Worksheet 1.6A, Inverse functions
MATH 1410
(SOLUTIONS)

1. Find the inverse function of $f(x)$. State the domain of the inverse function $f^{-1}(x)$.

   (a) $f(x) = 2x + 4$
   (b) $f(x) = 2(x + 4)$
   (c) $f(x) = 3(x - 7) + 5$
   (d) $f(x) = x^2 - 4$
   (e) $f(x) = x^3 + 7$
   (f) $f(x) = (x + 7)^3$

   Solution.

   (a) The inverse function of $f(x) = 2x + 4$ is $f^{-1}(x) = \dfrac{x-4}{2}$.
   The function $f^{-1}(x)$ has domain $(-\infty, \infty)$.

   (b) The inverse function of $f(x) = 2(x + 4)$ is $f^{-1}(x) = \dfrac{1}{2}(x - 8)$.
   The function $f^{-1}(x)$ has domain $(-\infty, \infty)$.

   (c) The inverse function of $f(x) = 3(x - 7) + 5$ is $f^{-1}(x) = \dfrac{x-5}{3} + 7$.
   The function $f^{-1}(x)$ has domain $(-\infty, \infty)$.

   (d) The inverse function of $f(x) = x^2 - 4$ is $f^{-1}(x) = \sqrt{x+4}$. In order to make $f(x)$ one-to-one, so that there is no ambiguity in $f^{-1}(x)$, we need to restrict the domain of $f(x)$ to $[0, \infty)$. This means that the range of $f^{-1}(x)$ is also $[0, \infty)$.
   The function $f^{-1}(x)$ has domain $[-4, \infty)$.

   (e) The inverse function of $f(x) = x^3 + 7$ is $f^{-1}(x) = \sqrt[3]{x-7}$.
   The function $f^{-1}(x)$ has domain $(-\infty, \infty)$.

   (f) The inverse function of $f(x) = (x + 7)^3$ is $f^{-1}(x) = \sqrt[3]{x} - 7$.
   The function $f^{-1}(x)$ has domain $(-\infty, \infty)$.

2. For each function $f(x)$, below, find a domain on which f is one-to-one and then create an inverse function $g(x)$ corresponding to f on that domain.

   (a) $f(x) = (x - 4)^2$
   (b) $f(x) = \sqrt{x+4}$
   (c) $f(x) = \dfrac{1}{x+1}$
   (d) $f(x) = \dfrac{x}{x+1}$
   (e) $f(x) = \sqrt{x^3 + 9}$
   (f) $f(x) = \dfrac{1}{\sqrt{x^2+1}}$

(a) The inverse function of $f(x) = (x-4)^2$ is $f^{-1}(x) = \sqrt{x} + 4$.
The range of $f(x)$ is $[0, \infty)$ which is the domain of f^{-1}. It is in this region that $f(x)$ is one-to-one so let us set the domain of $f(x)$ to the range of $f^{-1}(x)$, that is $[4, \infty)$.

$$\boxed{\text{Set the domain of } f(x) \text{ to } [4,\infty)\text{. Then } f(x) \text{ is one-to-one with inverse } f^{-1}(x) = \sqrt{x} + 4.}$$

(b) The inverse function of $f(x) = \sqrt{x+4}$ is $f^{-1}(x) = x^2 - 4$.
The range of $f(x)$ is $[0, \infty)$ which is the domain of f^{-1}. It is in this region that $f(x)$ is one-to-one so let us set the domain of $f(x)$ to the range of $f^{-1}(x)$ which is $[-4, \infty)$.

$$\boxed{\text{Set the domain of } f(x) \text{ to } [-4,\infty)\text{. Then } f(x) \text{ is one-to-one with inverse } f^{-1}(x) = x^2 - 4}$$

(c) If $y = \dfrac{1}{x+1}$ then the inverse function obeys the equation $x = \dfrac{1}{y+1}$.
Solve for y:

$$x = \frac{1}{y+1} \implies x(y+1) = 1 \implies xy + x = 1 \implies xy = 1 - x \implies y = \frac{1-x}{x}.$$

$$\boxed{\text{The domain of } f(x) \text{ to } (-\infty,-1)\cup(-1,\infty)\text{. } f(x) \text{ is one-to-one with inverse } f^{-1}(x) = \frac{1-x}{x}.}$$

(d) If $y = \dfrac{x}{x+1}$ then the inverse function obeys the equation $x = \dfrac{y}{y+1}$.
Solve for y:

$$x = \frac{y}{y+1} \implies x(y+1) = y \implies xy + x = y \implies xy - y = -x \implies y(x-1) = -x \implies y = \frac{x-1}{-x} = \frac{x}{1-x}.$$

$$\boxed{\text{The domain of } f(x) \text{ to } (-\infty,-1)\cup(-1,\infty)\text{. } f(x) \text{ is one-to-one with inverse } f^{-1}(x) = \frac{x}{1-x}.}$$

(e) The inverse function of $f(x) = \sqrt{x^3+9}$ is $f^{-1}(x) = x^3 - 9$. The domain of $f(x)$ is $[-\sqrt[3]{9}, \infty)$.

(f) If $y = \dfrac{1}{\sqrt{x^2+1}}$ then the inverse function involves the equation: $x = \dfrac{1}{\sqrt{y^2+1}}$.
Let us square both sides and then multiply both sides by $y^2 + 1$:

$$x = \frac{1}{\sqrt{y^2+1}} \implies x^2 = \frac{1}{y^2+1} \implies x^2(y^2+1) = 1.$$

Then divide both sides by x^2 and then subtract 1 from both sides:

$$x^2(y^2+1) = 1 \implies y^2 + 1 = \frac{1}{x^2} \implies y^2 = \frac{1}{x^2} - 1.$$

Then get a common denominator on the right side and then take square roots.

$$y^2 = \frac{1}{x^2} - 1 \implies y^2 = \frac{1-x^2}{x^2} \implies y = \pm\left(\frac{\sqrt{1-x^2}}{x}\right).$$

This solution does not give a function since there are two answers. So let us agree to restrict the domain of $f(x)$ to $[0, \infty)$ and then the inverse function will have range $[0, \infty)$. If we do that, then

$$\boxed{f^{-1}(x) = \frac{\sqrt{1-x^2}}{x}.}$$

3. Find the inverse function of $f(x)$. State the domain of the inverse function $f^{-1}(x)$.

   (a) $f(x) = x^3$
   (b) $f(x) = \sqrt[3]{x}$
   (c) $f(x) = 4x - 5$
   (d) $f(x) = \sqrt{x^3 + 5}$
   (e) $f(x) = x^3 + 5$

   (a) The inverse function of $f(x) = x^3$ is $f^{-1}(x) = \sqrt[3]{x}$. The domain of the inverse function is $(-\infty, \infty)$.
   (b) The inverse function of $f(x) = \sqrt[3]{x}$ is $f^{-1}(x) = x^3$. The domain of $f^{-1}(x)$ is $(-\infty, \infty)$.
   (c) The inverse function of $f(x) = 4x - 5$ is $f^{-1}(x) = \dfrac{x+5}{4}$. The domain of $f^{-1}(x)$ is $(-\infty, \infty)$.
   (d) The inverse function of $f(x) = \sqrt{x^3 + 5}$ is $f^{-1}(x) = \sqrt[3]{x^2 - 5}$. The domain of $f(x)$ is $[-\sqrt[3]{5}, \infty)$ and the range of $f(x)$ is $[0, \infty)$. Therefore the domain of $f^{-1}(x)$ is $[0, \infty)$.
   (e) The inverse function of $f(x) = x^3 + 5$ is $f^{-1}(x) = \sqrt[3]{x - 5}$. The domain of $f^{-1}(x)$ is $(-\infty, \infty)$.

4. Compose the function f and g and determine, from your answer, if the functions f and g are inverses of each other.

   (a) $f(x) = (x+3)^2 - 7,\ g(x) = \sqrt{x+7} - 3$. (Assume the domain of $f(x)$ is $[-3, \infty)$.)
   (b) $f(x) = 3x + 7,\ g(x) = \dfrac{1}{3}x - 7$.
   (c) $f(x) = 2\sqrt[3]{x-5} + 7,\ g(x) = (\dfrac{x-7}{2})^3 + 5$.
   (d) $f(x) = \dfrac{(x-7)^3}{2} + 5,\ g(x) = 2\sqrt[3]{x-5} + 7$.

   Solutions.

   (a) $(f \circ g)(x) = f(\sqrt{x+7} - 3) = (\sqrt{x+7} - 3 + 3)^2 - 7 = x$. (Similarly $(g \circ f)(x) = x$.) So, yes, these functions are inverses.
   (b) $(f \circ g)(x) = 3(\dfrac{1}{3}x - 7) + 7 = x - 21 + 7 = x - 14$. Since this answer is not equal to x, these functions are not inverses.
   (c) $(f \circ g)(x) = f((\dfrac{x-7}{2})^3 + 5) = 2\sqrt[3]{(\dfrac{x-7}{2})^3 + 5 - 5 + 7}) = 2\sqrt[3]{(\dfrac{x-7}{2})^3} + 7 = 2(\dfrac{x-7}{2}) + 7 = x$. (Similarly $(g \circ f)(x) = x$.) So, yes, these functions are inverses.
   (d) $(f \circ g)(x) = f(2\sqrt[3]{x-5} + 7) = \dfrac{(2\sqrt[3]{x-5} + 7 - 7)^3}{2} + 5 = \dfrac{(2\sqrt[3]{x-5})^3}{2} + 5 = \dfrac{8(x-5)}{2} + 5 = 4(x-5) + 5 = 4x - 15$.
   Since this answer is not equal to x, these functions are not inverses.