2.4 Complex numbers

2.4.1 A first look at complex numbers

The complex number system is an extension of the real number system. It unifies the mathematical number system and explains many mathematical phenomena.

We introduce a number \( i = \sqrt{-1} \) defined to satisfy the equation \( x^2 = -1 \). (As soon as we introduce this number, there is some ambiguity, for \( x = -i \) also satisfies \( x^2 = -1! \)) The complex numbers are defined as all numbers of the form \( a + bi \) where \( a \) and \( b \) are real numbers. We write

\[
\mathbb{C} := \{a + bi : a, b \in \mathbb{R}\}.
\]

A complex number of the form \( z = a + bi \) is said to have real part \( \Re = a \) and imaginary part \( \Im = b \).

Any “number” can be written in this form. The number \( i \) has real part 0 and is said to be “purely imaginary”; the number 5 has imaginary part 0 and is “real”. The real numbers are a subset of the complex numbers.

The conjugate of a complex number \( z = a + bi \) is created by changing the sign on the imaginary part: \( \bar{z} = a - bi \). Thus the conjugate of \( 2 + i \) is \( 2 - i \); the conjugate of \( \sqrt{3} - \pi i \) is \( \sqrt{3} + \pi i \). The conjugate of \( i \) is \( i = -i \) and the conjugate of the real number 5 is merely 5.

2.4.2 Motivation for the complex numbers

The nicest version of the Fundamental Theorem of Algebra says that every polynomial of degree \( n \) has exactly \( n \) zeroes. But this is not quite true.

Or is it?

Consider the functions \( f(x) = x^2 - 1 \), \( g(x) = x^2 \) and \( h(x) = x^2 + 1 \). We graph these functions below.

![Figure 14. Three quadratics](image)

It is obvious that the quadratic \( f(x) = x^2 - 1 \) (drawn in green in the figure) has two zeroes. Indeed, they are easy to find - set \( x^2 - 1 = 0 \), factor \( x^2 - 1 \) into \((x - 1)(x + 1)\) and so see that \( x = 1 \) and \( x = -1 \) are the zeroes. Or ... just look at the graph.

Now let’s move the green parabola up one unit, to graph \( y = x^2 \), drawn in blue. What happened to our two zeroes? They merged into the single \( x \)-intercept at the origin. But we can claim that \( g(x) = x^2 \) still has two zeroes, if we are willing to count multiplicities. This makes some sense because we can write

\[
x^2 = (x - 0)(x - 0)
\]
and since $x - 0$ is a factor twice, we could claim that $x = 0$ is a zero twice.

But what if we move the parabola up one more step and graph $y = x^2 + 1$ (drawn in red)? Now, suddenly, there are no solutions. The graph never touches the $x$-axis.

Algebraically, $h(x) = x^2 + 1$ does not have any zeroes because that would require that

$$x^2 + 1 = 0$$

which then requires

$$x^2 = -1.$$  

If we square any real number, the result is positive, so it is not possible for a square to be equal to $-1$.

In the late middle ages, mathematicians discovered that if one were willing to allow for a new number, one whose square was $-1$, quite a lot of mathematics got simpler! Indeed, in solutions to cubic equations, one could sometimes find all solutions by pretending for just a moment that there was a solution to $x^2 = -1$ and then, after a few steps, observing that this “imaginary” piece disappeared and one had the correct solutions to the cubic equation.

This “imaginary” number was therefore very useful, even if one didn’t quite believe in it.

Over time, the term “imaginary” has stuck, even though scientists and engineers now use complex numbers all the time. It is now common agreement to write $i$ as an entity that satisfies

$$i^2 = -1.$$  

Once we have done this, the equation

$$x^2 = -1$$

has two solutions,

$$x = i \text{ and } x = -i.$$  

So the polynomial $x^2 + 1$ factors as $(x + i)(x - i)$ and the function $h(x) = x^2 + 1$ has two zeroes, just like the other quadratics. It is just that the zeroes of $h(x)$ are imaginary and are not on the $x$-axis.

A brief article on applications of complex numbers is here at Wikipedia. Modern cell phone signals rely on sophisticated signal analysis; we would not have cell phones without the mathematics of complex numbers.

2.4.3 Complex numbers and the quadratic formula

Complex numbers appear naturally in quadratic equations. Suppose we wish to solve the quadratic equation

$$ax^2 + bx + c = 0 \quad (11)$$

By completing the square we can solve for $x$ and find that

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (12)$$

The expression $b^2 - 4ac$ under the radical sign is called the discriminant of the quadratic equation and is often abbreviated by $\Delta$.

If $\Delta = b^2 - 4ac$ is positive then the square root of $\Delta$ is a real number and so the quadratic equation has two real solutions:

$$x = \frac{-b + \sqrt{\Delta}}{2a} \text{ and } x = \frac{-b - \sqrt{\Delta}}{2a} \quad (13)$$
If \( \Delta \) is zero then there is only one solution since
\[
x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-b \pm \sqrt{0}}{2a} = -\frac{b}{2a}.
\]
This single solution occurs with multiplicity two.

But if \( \Delta \) is negative then \( \sqrt{\Delta} \) is imaginary and so our solutions are complex numbers which are \textit{not} real. To be explicit, if \( \Delta \) is negative then \( -\Delta \) is positive and so \( \sqrt{\Delta} = \sqrt{-\Delta} \ i \). The solutions to the quadratic formula are then
\[
x = \frac{-b + \sqrt{-\Delta} \ i}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{-\Delta} \ i}{2a}.
\]
In this case, the plus/minus sign (\( \pm \)) in front of \( \sqrt{\Delta} \) assures us that we will get two complex numbers as solutions.

These two complex solutions come in \textbf{conjugate pairs}. If one complex number is the root of a quadratic polynomial (with real coefficients) then its conjugate is also a root.

For example, the solutions to the quadratic equation
\[x^2 + x + 1 = 0\]
are
\[
1 \pm \sqrt{1^2 - 4(1)(1)} = 1 \pm \sqrt{-3} = 1 \pm \sqrt{-1} \sqrt{3} = 1 \pm \frac{1}{2} \pm \frac{\sqrt{3} \ i}{2}.
\]
Thus the two solutions to the equation \( x^2 + x + 1 = 0 \) are the complex conjugate pairs
\[
\frac{1}{2} + \frac{\sqrt{3}}{2} i \quad \text{and} \quad \frac{1}{2} - \frac{\sqrt{3}}{2} i.
\]
Since these are the two solutions to the equation \( x^2 + x + 1 = 0 \) then the polynomial \( x^2 + x + 1 \) factors as
\[
(x - \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right))(x - \left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)) = (x + \frac{1}{2} - \frac{\sqrt{3}}{2} i)(x + \frac{1}{2} + \frac{\sqrt{3}}{2} i)
\]

Some worked examples.

1. Solve the quadratic equation \( x^2 - x + 1 = 0 \). Also, factor \( x^2 - x + 1 \).

\textbf{Solution.} By the quadratic formula the solutions to \( x^2 - x + 1 = 0 \) are
\[
\frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm \sqrt{3} i}{2} = \frac{1}{2} \pm \frac{\sqrt{3} \ i}{2}.
\]
Since the two solutions to the equation \( x^2 - x + 1 = 0 \) are the complex numbers
\[
\frac{1}{2} + \frac{\sqrt{3}}{2} i \quad \text{and} \quad \frac{1}{2} - \frac{\sqrt{3}}{2} i.
\]
then the polynomial \( x^2 - x + 1 \) factors as
\[
(x - \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right))(x - \left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)) = (x - \frac{1}{2} - \frac{\sqrt{3}}{2} i)(x - \frac{1}{2} + \frac{\sqrt{3}}{2} i)
\]
2. Solve the quadratic equation
\[2x^2 + 5x + 7 = 0\]

**Solution.** According to the quadratic formula,
\[x = \frac{-5 \pm \sqrt{5^2 - 4(2)(7)}}{4} = \frac{-5 \pm \sqrt{-31}}{4} = \frac{-5 \pm \sqrt{31}i}{4} = \frac{-5}{4} \pm \frac{\sqrt{31}}{4}i.\]

Our two solutions are the conjugate pairs
\[x = -\frac{5}{4} + \frac{\sqrt{31}}{4}i \text{ and } x = -\frac{5}{4} - \frac{\sqrt{31}}{4}i.\]

3. Use the roots of \(2x^2 + 5x + 7\) to factor \(2x^2 + 5x + 7\).

**Solution.** Since the two solutions to the equation \(2x^2 + 5x + 7 = 0\) are
\[x = -\frac{5}{4} + \frac{\sqrt{31}}{4}i \text{ and } x = -\frac{5}{4} - \frac{\sqrt{31}}{4}i\]
and since \(c\) is a zero of a polynomial if and only if \(x - c\) is a factor, then
\[(x - (-\frac{5}{4} + \frac{\sqrt{31}}{4}i))(x - (-\frac{5}{4} - \frac{\sqrt{31}}{4}i))\]
must be a factor of \(2x^2 + 5x + 7\). But if we check the coefficient of \(x^2\) in the expression above, we see that we need to multiply by 2 to complete the factorization. So \(2x^2 + 5x + 7\) factors as
\[2(x - (-\frac{5}{4} + \frac{\sqrt{31}}{4}i))(x - (-\frac{5}{4} - \frac{\sqrt{31}}{4}i))\]
\[= 2(x + \frac{5}{4} - \frac{\sqrt{31}}{4}i)(x + \frac{5}{4} + \frac{\sqrt{31}}{4}i)\]

### 2.4.4 Geometric interpretation of complex numbers

Mathematicians began to recognize the value of complex numbers sometime back in the Renaissance period (fifteenth and sixteenth centuries) but it was not until there was a geometric interpretation of the complex numbers that people began to feel comfortable with them.

We may view the complex numbers as lying in the Cartesian plane. Let the traditional \(x\)-axis represent the real numbers and the traditional \(y\)-axis represent the numbers of the form \(yi\). We equate a complex number \(x + yi\) with the point \((x, y)\). (So the imaginary numbers \(yi\) are “perpendicular” to the real numbers!)

The complex plane is drawn on the next page.
The process of changing $a + bi$ into the point $(a, b)$ can be traced to Argand around 1800 and is sometimes called the “Argand diagram”. In the Argand diagram, the complex number $z = a + bi$ is equated with the point $(a, b)$ in the Cartesian plane. For example, $6 + 5i$ can be graphed as the point $(6, 5)$. The ordinary Cartesian plane then becomes a plane of complex numbers.

Thus $i = 0 + 1i$ is equated with the point $(0, 1)$ and the number $1 = 1 + 0i$ is equated with the point $(1, 0)$. The point $(2, 1)$ represents the number $2 + i$. The number $(\sqrt{3}+i)/2$ is equated with the point $(\sqrt{3}/2, 1/2)$. In the complex plane the $x$-axis is called the “real” axis and the $y$-axis is called the imaginary” axis.

2.4.5 The algebra of complex numbers

The complex numbers have a natural addition, subtraction and multiplication.

We add and subtract complex numbers just as we would polynomials, keeping up with the real and imaginary parts. For example,

$$(3 + 4i) + (7 + 11i) = 10 + 15i$$

and

$$(3 + 4i) - (7 + 11i) = -4 - 7i.$$ 

We multiply complex numbers $(a + bi)(c + di)$ just as we would the polynomials $(a + bx)(c + dx)$ except that we remember that $i^2 = -1$.

For example, since

$$(3 + 4x)(7 + 11x) = 21 + 61x + 44x^2.$$ 

then

$$(3 + 4i)(7 + 11i) = 21 + 61i + 44i^2 = 21 + 61i - 44 = -23 + 61i.$$
2.4.6 Division of complex numbers

We would like our complex numbers to be written in “Cartesian form” \( a + bi \) so there is a little twist involved in doing division with complex numbers. Note that if \( z = a + bi \) then \( z \overline{z} = (a + bi)(a - bi) = a^2 + b^2 \).

So if we are dividing by \( z \), we may view \( \frac{1}{z} \) as

\[
\frac{1}{z} = \frac{\overline{z}}{z \overline{z}} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i.
\]

Computationally, this means that anytime we have a fraction involving a complex number \( z \) in the denominator, we may multiply both numerator and denominator by \( \overline{z} \) and simplify.

For example,

\[
\frac{3 + 4i}{7 - 11i} = \frac{(3 + 4i)(7 + 11i)}{(7 - 11i)(7 + 11i)} = \frac{-23 + 61i}{170} = \frac{-23}{170} + \frac{61}{170}i.
\]

This process, multiplying the numerator and denominator of a fraction by the conjugate of the denominator, is called \textit{rationalizing the denominator}.

Some worked examples.

Write the complex fractions below into the “Cartesian” form \( z = a + bi \) where \( a, b \in \mathbb{R} \).

1. \( \frac{3 + 2i}{7 - 3i} \).
2. \( \frac{5}{3 + i} \).
3. \( \frac{2}{1 + i} \).
4. \( \frac{1}{i} \).

Solution.

1. \( \frac{3 + 2i}{7 - 3i} = \frac{(3 + 2i)(7 + 3i)}{(7 - 3i)(7 + 3i)} = \frac{15 + 23i}{58} = \frac{15}{58} + \frac{23}{58}i \) (So the real part is \( \frac{15}{58} \) and the imaginary part is \( \frac{23}{58} \)).

2. \( \frac{5}{3 + i} = \frac{5}{3 + i} \cdot \frac{3 - i}{3 - i} = \frac{15 - 5i}{10} = \frac{3}{2} - \frac{1}{2}i \).

3. \( \frac{2}{1 + i} = \frac{2}{1 + i} \cdot \frac{1 - i}{1 - i} = \frac{2(1 - i)}{2} = 1 - i \).

4. \( \frac{1}{i} = \frac{1}{i} \cdot \frac{-i}{-i} = \frac{-i}{1} = -i \) (Or \( 0 - i \)).

2.4.7 Using complex numbers and the factor theorem

We want to be ready to use complex numbers when factoring polynomials or solving polynomial equations.

For example, let’s factor the polynomial

\[ f(x) = x^3 - 2x^2 + 9x - 18. \]
Since this is a cubic polynomial and we don’t want to use the cubic formula\(^1\) then we need to find a zero. We could try some numbers (techniques in the next section will help us here) and discover by trial-and-error that \(f(2) = 0\). Or we could graph this polynomial on a graphing calculator and see that \(x = 2\) is a zero. Either way, if we want to factor this cubic, we need to find one root and then break the cubic down so that one piece is a quadratic. In this case, one we see that \(x = 2\) is a root, the rest of the problem is easy.

Since \(f(2) = 0\) (check this!) then \((x - 2)\) is a factor of \(x^3 - 2x^2 + 9x - 18\). Now divide \(x^2 - 2\) into \(f(x) = x^3 - 2x^2 + 9x - 18\) by synthetic division:

\[
\begin{array}{c|cccc}
1 & -2 & 9 & -18 \\
2 & 0 & 18 \\
2 & 0 & 9 & 0
\end{array}
\]

We see that \(f(x)\) factors as \((x - 2)(x^2 + 9)\). Since \(x^2 + 9 = 0\) implies that \(x = \pm 3i\) then \(x^2 + 9\) factors as \((x - 3i)(x + 3i)\). So \(f(x)\) factors as

\[x^3 - 2x^2 + 9x - 18 = (x - 2)(x - 3i)(x + 3i)\]

Note that if we have a complex root which is not real, that we have in fact two roots. Here both \(3i\) and its conjugate \(-3i\) are roots.

**Some worked examples.**

1. Factor the polynomial \(f(x) = x^3 - 8\) and then find all solutions to \(x^3 = 8\).

**Solution.** Since \(f(x) = x^3 - 8\) is zero when \(x^3 = 8\) then we know that one zero is \(x = 2\). Since \(f(2) = 0\) we know that \(x - 2\) is a factor of \(f(x)\). Dividing \(x^3 - 8\) by \(x - 2\) gives us \(x^3 - 8 = (x - 2)(x^2 + 2x + 4)\). So the solutions to \(x^3 - 8 = 0\) are the solutions to \((x - 2)(x^2 + 2x + 4)\). By the quadratic formula, the solutions to \(x^2 + 2x + 4 = 0\) are \(x = \frac{1}{2}(-2 \pm 2\sqrt{3}i) = -1 \pm \sqrt{3}i\). This implies that \((x^2 + 2x + 4)\) factors as

\[x^2 + 2x + 4 = (x - (-1 + \sqrt{3}i))(x - (-1 - \sqrt{3}i)) = (x + 1 + \sqrt{3}i)(x + 1 - \sqrt{3}i)\]

So the factoring of \(f(x)\) is

\[x^3 - 8 = (x - 2)(x + 1 - \sqrt{3}i)(x + 1 + \sqrt{3}i)\]

The full set of solutions to the equation \(x^3 = 8\) is the set of solutions to the equation \(x^3 - 8 = 0\). These are the zeroes of \(f(x)\):

\[x = 2, x = -1 + \sqrt{3}i, x = -1 - \sqrt{3}i\]

2. Factor completely \(g(x) = x^6 - 1\).

**Solution.** The Fundamental Theorem of Algebra tells us the \(x^6 - 1\) has six zeroes and therefore factors into six linear pieces. But it is not particularly easy to find this zeroes or the associated factors.

One way to begin factoring \(g(x) = x^6 - 1\) is to view this polynomial as a difference of squares:

\[g(x) = x^6 - 1 = (x^3 + 1)(x^3 - 1)\]

\(^1\)Yes, there is a cubic formula! But it is quite messy....
To further factor $x^3 - 1$, notice that $x = 1$ is surely a zero and after synthetic division we see that

$$x^3 - 1 = (x - 1)(x^2 + x + 1).$$

In a similar way we should notice that $x = -1$ is a zero of $x^3 + 1$ and so

$$x^3 + 1 = (x + 1)(x^2 - x + 1).$$

So we now have

$$g(x) = x^6 - 1 = (x^3 + 1)(x^3 - 1) = (x + 1)(x^2 - x + 1)(x - 1)(x^2 + x + 1).$$

We are not done; we need to factor the two quadratics $x^2 + x + 1$ and $x^2 - x + 1$. In an early example, we factored these using the quadratic formula and found that

$$x^2 + x + 1 = \left(x - \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\left(x - \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

and

$$x^2 - x + 1 = \left(x + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\left(x + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

So

$$g(x) = x^6 - 1 = (x + 1)(x^2 - x + 1)(x - 1)(x^2 + x + 1)$$

$$= \left(x + 1\right)\left(x + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\left(x + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(x - 1\right)\left(x - \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\left(x - \frac{1}{2} + \frac{\sqrt{3}}{2}i\right).$$

Notice that in this last example, relying on complex numbers and the formula for difference of squares, we were able to break the sixth degree polynomial $x^6 - 1$ down into six linear terms (!) as promised by the Fundamental Theorem of Algebra.

### 2.4.8 Other resources on complex numbers

In the free textbook, *Precalculus*, by Stitz and Zeager (version 3, July 2011, available at stitz-zeager.com) this material is covered in section 3.4.

In the free textbook, *Precalculus, An Investigation of Functions*, by Lippman and Rasmussen (Edition 1.3, available at www.opentextbookstore.com) this material is covered very briefly at the beginning of section 8.3.


There are lots of online resources for studying complex numbers. Here are some I recommend.

1. Wikipedia article on complex numbers.
2. Khan Academy videos on complex numbers.
3. Paul Dawkin’s online notes on complex numbers.
4. A Wikipedia article on applications of complex numbers

Worksheet to go with these notes.

As class homework, please complete Worksheet 2.4A, Complex numbers, available through the class webpage.