The algebra of functions

Given two functions, say \( f(x) = x^2 \) and \( g(x) = x + 1 \), we can, in *obvious* ways, add, subtract, multiply and divide these functions. For example, the function \( f + g \) is defined simply by

\[
(f + g)(x) = x^2 + (x + 1);
\]

the function \( f - g \) is defined simply by

\[
(f - g)(x) = x^2 - (x + 1).
\]

Similarly

\[
(f \cdot g)(x) = (x^2) \cdot (x + 1)
\]

and

\[
\left(\frac{f}{g}\right)(x) = \frac{x^2}{x + 1}.
\]

Function Composition

A more important operation between functions is the operation of *function composition*. If \( f \) is a function from \( X \) into \( Y \) and \( g \) is a function from \( Y \) into \( Z \), then \( g \circ f \) is a function from \( X \) into \( Z \) defined by first allowing \( f \) to map elements of \( X \) into \( Y \) and then allowing elements of \( Y \) to be mapped by \( g \) into \( Z \).
Function Composition

Notice that in \( g \circ f \), \( f \) is the first function involved while \( g \) is the second! We read function notation \((g \circ f)(x)\) from **right to left**.

In the example below, \( g \circ f \) maps the elements of \( X \) as follows:

\[
\begin{align*}
a & \mapsto @ \\
b & \mapsto @ \\
c & \mapsto # \\
d & \mapsto !!
\end{align*}
\]

Suppose \( f(x) = x^2 \) and \( g(x) = x + 1 \).
The function \((g \circ f)\) maps 3 to 10 since \( f(3) = 3^2 = 9 \) and \( g(9) = 10 \).
If \( f \) and \( g \) are described by an equation then often \((g \circ f)\) can be described by an equation.
Here \((g \circ f)(x) = g(f(x)) = g(x^2) = x^2 + 1 \).
So \((g \circ f)(x) = x^2 + 1 \).

Function Composition

View a function as a “machine”, taking inputs and generating outputs.

The composition of two functions will be a **sequence** of function machines:

\[
\begin{align*}
\text{INPUT } x \\
\text{FUNCTION } f: & \\
\text{OUTPUT } f(x) \\
& \\
\text{INPUT } x^2 \\
\text{FUNCTION } g: & \\
\text{OUTPUT } g(f(x)) = 10 \\
& \\
\end{align*}
\]

If the codomain of the function \( f \) is the same as the domain of the function \( g \), then we can compose first \( f \) then \( g \) to create \((g \circ f)\).
Or we can compose first \( g \) then \( f \) to create \((f \circ g)\).
But the **order** of composition is important!
The function \((f \circ g)\) is probably **not** the same function as \((g \circ f)\)!
For example, if \( f(x) = x^2 \) and \( g(x) = x + 1 \) then (as done above) we have
\[
(g \circ f)(x) = x^2 + 1.
\]
On the other hand
\[
(f \circ g)(x) = f(g(x)) = f(x + 1) = (x + 1)^2.
\]
So \((g \circ f)(x) = x^2 + 1 \) but \((f \circ g)(x) = (x + 1)^2 \).
Function composition examples

\[ f(x) = x^2, \ g(x) = x + 1 \]

In elementary algebra, we learned the importance of parentheses – that \( 1 + x^2 \) is quite different from \( (1 + x)^2 \).

The use of parentheses and the order of operations is especially important in the composition of functions.

Here squaring and then adding one \((g \circ f)\) is different from adding one and then squaring \((f \circ g)\).

\[ (g \circ f)(x) = x^2 + 1 \] but \[ (f \circ g)(x) = (x + 1)^2 \].

In the next lesson we will work some problems involving function composition.

(END)

Function Composition Exercises

Some worked examples. Given the functions \( f(x) = x^2 - 1 \) and \( g(x) = x + 2 \), create the following composition functions:

1. \((f \circ g)(x)\)
2. \((g \circ f)(x)\).

Solutions.

1. \((f \circ g)(x) = f(g(x)) = f(x + 2) = (x + 2)^2 - 1 = x^2 + 4x + 4 - 1 = x^2 + 4x + 3.\)
2. \((g \circ f)(x) = g(f(x)) = g(x^2 - 1) = (x^2 - 1) + 2 = x^2 + 1.\)

Given the functions \( f \) and \( g \), below, find the composition functions \( f \circ g \) and \( g \circ f \). The function \((f \circ g)(x)\) is the same as \( f(g(x))\); \((g \circ f)(x)\) is the same as \( g(f(x))\).

1. \( f(x) = x^2 + 1 \) and \( g(x) = \sqrt{3} \).

Solution.

1. \( f(x) = x^2 + 1 \) and \( g(x) = \sqrt{3} \).
\[ (f \circ g)(x) = f(g(x)) = f(\sqrt{3}) = \sqrt{3}^2 + 1 = 3 + 1 = 4. \]
\[ (g \circ f)(x) = g(f(x)). \] But \( g(\text{anything}) = \sqrt{3} \), so the answer is \( \sqrt{3} \).

\[ (f \circ g)(x) = 4 \] and \( (g \circ f)(x) = \sqrt{3}. \)
Function Composition Exercises

3. \( f(x) = x^2 + 9 \) and \( g(x) = \sqrt{x} \).

4. \( f(x) = x^2 + 5 \) and \( g(x) = \sqrt{x-5} \).

Solutions.

3. \( f(x) = x^2 + 9 \) and \( g(x) = \sqrt{x} \).
\[
(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 + 9 = x + 9.
\]
\[
(g \circ f)(x) = g(f(x)) = g(x^2 + 9) = \sqrt{x^2 + 9}.
\]
\[
(f \circ g)(x) = x + 9 \quad \text{and} \quad (g \circ f)(x) = \sqrt{x^2 + 9}.
\]

4. \( f(x) = x^2 + 5 \) and \( g(x) = \sqrt{x-5} \).
\[
(f \circ g)(x) = f(g(x)) = f(\sqrt{x-5}) = (\sqrt{x-5})^2 + 5 = (x-5) + 5 = x
\]
\[
(g \circ f)(x) = g(f(x)) = g(x^2 + 5) = \sqrt{x^2 + 5 - 5} = \sqrt{x^2} = |x|.
\]
\[
(f \circ g)(x) = x \quad \text{and} \quad (g \circ f)(x) = |x|.
\]

Next we will look at “chaining” functions together with function composition.

(END)

Breaking a function down into components

It is convenient at times to break a function down into pieces, so that we may view the function itself as a composition of two or more functions. For example, suppose

\[ h(x) = \sqrt{3x + 4}. \]

If we input an \( x \)-value into \( h \), we first compute \( 3x + 4 \) and then we take the square root. So we may view the function \( h \) as a composition of a function \( g(x) = 3x + 4 \) and \( f(x) = \sqrt{x} \).

\[ h = (f \circ g) \quad \text{where} \quad g(x) = 3x + 4 \quad \text{and} \quad f(x) = \sqrt{x}. \]

This is an example of taking a complicated function and breaking it down into its simple pieces.
Some more worked examples.
For each of the functions $f(x)$ and $h(x)$ below, find a function $g(x)$ such that $h(x) = (f \circ g)(x)$.

1. $f(x) = 10^x$, $h(x) = 10(x^2-17).
2. $f(x) = \sqrt{x}$, $h(x) = \sqrt{x^2 + 4}$.

Solution.
1. $h(x) = 10(x^2-17) = (f \circ g)(x)$ if $g(x) = x^2 - 17$.
2. $h(x) = \sqrt{x^2 + 4} = (f \circ g)(x)$ if $g(x) = x^2 + 4$.

Breaking a function down....

For each function $h$ given below, decompose $h$ into the composition of two functions $f$ and $g$ so that $h = f \circ g$.

1. $h(x) = (x + 5)^2$
2. $h(x) = \sqrt[3]{5x^2 + 1}$
3. $h(x) = 2^\cos{x}$

Solutions.
1. $h(x) = (x + 5)^2$ is the composition of $g(x) = x + 5$ and $f(x) = x^2$.
2. $h(x) = \sqrt[3]{5x^2 + 1}$ is the composition of $g(x) = 5x^2 + 1$ and $f(x) = \sqrt[3]{x}$.
3. $h(x) = 2^\cos{x}$ is the composition of $g(x) = \cos{x}$ and $f(x) = 2^x$.
(We can find the functions $g$ and $f$, even if we have not yet studied the function $\cos{x}$ — the notation leads us to the answer!)

Function Chaining

Once we understand function composition, there is no reason to stop at composing just two functions! We can compose a chain of functions, running an input $x$ through one function after another.

For example, suppose that $f(x) = x^2$, $g(x) = 3x + 5$ and $h(x) = \sqrt{x}$.

If we run $x$ through $f$, $g$ and $h$ in that order we get

\[(h \circ g \circ f)(x) = h(g(f(x))) = h(3x^2 + 5) = \sqrt{3x^2 + 5}.
\]

There is no limit to the number of functions we can “chain” together!

For example, suppose that $f(x) = x^2$, $g(x) = 3x + 5$, $h(x) = \sqrt{x}$ and $j(x) = \cos{x}$.

If we run $x$ through $f$, $g$, $h$ and $j$ in that order we get

\[(j \circ h \circ g \circ f)(x) = j(h(g(f(x)))) = j(h(g(x^2))) = j(h(3x^2 + 5)) = j(\sqrt{3x^2 + 5}) = \cos(\sqrt{3x^2 + 5}).
\]
(We can do this even if we have not yet studied the cosine function $\cos{x}$ — we just follow our notation!)

In calculus, after we study the derivative of a function, we will learn to take the derivative of a “chain” of functions composed together in this manner. The method we develop there is called “The Chain Rule” for derivatives.