0 & -2 & 10 \\
-2 & -6 & 12 & -8 & 16 & -28 \\
\hline
   & 3 & -6 & 4 & -8 & 14 & -18 \\
   \end{array}
   \]

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Answer: \( 3x^5 - 8x^3 - 2x + 10 = (3x^4 - 6x^3 + 4x^2 - 8x + 14)(x - 2) - 18 \)

3. Let \( f(x) = 3x^5 - 8x^3 - 2x + 10 \). Compute

(a) \( f(2) \)
(b) \( f(-2) \)

**Solution.**

(a) From our work in problem 1, and our understanding of the Remainder Theorem, we see that \( f(2) = 38 \), the remainder when \( f(x) \) is divided by \( x - 2 \).

(b) From our work in problem 2, and our understanding of the Remainder Theorem, we see that \( f(-2) = -18 \).

### 2.3.5 Other resources on long division

In the free textbook, *Precalculus*, by Stitz and Zeager (version 3, July 2011, available at stitz-zeager.com) this material is covered in section 3.2.


There are lots of online resources for studying polynomial division and results about the zeroes of polynomials. Here are some I recommend.

3. [Khan Academy videos on synthetic division.](https://www.khanacademy.org/math/algebra2/precalculus)
4. [Paul Dawkin’s online notes on dividing polynomials.](http://tutorial.math.lamar.edu hạng6/Polynomials/Polynomials.aspx)

**Worksheet to go with these notes.**

As class homework, please complete **Worksheet 2.3, Zeroes of polynomials**, available through the class webpage.
2.4 Complex numbers

2.4.1 A first look at complex numbers

The complex number system is an extension of the real number system. It unifies the mathematical number system and explains many mathematical phenomena.

We introduce a number $i = \sqrt{-1}$ defined to satisfy the equation $x^2 = -1$. (As soon as we introduce this number, there is some ambiguity, for $x = -i$ also satisfies $x^2 = -1$!) The complex numbers are defined as all numbers of the form $a + bi$ where $a$ and $b$ are real numbers. We write

$$\mathbb{C} := \{a + bi : a, b \in \mathbb{R}\}.$$ 

A complex number of the form $z = a + bi$ is said to have real part $\Re = a$ and imaginary part $\Im = b$.

Any “number” can be written in this form. The number $i$ has real part 0 and is said to be “purely imaginary”; the number 5 has imaginary part 0 and is “real”. The real numbers are a subset of the complex numbers.

The conjugate of a complex number $z = a + bi$ is created by changing the sign on the imaginary part: $\bar{z} = a - bi$. Thus the conjugate of $2 + i$ is $2 + i = 2 - i$; the conjugate of $\sqrt{3} - \pi i$ is $\sqrt{3} + \pi i$. The conjugate of $i$ is $i = -i$ and the conjugate of the real number 5 is merely 5.

2.4.2 Motivation for the complex numbers

The nicest version of the Fundamental Theorem of Algebra says that every polynomial of degree $n$ has exactly $n$ zeroes. But this is not quite true.

Or is it?

Consider the functions $f(x) = x^2 - 1$, $g(x) = x^2$ and $h(x) = x^2 + 1$. We graph these functions below.

![Figure 14. Three quadratics](image)

It is obvious that the quadratic $f(x) = x^2 - 1$ (drawn in green in the figure) has two zeroes. Indeed, they are easy to find – set $x^2 - 1 = 0$, factor $x^2 - 1$ into $(x - 1)(x + 1)$ and so see that $x = 1$ and $x = -1$ are the zeroes. Or ... just look at the graph.

Now let’s move the green parabola up one unit, to graph $y = x^2$, drawn in blue. What happened to our two zeroes? They merged into the single $x$-intercept at the origin. But we can claim that $g(x) = x^2$ still has two zeroes, if we are willing to count multiplicities. This makes some sense because we can write

$$x^2 = (x - 0)(x - 0).$$
and since $x - 0$ is a factor twice, we could claim that $x = 0$ is a zero twice.

But what if we move the parabola up one more step and graph $y = x^2 + 1$ (drawn in red)? Now, suddenly, there are no solutions. The graph never touches the $x$-axis.

Algebraically, $h(x) = x^2 + 1$ does not have any zeroes because that would require that

$$x^2 + 1 = 0$$

which then requires

$$x^2 = -1.$$  

If we square any real number, the result is positive, so it is not possible for a square to be equal to $-1$.

In the late middle ages, mathematicians discovered that if one were willing to allow for a new number, one whose square was $-1$, quite a lot of mathematics got simpler! Indeed, in solutions to cubic equations, one could sometimes find all solutions by pretending for just a moment that there was a solution to $x^2 = -1$ and then, after a few steps, observing that this “imaginary” piece disappeared and one had the correct solutions to the cubic equation.

This “imaginary” number was therefore very useful, even if one didn’t quite believe in it.

Over time, the term “imaginary” has stuck, even though scientists and engineers now use complex numbers all the time. It is now common agreement to write $i$ as an entity that satisfies

$$i^2 = -1.$$  

Once we have done this, the equation

$$x^2 = -1$$

has two solutions,

$$x = i \text{ and } x = -i.$$  

So the polynomial $x^2 + 1$ factors as $(x + i)(x - i)$ and the function $h(x) = x^2 + 1$ has two zeroes, just like the other quadratics. It is just that the zeroes of $h(x)$ are imaginary and are not on the $x$-axis.

A brief article on applications of complex numbers is [here at Wikipedia](https://en.wikipedia.org/wiki/Complex_number). Modern cell phone signals rely on sophisticated signal analysis; we would not have cell phones without the mathematics of complex numbers.

### 2.4.3 Complex numbers and the quadratic formula

Complex numbers appear naturally in quadratic equations. Suppose we wish to solve the quadratic equation

$$ax^2 + bx + c = 0$$

By completing the square we can solve for $x$ and find that

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression $b^2 - 4ac$ under the radical sign is called the **discriminant** of the quadratic equation and is often abbreviated by $\Delta$.

If $\Delta = b^2 - 4ac$ is positive then the square root of $\Delta$ is a real number and so the quadratic equation has two real solutions:

$$x = \frac{-b + \sqrt{\Delta}}{2a} \text{ and } x = \frac{-b - \sqrt{\Delta}}{2a}$$
If $\Delta$ is zero then there is only one solution since
\[ x = \frac{-b \pm \sqrt{\Delta}}{2a} = -\frac{b \pm \sqrt{0}}{2a} = -\frac{b}{2a}. \]
This single solution occurs with multiplicity two.

But if $\Delta$ is negative then $\sqrt{\Delta}$ is imaginary and so our solutions are complex numbers which are not real. To be explicit, if $\Delta$ is negative then $-\Delta$ is positive and so $\sqrt{\Delta} = \sqrt{-\Delta} i$. The solutions to the quadratic formula are then
\[ x = \frac{-b + \sqrt{-\Delta} i}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{-\Delta} i}{2a}. \]
In this case, the plus/minus sign ($\pm$) in front of $\sqrt{\Delta}$ assures us that we will get two complex numbers as solutions.

These two complex solutions come in conjugate pairs. If one complex number is the root of a quadratic polynomial (with real coefficients) then its conjugate is also a root.

For example, the solutions to the quadratic equation
\[ x^2 + x + 1 = 0 \]
are
\[ \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2} = -\frac{1 \pm \sqrt{-3}}{2} = -\frac{1 \pm \sqrt{3}i}{2} = -\frac{1 \pm \sqrt{3}i}{2}. \]
Thus the two solutions to the equation $x^2 + x + 1 = 0$ are the complex conjugate pairs
\[ -\frac{1}{2} + \frac{\sqrt{3}}{2}i \quad \text{and} \quad -\frac{1}{2} - \frac{\sqrt{3}}{2}i. \]
Since these are the two solutions to the equation $x^2 + x + 1 = 0$ then the polynomial $x^2 + x + 1$ factors as
\[ (x - (-\frac{1}{2} + \frac{\sqrt{3}}{2}i))(x - (-\frac{1}{2} - \frac{\sqrt{3}}{2}i)) \]
\[ = (x + \frac{1}{2} - \frac{\sqrt{3}}{2}i)(x + \frac{1}{2} + \frac{\sqrt{3}}{2}i) \]

Some worked examples.

1. Solve the quadratic equation $x^2 - x + 1 = 0$. Also, factor $x^2 - x + 1$.

   \textbf{Solution.} By the quadratic formula the solutions to $x^2 - x + 1 = 0$ are
   \[ \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm \sqrt{3}i}{2}. \]
   Since the two solutions to the equation $x^2 - x + 1 = 0$ are the complex numbers
   \[ \frac{1}{2} + \frac{\sqrt{3}}{2}i \quad \text{and} \quad \frac{1}{2} - \frac{\sqrt{3}}{2}i. \]
   then the polynomial $x^2 - x + 1$ factors as
   \[ (x - (\frac{1}{2} + \frac{\sqrt{3}}{2}i))(x - (\frac{1}{2} - \frac{\sqrt{3}}{2}i)) \]
   \[ = (x - \frac{1}{2} - \frac{\sqrt{3}}{2}i)(x - \frac{1}{2} + \frac{\sqrt{3}}{2}i) \]
2. Solve the quadratic equation
\[ 2x^2 + 5x + 7 = 0 \]

**Solution.** According to the quadratic formula,
\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-5 \pm \sqrt{5^2 - 4(2)(7)}}{4} = \frac{-5 \pm \sqrt{-31}}{4} = \frac{-5 \pm \sqrt{31}i}{4} = \frac{-5}{4} \pm \frac{\sqrt{31}}{4}i. \]

Our two solutions are the conjugate pairs
\[ x = -\frac{5}{4} + \frac{\sqrt{31}}{4}i \text{ and } x = -\frac{5}{4} - \frac{\sqrt{31}}{4}i. \]

3. Use the roots of \(2x^2 + 5x + 7\) to factor \(2x^2 + 5x + 7\).

**Solution.** Since the two solutions to the equation \(2x^2 + 5x + 7 = 0\) are
\[ x = -\frac{5}{4} + \frac{\sqrt{31}}{4}i \text{ and } x = -\frac{5}{4} - \frac{\sqrt{31}}{4}i \]

and since \(c\) is a zero of a polynomial if and only if \(x - c\) is a factor, then
\[(x - \left(-\frac{5}{4} + \frac{\sqrt{31}}{4}i\right))(x - \left(-\frac{5}{4} - \frac{\sqrt{31}}{4}i\right))\]
must be a factor of \(2x^2 + 5x + 7\). But if we check the coefficient of \(x^2\) in the expression above, we see that we need to multiply by 2 to complete the factorization. So \(2x^2 + 5x + 7\) factors as
\[ 2x^2 - 5x + 7 = 2\left(x + \frac{5}{4} - \frac{\sqrt{31}}{4}i\right)\left(x + \frac{5}{4} + \frac{\sqrt{31}}{4}i\right) \]

2.4.4 Geometric interpretation of complex numbers

Mathematicians began to recognize the value of complex numbers sometime back in the Renaissance period (fifteenth and sixteenth centuries) but it was not until there was a geometric interpretation of the complex numbers that people began to feel comfortable with them.

We may view the complex numbers as lying in the Cartesian plane. Let the traditional \(x\)-axis represent the real numbers and the traditional \(y\)-axis represent the numbers of the form \(yi\). We equate a complex number \(x + yi\) with the point \((x, y)\). (So the imaginary numbers \(yi\) are “perpendicular” to the real numbers!)

The complex plane is drawn on the next page.
The process of changing \( a + bi \) into the point \((a, b)\) can be traced to Argand around 1800 and is sometimes called the “Argand diagram”. In the Argand diagram, the complex number \( z = a + bi \) is equated with the point \((a, b)\) in the Cartesian plane. For example, \(6 + 5i\) can be graphed as the point \((6, 5)\). The ordinary Cartesian plane then becomes a plane of complex numbers.

Thus \( i = 0 + 1i \) is equated with the point \((0, 1)\) and the number \(1 = 1 + 0i\) is equated with the point \((1, 0)\). The point \((2, 1)\) represents the number \(2 + i\). The number \((\sqrt{3} + i)/2\) is equated with the point \((\sqrt{3}/2, 1/2)\).

In the complex plane the \(x\)-axis is called the “real” axis and the \(y\)-axis is called the imaginary” axis.

2.4.5 The algebra of complex numbers

The complex numbers have a natural addition, subtraction and multiplication.

We add and subtract complex numbers just as we would polynomials, keeping up with the real and imaginary parts. For example,

\[(3 + 4i) + (7 + 11i) = 10 + 15i\]

and

\[(3 + 4i) - (7 + 11i) = -4 - 7i.\]

We multiply complex numbers \((a + bi)(c + di)\) just as we would the polynomials \((a + bx)(c + dx)\) except that we remember that \(i^2 = -1\).

For example, since

\[(3 + 4x)(7 + 11x) = 21 + 61x + 44x^2.\]

then

\[(3 + 4i)(7 + 11i) = 21 + 61i + 44i^2 = 21 + 61i - 44 = -23 + 61i.\]
2.4.6 Division of complex numbers

We would like our complex numbers to be written in “Cartesian form” $a + bi$ so there is a little twist involved in doing division with complex numbers. Note that if $z = a + bi$ then $z\bar{z} = (a + bi)(a - bi) = a^2 + b^2$. So if we are dividing by $z$, we may view $\frac{1}{z}$ as

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i.$$

Computationally, this means that anytime we have a fraction involving a complex number $z$ in the denominator, we may multiply both numerator and denominator by $\bar{z}$ and simplify. For example,

$$\frac{3 + 4i}{7 - 11i} = \frac{(3 + 4i)(7 + 11i)}{(7 - 11i)(7 + 11i)} = \frac{-23 + 61i}{170} = -\frac{23}{170} + i\frac{61}{170}.$$

This process, multiplying the numerator and denominator of a fraction by the conjugate of the denominator, is called **rationalizing the denominator**.

**Some worked examples.**

Write the complex fractions below into the “Cartesian” form $z = a + bi$ where $a, b \in \mathbb{R}$.

1. $\frac{3 + 2i}{7 - 3i}$
2. $\frac{5}{3 + i}$
3. $\frac{2}{1 + i}$
4. $\frac{1}{i}$

**Solution.**

1. $\frac{3 + 2i}{7 - 3i} = \frac{(3 + 2i)(7 - 3i)}{(7 - 3i)(7 + 3i)} = \frac{15 + 23i}{58} = \frac{15}{58} + i\frac{23}{58}$
   (So the real part is $\frac{15}{58}$ and the imaginary part is $\frac{23}{58}$.)

2. $\frac{5}{3 + i} = \frac{5}{3 + i} \cdot \frac{3 - i}{3 - i} = \frac{15 - 5i}{10} = \frac{3}{2} - i\frac{1}{2}$

3. $\frac{2}{1 + i} = \frac{2}{1 + i} \cdot \frac{1 - i}{1 - i} = \frac{2(1 - i)}{2} = 1 - i$

4. $\frac{1}{i} = \frac{1}{i} \cdot \frac{-i}{-i} = \frac{-i}{1} = -i$ (Or $0 - i$)

2.4.7 Using complex numbers and the factor theorem

We want to be ready to use complex numbers when factoring polynomials or solving polynomial equations. For example, let’s factor the polynomial

$$f(x) = x^3 - 2x^2 + 9x - 18.$$
Since this is a cubic polynomial and we don’t want to use the cubic formula, then we need to find a zero. We could try some numbers (techniques in the next section will help us here) and discover by trial-and-error that \( f(2) = 0 \). Or we could graph this polynomial on a graphing calculator and see that \( x = 2 \) is a zero. Either way, if we want to factor this cubic, we need to find one root and then break the cubic down so that one piece is a quadratic. In this case, one we see that \( x = 2 \) is a root, the rest of the problem is easy.

Since \( f(2) = 0 \) (check this!) then \( x - 2 \) is a factor of \( x^3 - 2x^2 + 9x - 18 \). Now divide \( x - 2 \) into \( f(x) = x^3 - 2x^2 + 9x - 18 \) by synthetic division:

\[
\begin{array}{cccc}
1 & -2 & 9 & -18 \\
2 & 2 & 0 & 18 \\
1 & 0 & 9 & 0 \\
\end{array}
\]

We see that \( f(x) \) factors as \((x - 2)(x^2 + 9)\). Since \( x^2 + 9 = 0 \) implies that \( x = \pm 3i \) then \( x^2 + 9 \) factors as \((x - 3i)(x + 3i)\). So \( f(x) \) factors as

\[
x^3 - 2x^2 + 9x - 18 = (x - 2)(x - 3i)(x + 3i).
\]

Note that if we have a complex root which is not real, that we have in fact two roots. Here both \( 3i \) and its conjugate \( -3i \) are roots.

Some worked examples.

1. Factor the polynomial \( f(x) = x^3 - 8 \) and then find all solutions to \( x^3 = 8 \).

   **Solution.** Since \( f(x) = x^3 - 8 \) is zero when \( x = 2 \) then we know that one zero is \( x = 2 \). Since \( f(2) = 0 \) we know that \( x - 2 \) is a factor of \( f(x) \). Dividing \( x^3 - 8 \) by \( x - 2 \) gives us \( x^3 - 8 = (x - 2)(x^2 + 2x + 4) \). So the solutions to \( x^3 - 8 = 0 \) are the solutions to \( (x - 2)(x^2 + 2x + 4) = 0 \). By the quadratic formula, the solutions to \( x^2 + 2x + 4 = 0 \) are \( x = \frac{1}{2}(-2 \pm 2\sqrt{3}i) = -1 \pm \sqrt{3}i \). This implies that \((x^2 + 2x + 4)\) factors as

\[
x^2 + 2x + 4 = (x - (-1 + \sqrt{3}i))(x - (-1 - \sqrt{3}i)) = (x + 1 - \sqrt{3}i)(x + 1 + \sqrt{3}i).
\]

So the factoring of \( f(x) \) is

\[
x^3 - 8 = (x - 2)(x + 1 - \sqrt{3}i)(x + 1 + \sqrt{3}i).
\]

The full set of solutions to the equation \( x^3 = 8 \) is the set of solutions to the equation \( x^3 - 8 = 0 \). These are the zeroes of \( f(x) \):

\[
x = 2, x = -1 + \sqrt{3}i, x = -1 - \sqrt{3}i.
\]

2. Factor completely \( g(x) = x^6 - 1 \).

   **Solution.** The Fundamental Theorem of Algebra tells us the \( x^6 - 1 \) has six zeroes and therefore factors into six linear pieces. But it is not particularly easy to find this zeroes or the associated factors.

   One way to begin factoring \( g(x) = x^6 - 1 \) is to view this polynomial as a difference of squares:

\[
g(x) = x^6 - 1 = (x^3 + 1)(x^3 - 1).
\]

\[1\text{Yes, there is a cubic formula! But it is quite messy.}...
To further factor \( x^3 - 1 \), notice that \( x = 1 \) is surely a zero and after synthetic division we see that

\[
x^3 - 1 = (x - 1)(x^2 + x + 1).
\]

In a similar way we should notice that \( x = -1 \) is a zero of \( x^3 + 1 \) and so

\[
x^3 + 1 = (x + 1)(x^2 - x + 1).
\]

So we now have

\[
g(x) = x^6 - 1 = (x^3 + 1)(x^3 - 1) = (x + 1)(x^2 - x + 1)(x^2 + x + 1)\]

We are not done; we need to factor the two quadratics \( x^2 - x + 1 \) and \( x^2 + x + 1 \). In an early example, we factored these using the quadratic formula and found that

\[
x^2 + x + 1 = (x - \frac{1}{2} - \frac{\sqrt{3}}{2} i)(x - \frac{1}{2} + \frac{\sqrt{3}}{2} i)
\]

and

\[
x^2 - x + 1 = (x + \frac{1}{2} - \frac{\sqrt{3}}{2} i)(x + \frac{1}{2} + \frac{\sqrt{3}}{2} i)
\]

So

\[
g(x) = x^6 - 1 = (x + 1)(x^2 - x + 1)(x^2 + x + 1) = \left( (x + 1)(x - \frac{1}{2} - \frac{\sqrt{3}}{2} i)(x + \frac{1}{2} - \frac{\sqrt{3}}{2} i)(x - \frac{1}{2} + \frac{\sqrt{3}}{2} i) \right)
\]

Notice that in this last example, relying on complex numbers and the formula for difference of squares, we were able to break the sixth degree polynomial \( x^6 - 1 \) down into six linear terms (!) as promised by the Fundamental Theorem of Algebra.

### 2.4.8 Other resources on complex numbers

In the free textbook, *Precalculus*, by Stitz and Zeager (version 3, July 2011, available at [stitz-zeager.com](http://stitz-zeager.com)), this material is covered in section 3.4.

In the free textbook, *Precalculus, An Investigation of Functions*, by Lippman and Rasmussen (Edition 1.3, available at [www.opentextbookstore.com](http://www.opentextbookstore.com)), this material is covered very briefly at the beginning of section 8.3.


There are lots of online resources for studying complex numbers. Here are some I recommend.

2. [Khan Academy videos](https://www.khanacademy.org) on complex numbers.
3. [Paul Dawkin’s online notes](http://tutorial.math.lamar.edu) on complex numbers.
4. A Wikipedia article [on applications of complex numbers](https://en.wikipedia.org/wiki/Complex_number)

**Worksheet to go with these notes.**

As class homework, please complete **Worksheet 2.4A, Complex numbers**, available through the class webpage.
2.5 More on zeroes of polynomials

Recall: a zero of a polynomial is sometimes called a “root”.

Our goal in this section is to set up a strategy for attempting to find (if possible) all the zeroes of a given polynomial. We will assume, for this section, that our polynomial has coefficients which are integers. We will then set up some tests to run on the polynomial so that we can make some guesses at possible roots of the polynomial and begin to factor it.

The Fundamental Theorem of Algebra tells us that a polynomial of degree $n$ will have $n$ zeroes, if we include complex roots and if we count the multiplicity of the roots. We will be particularly interested in finding all the zeroes for various polynomials of small degree, $n = 3, 4$ or maybe $n = 5$.

2.5.1 Rational Root Test

A rational number is a number which can be written as a ratio $\frac{b}{d}$ where both the numerator $b$ and the denominator $d$ are integers. In this part of our lecture, we describe the set of all possible rational numbers which might be the root of our polynomial. We will call this set of all possible rational numbers the rational test set; it will be a list of numbers to examine in our hunt for roots.

Consider the simple linear polynomial $3x - 5$. It has one zero, $x = \frac{5}{3}$.

This zero, $\frac{5}{3}$, is a rational number with numerator given by the constant term 5 and denominator given by the leading coefficient 3 of this (small) polynomial.

This concept generalizes. If we are factoring a polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$$

then when we eventually write out the factoring

$$f(x) = (d_1 x - b_1)(d_2 x - b_2) \cdots (d_n x - b_n)$$

the products of the coefficients $d_1 d_2 \cdots d_n$ must equal the leading coefficient $a_n$ and the products of the constants $b_1 b_2 \cdots b_n$ must equal the constant $a_0$.

This leads to the Rational Root Test.

Rational Root Test.

If $x = \frac{b}{d}$ is a rational number that is the root (zero) of the polynomial $f(x) = a_n x^n + ... + a_1 x + a_0$ then the numerator $b$ is a factor of the constant term $a_0$ and the denominator $d$ is a factor of the leading coefficient $a_n$.

The effect of the Rational Root Test is that given a polynomial $f(x)$ we can create a “Test Set” of rational numbers to try as zeroes.

Some Worked Examples.

Find the set of all possible rational zeroes of the given function, as given by the Rational Root Theorem.

1. $f(x) = 2x^3 + 5x^2 - 4x - 3$
2. $f(x) = 3x^3 - 4x^2 + 5$
3. $f(x) = 6x^6 + 5x^2 + x - 35$. 
Solutions.

1. The set of rational zeroes of \( f(x) = 2x^3 + 5x^2 - 4x - 3 \) is limited to fractions whose numerator divides 3 and whose denominator divides 2:

\[
\text{Rational Test Set} = \{\pm 1, \pm \frac{1}{2}, \pm \frac{3}{2}\}.
\]

2. The set of rational zeroes of \( f(x) = 3x^3 - 4x^2 + 5 \) is limited to fractions whose numerator divides 5 and whose denominator divides 3:

\[
\text{Rational Test Set} = \{\pm 1, \pm \frac{1}{3}, \pm \frac{5}{3}\}.
\]

3. The set of rational zeroes of \( f(x) = 6x^6 + 5x^2 + x - 35 \) is limited to fractions whose numerator divides 35 and whose denominator divides 6:

\[
\text{Rational Test Set} = \{\pm 1, \pm 5, \pm \frac{1}{2}, \pm \frac{5}{2}, \pm 3, \pm \frac{7}{3}, \pm \frac{35}{3}, \pm \frac{1}{6}, \pm \frac{5}{6}, \pm \frac{7}{6}, \pm \frac{35}{6}\}.
\]

2.5.2 Bounds to the set of zeroes

In this section we work through the details of trying to compute (exactly) the zeroes of a polynomials. These techniques, over three centuries old, are now aided by tools such as graphing calculators.

We work though an example in detail. Suppose we wish to factor completely the polynomial \( f(x) = 2x^5 - 3x^4 + 14x^3 + 15x^2 - 34x - 30 \).

We first create a “test set” of rational roots to try. Since the constant term 30 has 1, 2, 3, 5, 6, 10, 15, 30 as factors and the leading coefficient 2 has factors 1 and 2 then by the Rational Root Test, our test set of possible rational roots is

\[
\text{Rational Test Set} = \{\pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \pm 2, \pm \frac{5}{2}, \pm 3, \pm 5, \pm 6, \pm \frac{15}{2}, \pm 10, \pm 15, \pm 30\}.
\]

This is a large set of rational numbers to try!

But let us get started. We might begin by trying the easier numbers, the integers. Let us first divide \( f(x) \) by \( x - 1 \), using synthetic division with \( c = 1 \).

\[
\begin{array}{cccccc}
2 & -3 & 14 & 15 & -34 & -30 \\
1 & & 2 & -1 & 13 & 28 & -6 \\
\hline
2 & -1 & 13 & 28 & -6 & -36
\end{array}
\]

So \( f(1) = -36 \) and so \( x = 1 \) is not a zero. This might be discouraging, but doing synthetic division with \( c = 1 \) was pretty easy!

Let’s try \( c = 2 \).

\[
\begin{array}{cccccc}
2 & -3 & 14 & 15 & -34 & -30 \\
2 & & 4 & 2 & 32 & 94 & 120 \\
\hline
2 & 1 & 16 & 47 & 60 & 90
\end{array}
\]

So \( x = 2 \) is not a zero of \( f(x) \).

But notice two things here. First notice that the remainder is positive; \( f(2) = 90 \). In our earlier work, we discovered that \( f(1) = -36 \) and so, by the Intermediate Value Theorem, the graph of the function
\( f(x) \) crosses the \( x \)-axis between \( x = 1 \) and \( x = 2! \) Since \( f(1) \) is negative and \( f(2) \) is positive then there is a zero somewhere between 1 and 2! This is important information!

Also notice that the bottom row in our synthetic division is all positive numbers. We can conclude from our understanding of synthetic division that if we were to try a larger positive number \( c \) greater than \( c = 2 \) then the numbers on the bottom row would get even larger still and so there is no chance of a zero to the right of \( x = 2 \). We have found an upper bound on the zeroes of \( f(x) \).

**Upper Bound**

If, upon doing synthetic division with a positive value \( c \), the bottom row in our computation of \( f(c) \) consists of all positive numbers (or zero) then \( c \) is an upper bound for the zeroes of \( f(x) \). We should not look for zeroes further to the right of \( c \).

In our case, this immediately rules out \( \frac{5}{2}, 3, 5, 6, \frac{15}{2}, 10, 15, 30 \). We need not try any of these.

Let us go back to our observation that there is a zero between \( x = 1 \) and \( x = 2 \). This suggests that we try \( x = \frac{3}{2} \) as a root.

We do the synthetic division.

\[
\begin{array}{cccccc}
2 & -3 & 14 & 15 & -34 & -30 \\
3 & 0 & 21 & 54 & 30 & \\
2 & 0 & 14 & 36 & 20 & 0
\end{array}
\]

Success!! So \( x = \frac{3}{2} \) is a root of \( f(x) \) and \( f(x) \) factors as

\[
2x^5 - 3x^4 + 14x^3 + 15x^2 - 34x - 30 = (x - \frac{3}{2})(2x^4 + 14x^3 + 36x + 20).
\]

It is probably better if we factor a 2 out of the right-hand factor and multiply it into the linear term and rewrite this as

\[
2x^5 - 3x^4 + 14x^3 + 15x^2 - 34x - 30 = (2x - 3)(x^4 + 7x^2 + 18x + 10) \quad (14)
\]

We want to find more roots of \( f(x) \) but since we have factored out a linear term, let us now focus on factoring \( x^4 + 7x^2 + 18x + 10 \).

There is an important principle here: *once we have found a factor, concentrate on the quotient that remains.* Do not waste time by returning to the original polynomial.

Is it clear that this new polynomial \( (x^4 + 7x^2 + 18x + 10) \) has no positive zeroes? If we try synthetic division with \( c = 0 \) we would just get, as bottom row, the coefficients 1, 0, 7, 18, 10 which are already positive. Anything to the right of zero will only makes these numbers bigger.

So we should try some negative numbers. At this point, since no positive numbers could give a zero and since this polynomial has constant term 10 and leading coefficient 1, the Test Set of possible rational roots has shrunk to \( \{-10, -5, -2, -1\} \).

Let us try \( c = -1 \).

\[
\begin{array}{cccccc}
1 & 0 & 7 & 18 & 10 \\
-1 & -1 & 1 & -8 & -10 & \\
1 & -1 & 8 & 10 & 0
\end{array}
\]
We have found another factor! So

\[ x^4 + 7x^2 + 18x + 10 = (x + 1)(x^3 - x^2 + 8x + 10) \]

and so

\[ 2x^5 - 3x^4 + 14x^3 + 15x^2 - 34x - 30 = (2x - 3)(x + 1)(x^3 - x^2 + 8x + 10). \]  

(15)

We continue on with our factoring by trying to factor \( x^3 - x^2 + 8x + 10 \). Let’s try \( c = -2 \).

\[
\begin{array}{cccc}
1 & -1 & 8 & 10 \\
-2 & & \\
1 & -3 & 14 & -18 \\
\end{array}
\]

So \( f(-2) = -18 \) and so \( x = -2 \) is not a zero.

Notice the pattern across the bottom row in our synthetic division. It alternates, positive 1, negative 3, positive 14, negative 18. If we were try a negative number to the left of \( x = -2 \) on the real line, it would make the negative 3 more negative, which in turn would give a larger positive value to the next entry, leading to a bottom line entry larger than positive 14 and then, in the next step, a number more negative than negative 18. The situation would get even worse, leading to a final value further from zero.

For example, to illustrate this, I provide here the synthetic division with \( c = -3 \) and \( c = -4 \).

\[
\begin{array}{cccc}
1 & -1 & 8 & 10 \\
-3 & & \\
1 & -4 & 20 & -50 \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & -1 & 8 & 10 \\
-4 & & \\
1 & -5 & 28 & -102 \\
\end{array}
\]

Notice how the bottom rows continued to alternate, with larger and larger absolute value. So \( x = -2 \) is a lower bound for our possible roots; there is no reason to try anything smaller.

We summarize what we have learned here by describing when we know we have a lower bound for our roots.

**Lower Bound**

If, upon doing synthetic division with a negative value \( c \), the bottom row in our computation of \( f(c) \) consists of numbers alternating in sign then \( c \) is an lower bound for the zeroes of \( f(x) \). We should not look for zeroes further to the left of \( c \) on the number line.

(For the purpose of this result, we can treat zero as positive or negative, giving it whatever sign we wish.)

Returning to our earlier factoring problem: We are able to conclude the \(-2\) is a lower bound of the roots of this polynomial.

We have now ruled out everything else is our Test Set, while discovering that \( x = \frac{3}{2} \) and \( x = -1 \) are zeroes of our polynomial. Now what do we do?

We summarize what we know to this point.

\[ 2x^5 - 3x^4 + 14x^3 + 15x^2 - 34x - 30 = (2x - 3)(x + 1)(x^3 - x^2 + 8x + 10). \]
Let’s go back and look at our cubic \( g(x) = x^3 - x^2 + 8x + 10 \). It has \( y \)-intercept \((0, 10)\). It is a cubic polynomial with end behavior \( \nearrow \searrow \) so we know that although \( g(0) = 10 \), eventually to the left of \( x = 0 \) the function becomes negative. By the intermediate value theorem, this cubic polynomial has a root which is negative, which we have not yet found.

Did we try everything? Almost. We tried \( x = -1 \), which was a zero of \( f(x) \) and then we agreed that \( x = -2 \) was a lower bound on zeroes of \( f(x) \).

What we did not do is test \( x = -1 \) twice! Recall that a polynomial can have a zero with multiplicity two or more....

Let us test \( x = -1 \), using synthetic division, with the cubic \( x^3 - x^2 + 8x + 10 \).

\[
\begin{array}{c|cccc}
1 & 1 & -1 & 8 & 10 \\
\hline 
& & -1 & 2 & -10 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
1 & 0 & -2 & 10 & 0 \\
\end{array}
\]

So \( x = -1 \) is a zero a second time (and \( x + 1 \) is a factor, twice, of \( f(x) \).

We now have

\[
2x^5 - 3x^4 + 14x^3 + 15x^2 - 34x - 30 = (2x - 3)(x + 1)^2(x^2 - 2x + 10) \tag{16}
\]

Once we reach a quadratic polynomial, we are almost done. Factoring quadratics are easy! We can use the quadratic formula if we don’t see an obvious factoring. In this case, if

\[
x^2 - 2x + 10 = 0
\]

then

\[
x = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i.
\]

Thus the quadratic equation \( x^2 - 2x + 10 \) has two complex roots (appearing, of course, as conjugate pairs.)

So \( x^2 - 2x + 10 \) factors into

\[
x^2 - 2x + 10 = (x - (2 + 3i))(x - (2 - 3i)).
\]

Our final factoring of the fifth degree polynomial \( f(x) \) is then

\[
f(x) = 2x^5 - 3x^4 + 14x^3 + 15x^2 - 34x - 30 = (2x - 3)(x + 1)^2(x - (2 + 3i))(x - (2 - 3i)) \tag{17}
\]

Here is the graph of \( y = f(x) \).

![Figure 17. A polynomial of degree 5 with two complex roots.](image)
We can see that $x = -1$ is a root with multiplicity 2 and that $x = 3/2$ is also a real root.

We summarize what we learned about the upper and lower bounds for our set of real zeroes.

**Upper Bound**
If, upon doing synthetic division with a positive value $c$, the bottom row in our computation of $f(c)$ consists of all positive numbers (or zero) then $c$ is an upper bound for the zeroes of $f(x)$. We should not look for zeroes further to the right of $c$.

(For the purpose of this result, we can treat zero as positive or negative, giving it whatever sign we wish.)

**Lower Bound**
If, upon doing synthetic division with a negative value $c$, the bottom row in our computation of $f(c)$ consists of numbers alternating in sign then $c$ is an lower bound for the zeroes of $f(x)$. We should not look for zeroes further to the left of $c$ on the number line.

(For the purpose of this result, we can treat zero as positive or negative, giving it whatever sign we wish.)

### 2.5.3 Descartes’ Rule of Signs

We have one more guide in our search for roots of a polynomial. It is a “rule” which is four centuries old, discovered by René Descartes.

**Descartes' Rule of Signs** (Positive version)
List the coefficients of a polynomial $f(x)$, from leading coefficient to the constant term. Count the change of signs. This is an upper bound on the number of positive roots.

The true number of positive roots may vary from this upper bound by a multiple of two (since complex number occur in conjugate pairs.)

**Examples.**

1. The polynomial $x^3 - 8$ has coefficients $1$, $(0, 0, 0) - 8$. Ignore the zeroes; there is one change of sign, from 1 to -8. So the polynomial has 1 positive root.

2. The polynomial $f(x) = 2x^5 - 3x^4 + 14x^3 + 15x^2 - 34x - 30$, studied above, has coefficients $2, -3, 14, 15, -34, -30$. This changes sign 3 times (from 2 to -3, from -3 to 14 and from 14 to -34.) An upper bound for the number of positive roots of $f(x)$ is 3. The polynomial either has 3 positive roots or 1. (As we saw in our work, there was a pair of complex numbers, and so there was only one positive root.)

**Descartes’ Rule of Signs** (Negative version)
Given the polynomial $f(x)$, list the coefficients of $f(-x)$ (note the insertion of $-x$!), from leading coefficient to the constant term. Count the change of signs. This is an upper bound on the number of negative roots.

The true number of negative roots may vary from this upper bound by a multiple of two (since complex number occur in conjugate pairs.)

**Examples.**

1. Consider the polynomial $g(x) = x^3 - 8$. $g(-x) = -x^3 - 8$ has coefficients $-1$, $(0, 0, 0) - 8$. Ignore the zeroes; there is no change of sign so the polynomial has no negative roots.

2. Or consider the polynomial $f(x) = 2x^5 - 3x^4 + 14x^3 + 15x^2 - 34x - 30$, studied above. $f(-x) = -2x^5 - 3x^4 - 14x^3 + 15x^2 + 34x - 30$, has coefficients $-2, -3, -14, 15, 34, -30$ which has 2
changes of sign. An upper bound for the number of negative roots of \( f(x) \) is two. The polynomial either has two positive roots or none. (As we saw in our work, \( x = -1 \) was a root twice.)

Descartes’ Rule of Signs narrows our search for roots of a polynomial. Earlier we searched for roots of \( x^3 - 8 \). Descartes’ Rule of Signs tells us that that polynomial has 1 positive real root and 0 negative real roots. If we expect 3 roots then we know that the other two roots must come in complex conjugate pairs.

2.5.4 The Fundamental Theorem of Algebra

After the material on complex numbers, we are now able to state the Fundamental Theorem of Algebra more precisely.

The Fundamental Theorem of Algebra

A polynomial \( f(x) = a_n x^n + a_{n-1} x^{n-1} + a_2 x^2 + a_1 x + a_0 \) with real coefficients \( a_j \), has exactly \( n \) zeroes, if we include complex zeroes and also count the multiplicity of zeroes. Complex solutions come in conjugate pairs.

Since a zero \( x = c \) of a polynomial gives a factor \( x - c \), we can restate this in terms of factors.

The Fundamental Theorem of Algebra (Second version)

A polynomial \( f(x) = a_n x^n + a_{n-1} x^{n-1} + a_2 x^2 + a_1 x + a_0 \) with real coefficients \( a_j \), factors completely into \( n \) linear terms, if we allow factoring involving complex numbers.

2.5.5 Other resources for zeroes of polynomials

In the free textbook, *Precalculus*, by Stitz and Zeager (version 3, July 2011, available at stitz-zeager.com) this material is covered in section 3.3. Their the “Rational Root Test” is Theorem 3.9. (We do not cover Theorem 3.8, Cauchy’s bound, in the precalculus class at Sam Houston State University.)


There are lots of online resources for studying zeroes of polynomials. Here are some I recommend.

1. Dr. Paul’s online math notes on finding zeroes of polynomials.
2. Video on Descartes’ Rule of Signs from Khan Academy,
3. Wikipedia article on Descartes’ Rule of Signs.

Homework.

As class homework, please complete **Worksheet 2.5, More Zeroes of Polynomials** available through the class webpage.
2.6 Rational Functions

A rational function $f(x)$ is a function which is the ratio of two polynomials, that is,

$$f(x) = \frac{n(x)}{d(x)}$$

where $n(x)$ and $d(x)$ are polynomials.

For example, $f(x) = \frac{3x^2 - x - 4}{x^2 - 2x - 8}$ is a rational function. In this case, both the numerator and denominator are quadratic polynomials.

2.6.1 Algebra with mixed fractions

Earlier in these notes we looked at the function

$$g(x) := \frac{1}{x + 2} + \frac{2x - 3}{2x + 1} + x - 5.$$  \hspace{1cm} (19)

This function, $g$, is a rational function. We can put $g$ into a fraction form, as the ratio of two polynomials, by finding a common denominator. The least common multiple of the denominators $x + 2$ and $2x + 1$ is simply their product, $(x + 2)(2x + 1)$. We may write $g(x)$ as a fraction with this denominator if we multiply the first term by $1 = \frac{2x + 1}{2x + 1}$, multiply the second term by $1 = \frac{x + 2}{x + 2}$ and multiply the third term by $1 = \frac{(2x + 1)(x + 2)}{(2x + 1)(x + 2)}$. This allows us to rewrite $g$ as

$$g(x) = \left(\frac{1}{x + 2}\right)\frac{(2x + 1)}{(2x + 1)} + \left(\frac{2x - 3}{2x + 1}\right)\frac{(x + 2)}{(x + 2)} + \left(x - 5\right)\frac{(2x + 1)(x + 2)}{(2x + 1)(x + 2)}$$  \hspace{1cm} (20)

and then combine the numerators

$$g(x) = \frac{(2x + 1) + (2x - 3)(x + 2) + (x - 5)(2x + 1)(x + 2)}{(2x + 1)(x + 2)}.$$  \hspace{1cm} (21)

The numerator is a polynomial of degree 3 (it can be expanded out to $2x^3 - 3x^2 - 20x - 15$) and the denominator is a polynomial of degree 2.

The algebra of mixed fractions, including the use of a common denominator, is an important tool when working with rational functions.

2.6.2 Zeroes of rational functions

Given a rational function $f(x) = \frac{n(x)}{d(x)}$ we will, as always, be interested in $y$- and $x$- intercepts.

The $y$-intercept occurs where $x$ is zero and it is usually very easy to compute $f(0) = \frac{n(0)}{d(0)}$.

However, the $x$-intercepts occur where $y = 0$, that is, where

$$0 = \frac{n(x)}{d(x)}.$$  \hspace{1cm} (22)

As a first step to solving this equation, we may multiply both sides by $d(x)$ and so concentrate on the zeroes of the numerator, solving the equation

$$0 = n(x).$$  \hspace{1cm} (23)

At this point, we have reduced the problem to finding the zeroes of a polynomial, exercises from a previous section!
For example, suppose

\[ h_1(x) = \frac{x^2 - 6x + 8}{x^2 + x - 12}. \]  

(24)

The \( y \)-intercept is \((-\frac{2}{3}, 0)\) since \( h_1(0) = \frac{8}{-12} = -\frac{2}{3} \).

The \( x \)-intercepts occur where \( x^2 - 6x + 8 = 0 \). Factoring \( x^2 - 6x + 8 = (x - 4)(x - 2) \) tells us that \( x = 4 \) and \( x = 2 \) should be zeroes. (We do need to check that they do not make the denominator zero — but they do not.)

### 2.6.3 Poles and holes

Since rational functions have a denominator which is a polynomial, we must worry about the domain of the rational function. In particular, any real number which makes the denominator zero, cannot be in the domain.

There are two types of zeroes in the denominator. One common type is a zero of the denominator which is not a zero of the numerator. In that case, the real number which makes the denominator zero is a “pole” and creates, in the graph, a vertical asymptote.

For example, using \( h_1(x) = \frac{x^2 - 6x + 8}{x^2 + x - 12} \) from before, we see that \( x^2 + x - 12 = (x + 4)(x - 3) \) has zeroes at \( x = -4 \) and \( x = 3 \).

Since neither \( x = -4 \) and \( x = 3 \) are zeroes of the numerator, these values give poles of the function \( h_1(x) \) and in the graph we will see vertical lines \( x = -4 \) and \( x = 3 \) that are “approached” by the graph. The lines are called asymptotes, in this case we have vertical asymptotes with equations \( x = -4 \) and \( x = 3 \).

The figure below graphs our function in blue and shows the asymptotes. (The graph is in blue; the asymptotes, which are not part of the graph, are in red.)

![Figure 18. Rational function (blue) with vertical asymptotes (red)](image)

If we change our function just slightly, so that it is

\[ h_2(x) = \frac{x^2 - 6x + 8}{x^2 - x - 12} = \frac{(x - 4)(x - 2)}{(x - 4)(x + 3)}. \]  

(25)
something very different occurs.

The rational function $h_2(x)$ here is still undefined at $x = 4$. If one attempts to evaluate $h_2(4)$ one gets the fraction $\frac{0}{0}$ which is undefined.

But, as long as $x$ is not zero, we can cancel the term $x - 4$ occurring both in the numerator and denominator and write

$$h_2(x) = \frac{x - 2}{x + 3}, \quad x \neq 4.$$ 

In this case there is a pole at $x = -3$, represented in the graph by a vertical asymptote (in red) and there is a hole (“removable singularity”) at $x = 4$ where, for just a moment, the function is undefined.

$$\text{Figure 19. Rational function (blue) with vertical asymptotes (red)}$$

2.6.4 The sign diagram of a rational function

When we looked at graphs of polynomials, we viewed the zeroes of the polynomial as dividers or fences, separating regions of the $x$-axis from one another. Within a particular region, between the zeroes, the polynomial has a fixed sign, (+) or (−), since changing sign requires crossing the $x$-axis. We used this idea to create the sign diagram of a polynomial, a useful tool to guide us in the drawing of the graph of the polynomial.

Just as we did with polynomials, we can create a sign diagram for a rational function. In this case, we need to use both the zeroes of the rational function and the vertical asymptotes as our dividers, our “fences”.

To create a signed diagram of rational function, list all the $x$-values which give a zero or a vertical asymptote. Put them in order. Then between these $x$-values, test the function to see if it is positive or negative and indicate that by a plus sign or minus sign.

For example, consider the function $h_2(x) = \frac{x^2 - 6x + 8}{x^2 - x - 12} = \frac{(x - 4)(x - 2)}{(x - 4)(x + 3)}$ from above. It has a zero at $x = 2$ and a vertical asymptote $x = -3$. The sign diagram then represents the values of $h_2(x)$ in the regions divided by $x = -3$ and $x = 2$. (For the purpose of a sign diagram, the hole at $x = 4$ is irrelevant since it does not effect the sign of the rational function.) To the left of $x = -3$, $h_2(x)$ is positive. Between $x = -3$ and $x = 1$, $h_2(x)$ is negative. Finally, to the right of $x = 1$, $h_2(x)$ is positive. So the sign diagram of $h_2(x)$ is
2.6.5 Horizontal asymptotes

Just as we did with polynomials, we ask questions about the “end behavior” of rational functions: what happens for \( x \)-values far away from 0, towards the “ends” of our graph?

In many cases this leads to questions about horizontal asymptotes and oblique asymptotes.

Before we go very far into discussing end-behavior of rational functions, we need to agree on a basic fact. Recall that in our discussion of polynomials earlier, we agreed that as long as \( p(x) \) has degree at least one (and so was not just a constant) then as \( x \) grew large then \( p(x) \) also grew large in absolute value. This implies then that as \( x \) goes to infinity, \( \frac{1}{p(x)} \) becomes close to zero.

We explicitly list this as a lemma, a mathematical fact we will often use.

**Lemma.** Suppose that \( p(x) \) is a polynomial of degree at least 1. Then the rational function \( \frac{1}{p(x)} \) tends to zero as \( x \) gets large in absolute value.

In calculus terms, the limit, as \( x \) goes to infinity, of \( \frac{1}{p(x)} \) is zero.

**Corollary.** Suppose that \( f(x) = \frac{n(x)}{d(x)} \) is a rational function where the degree of \( n(x) \) is smaller than the degree of \( d(x) \). Then \( f(x) \) tends to zero as \( x \) grows large in absolute value.

This means that if \( f(x) = \frac{n(x)}{d(x)} \) is a rational function where the degree of \( n(x) \) is smaller than the degree of \( d(x) \) then as \( x \) gets large in absolute value, the graph approaches the \( x \)-axis. The \( x \)-axis, \( y = 0 \) is a **horizontal asymptote**.

**Horizontal asymptotes: some worked examples.**

**Example 1.** Consider the rational function

\[
 f(x) = \frac{x^2 - 9}{x^3 - 4x}.
\]

Since the numerator has degree 2 and the denominator has degree 3 then as \( x \) gets large in absolute value (say \( x \) is equal to one million ... or \( x \) is equal to negative one million) then the denominator is much larger in absolute value than the numerator and so \( f(x) \) is close to zero.

This means that as \( x \to \infty \) or \( x \to -\infty \), \( f(x) \to 0 \). So \( y = 0 \) is a horizontal asymptote of \( f(x) \).

**Example 2.** If instead the degree of \( n(x) \) is equal to the degree of \( d(x) \), then the highest power terms dominate. For example, in equation \[24\] we defined a certain rational function

\[
 h_1(x) = \frac{x^2 - 6x + 8}{x^2 + x - 12}.
\]

As \( x \) gets large in absolute value, the quadratic terms \( x^2 \) begin to dominate. For example, if \( x = 1,000,000 \) then the denominator \( x^2 + x - 12 \) is equal to 1,000,000,000,000 + 1,000,000 - 12 = 1,000,000,099,988, which for all practical purposes can be approximated by 1,000,000,000,000. Similarly, if \( x \) is a million, the numerator is equal to 1,000,000,000,000 - 6,000,000 + 8 = 999,999,400,008 which can also be approximated by 1,000,000,000,000. Thus

\[
 f(1000000) = \frac{1,000,000,099,988}{999,999,400,008} \approx 1.0000007 \approx 1.
\]
The same result occurs if we set $x$ equal to negative numbers which are large in absolute value, such as $x = -1000000$. More generally, as $x$ gets large in absolute value, \[
\frac{x^2 - 6x + 8}{x^2 + x - 12}
\] begins to look like $\frac{x^2}{x^2} = 1$.

We conclude then that as $x$ gets large in absolute value, $f(x)$ approaches 1 and so $y = 1$ is a horizontal asymptote of $f(x)$.

Here, below, is a graph of $y = h_1(x)$, with the function drawn in blue and the various asymptotes drawn in green or red.

**Figure 20.** The graph of rational function $h_1(x)$ with vertical asymptotes (red) and horizontal asymptote (green).

**Example 3.** In equation 25 above, we considered the rational function $h_2(x) = \frac{(x - 4)(x - 2)}{(x - 4)(x + 3)}$. Like $h_1(x)$, this function has a horizontal asymptote $y = 1$.

**Figure 21.** The graph of $h_2(x)$ (in blue) with vertical asymptotes (red) and horizontal asymptote (green).

**Example 4.** Find the zeroes and vertical asymptotes of the rational function

\[
g(x) = \frac{3(x + 1)(x - 2)}{4(x + 3)(x - 1)}
\]
and draw the sign diagram. Then find the horizontal asymptotes.

**Solution.**

Looking at the numerator of \( g(x) = \frac{3(x + 1)(x - 2)}{4(x + 3)(x - 1)} \), we see that the zeroes occur at \( x = -1 \) and \( x = 2 \).

Looking at the denominator of we can see that the vertical asymptotes of \( g(x) \) are the lines \( x = -3 \) and \( x = 1 \).

The sign diagram is

\[
\begin{array}{c|c|c|c|c|c}
(+)&(-)&(+)&(−)&(+)\\
\hline
−3&−1&1&1&2\\
\end{array}
\]

There is one horizontal asymptote found by considering the end behavior of \( g(x) \). As \( x \) goes to infinity, \( g(x) \) goes to \( \frac{3}{4} \) so the horizontal asymptote is the line \( y = \frac{3}{4} \).

### 2.6.6 More on the end-behavior of rational functions

We can tell the end behavior of a rational function by doing long division and writing \( \frac{n(x)}{d(x)} \) in the form \( q(x) + \frac{r(x)}{d(x)} \) where \( q(x) \) is the quotient and \( r(x) \) is the remainder given by long division.

For example, consider the rational function \( h_1(x) = \frac{x^2 - 6x + 8}{x^2 + x - 12} \) mentioned earlier. If we did the appropriate long division, we could write \( h_1(x) = 1 + \frac{r(x)}{x^2 + x - 12} \) where \( r(x) \) is some polynomial of degree smaller than two. Therefore, as \( x \) grows to infinity, \( h_1(x) \) tends to 1.

This means that \( h_1(x) \) has a horizontal asymptote \( y = 1 \).

In equation 19 at the beginning of this section we defined a certain rational function

\[
g(x) := \frac{1}{x + 2} + \frac{2x - 3}{2x + 1} + x - 5.
\]

Notice that as \( x \) grows large in absolute value, the two fractions shrink to zero and the graph begins to look like that of \( y = x - 5 \). So the line \( y = x - 5 \) is an asymptote for \( g(x) \). In this case this asymptote is neither vertical nor horizontal; it is an slant asymptote. Slant asymptotes are also called “oblique” asymptotes.

### 2.6.7 Putting it all together – the six steps

The textbook *Precalculus, by Stitz and Zeager* suggests six steps to graphing a rational function \( f(x) \). Here (from page 321 in the third edition) are the six steps.

1. Find the domain of the rational function \( f(x) \):
2. Reduce \( f(x) \) to lowest terms, if applicable.
3. Find the \( x \)- and \( y \)-intercepts of the graph of \( y = f(x) \), if they exist.
4. Determine the location of any vertical asymptotes or holes in the graph, if they exist. Analyze the behavior of \( r \) on either side of the vertical asymptotes, if applicable.
5. Analyze the end behavior of \( r \). Find the horizontal or slant asymptote, if one exists.
6. Use a sign diagram and plot additional points, as needed, to sketch the graph of \( y = f(x) \).
Two worked examples.

Example 1. Use the six steps, above, to graph the rational function $h(x) = \frac{10x^2 - 250}{x^2 + 6x + 8}$.

Solution.

1. We factor the numerator and denominator to rewrite

$$h(x) = \frac{10x^2 - 250}{x^2 + 6x + 8} = \frac{10(x - 5)(x + 5)}{(x + 2)(x + 4)}.$$ 

The domain is the set of all real numbers except $x = -2$ and $x = -4$. In interval notation this is

$$(-\infty, -4) \cup (-4, -2) \cup (-2, \infty).$$

2. Since the numerator and the denominator have no common factors then $h(x)$ is in lowest terms. This means that there are no holes (removable singularities) in the graph.

3. The $y$-intercept is when $y = h(0) = \frac{-250}{8} = -\frac{125}{4} = -31.25$.

To find the $x$-intercepts we set the numerator equal to zero:

$$0 = 10x^2 - 250.$$ 

Divide both sides by 10 and factor

$$0 = x^2 - 25.$$ 

$$0 = (x - 5)(x + 5),$$ 

so $x = 5$ and $x = -5$. So our intercepts are $(-5, 0), (5, 0)$.

4. Since $h(x)$ is in lowest terms, there are no holes.

Since the denominator factors as $x^2 + 6x + 8 = (x + 4)(x + 2)$ then the denominator is zero when $x = -4$ or $x = -2$. So the vertical asymptotes are $x = -4$ and $x = -2$.

5. As $x$ gets large in absolute value (and so far away from the $y$-axis), $h(x)$ begins to look like

$$h(x) = \frac{10x^2}{x^2} = 10.$$ 

So the horizontal asymptote is $y = 10$.

6. The sign diagram is

<table>
<thead>
<tr>
<th>( )</th>
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<th>( )</th>
<th>( )</th>
<th>( )</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-5$</td>
<td>$-4$</td>
<td>$-2$</td>
<td>$5$</td>
<td></td>
</tr>
</tbody>
</table>

The graph is given below. The graph is in blue; the vertical asymptotes are in red and the horizontal asymptote is in green.
Example 2. Let \( f(x) = \frac{3x^3 + x^2 - 12x - 4}{x^2 - 2x - 8} \) Find all intercepts, zeroes and then graph this function, displaying the features found.

Solution. We will work through the six steps.

1. The denominator factors as \( x^2 - 2x - 8 = (x - 4)(x + 2) \). So the domain is the set of real numbers where the denominator is not zero, that is,
   \[ (-\infty, -2) \cup (-2, 4) \cup (4, \infty). \]

2. Notice that when we evaluate the numerator at \( x = -2 \), we get zero. So \( x + 2 \) is a factor of both the numerator and the denominator.

   Recognizing that \( x + 2 \) is a factor of the numerator, we can further factor the numerator using the techniques we learned in the sections on polynomial zeroes. The numerator factors as \( (x + 2)(x - 2)(3x + 1) \).

   So
   \[ f(x) = \frac{(x + 2)(x - 2)(3x + 1)}{(x + 2)(x - 4)}. \]

   The point where \( x = -2 \) is a hole (removable singularity.) If we put the rational function into lowest terms, it becomes
   \[ f(x) = \frac{(x - 2)(3x + 1)}{(x - 4)}, \quad x \neq -2 \]

3. The \( y \)-intercept is when \( y = f(0) = \frac{-2}{-4} = 2 \).

   The \( x \)-intercepts occur when we set \( (x - 2)(3x + 1) \) equal to zero and so these occur when \( x = 2 \) and when \( x = -\frac{1}{3} \).
4. A hole occurs when \( x = -2 \). Looking at the reduced form, we see that the hole has \( y \)-value

\[
\frac{(-2 - 2)(3(-2) + 1)}{(-2 - 4)} = \frac{(-4)(-5)}{-6} = \frac{-10}{3}.
\]

There is one vertical asymptote; it is \( x = 4 \).

5. Analyze the end behavior of \( r \). Find the horizontal or slant asymptote, if one exists. To analyze the end behavior, we do long division:

\[
x - 1 \left| \begin{array}{c}
3x^2 - 5x - 2 \\
3x^2 + 3x \\
- 2x - 2 \\
2x - 2 \\
- 4
\end{array} \right.
\]

and write \( f(x) = 3x - 2 - \frac{4}{x-1} \) and so there is an oblique (slant) asymptote at \( y = 3x - 2 \).

6. Use a sign diagram and plot additional points, as needed, to sketch the graph of \( y = f(x) \).

The sign diagram is

\[
\begin{array}{c|c|c|c|c}
-\frac{1}{3} & (+) & (-) & (+) \\
2 & 4
\end{array}
\]

The graph is drawn below in blue (with asymptotes, again, in colors red and green.)

**Figure 23.** The graph of \( f(x) \) (in blue) with vertical asymptotes (red) and horizontal asymptote (green). The hole at \((-2, -\frac{10}{3})\) is not shown.
2.6.8 Other resources for rational functions

In the free textbook, *Precalculus*, by Stitz and Zeager (version 3, July 2011, available at [stitz-zeager.com](http://stitz-zeager.com)) chapter 4 (two sections) is devoted to rational functions. The six-step process used in these notes is taken from that text (p. 321.)


There are lots of online resources on rational functions. Here are some I recommend.

1. Dr. Paul’s online math notes on polynomials,
2. Videos on polynomials from Khan Academy,

**Homework.**

As class homework, please complete **Worksheet 2.6, Rational Functions** available through the class webpage.