Worksheet 1.1A, Definition of a function, domain of a function
MATH 1410
(SOLUTIONS)

1. A function \( f \), from the set \( X = \{1, 2, 3, 4\} \) to the set \( \mathbb{R} \) is given by the following table:

<table>
<thead>
<tr>
<th>( X )</th>
<th>( \mathbb{R} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
</tr>
</tbody>
</table>

(a) Write the function \( f \) as a set of ordered pairs.
(b) Can you create an equation to describe the function \( f \)?

Solution.
As a set, we can write \( f := \{(1, 4), (2, 6), (3, 8), (4, 10)\} \)

As an equation, we can describe the output as \( f(x) = 2x + 2 \).

2. A function \( f \), from the set \( X = \{0, 1, 2, 3, 4, 5\} \) to the set \( \mathbb{R} \) is given by the following table:

<table>
<thead>
<tr>
<th>( X )</th>
<th>( \mathbb{R} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>−1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
</tr>
<tr>
<td>5</td>
<td>24</td>
</tr>
</tbody>
</table>

(a) Write the function \( f \) as a set of ordered pairs.
(b) Can you create an equation to describe the function \( f \)?

Solution.
As a set, we can write \( f := \{(0, −1), (1, 0), (2, 3), (3, 8), (4, 15), (5, 24)\} \)

As an equation, we can describe the output as \( f(x) = x^2 − 1 \).

3. Determine whether the equations below define \( y \) as a function of \( x \). (If possible, choose the domain so that you can answer “Yes.” If the equation does not define a function, give a value of \( x \) which yields multiple values of \( y \).)

(a) \( xy = 1 \)
(b) \(|xy| = 1\).
(c) \( y = |x − 2| \)
(d) \( y\sqrt{x} = 2 \)
(e) \( xy^2 = 0 \)
(f) \( xy^2 = 1 \).
(g) \( x^2 + y^3 = 8 \).
Solution.

(a) \( xy = 1 \implies y = \frac{1}{x} \). YES, this IS a function.

(b) \(|xy| = 1\). If \( x = 1 \) then \( y \) could be 1 or \(-1\). NO, NOT a function.

(c) \( y = |x - 2| \). YES, this IS a function. We have already defined \( y \) in a precise manner.

(d) \( y\sqrt{x} = 2 \). Solve for \( y \):
\[ y = \frac{2}{\sqrt{x}}. \]
This looks like we have defined \( y \) carefully but what if \( x = 0 \)? If \( x = 0 \) then could \( y\sqrt{0} = 2 \)? No. So \( x \) is never zero in the original problem. Since \( x \) is never zero, \( y = \frac{2}{\sqrt{x}} \) does a perfectly good job of defining \( y \) in terms of \( x \). YES, this is a function.

(e) \( yx^2 = 0 \). If \( x = 0 \) then \( y \) can be anything. NO, NOT a function.

(f) \( xy^2 = 1 \). If \( x = 1 \) then \( y \) can be either 1 or \(-1\). NO, NOT a function.

(g) \( x^2 + y^3 = 8 \). Solve for \( y \):
\[ y = \sqrt[3]{8 - x^2} \] so yes, this is a function.

4. Let \( f(x) = x^2 + 1 \). Find

(a) \( f(0), f(-1), f(1), f(-5), f(-x) \)

(b) \( f(x + h) \),

(c) \( f(\sqrt{x}) - 1 \),

(d) \( f(a + b) \),

(e) \( -f(x) + 2 \)

(f) \( f(x + h) - f(x) \)

(g) \( \frac{f(x + h) - f(x)}{h} \)

Solutions.

(a) \( f(0) = 1, f(-1) = 2, f(1) = 2, f(-5) = 26, f(-x) = x^2 + 1 \)

(b) \( f(x + h) = x^2 + 2xh + h^2 + 1, f(\sqrt{x}) - 1 = x, f(a + b) + a^2 + 2ab + b^2 + 1, -f(x) + 2 = -x^2 - 1 + 2 = -x^2 + 1 \)

(c) \( f(\sqrt{x}) - 1 = \sqrt{x^2} + 1 - 1 = x \).

(d) \( f(a + b) = (a + b)^2 + 1 = a^2 + 2ab + b^2 + 1 \)

(e) \( -f(x) + 2 = -x^2 + 1 + 2 = -x^2 + 3 \).

(f) \( f(x + h) - f(x) = (x + h)^2 + 1 - (x^2 + 1) = x^2 + 2xh + h^2 + 1 - x^2 - 1 = 2xh + h^2 \).

(g) \( \frac{f(x + h) - f(x)}{h} = \frac{2xh + h^2}{h} = \frac{h(2x + h)}{h} = 2x + h \).

5. Let \( g(x) = \sqrt{3 - x} \).

(a) \( g(0), g(-1), g(1), g(-5), g(-x) \)

(b) \( g(x + h) \),

(c) \( g(x^2 - 3) \),

(d) \( -g(x) + 3 \)

(e) \( g(x + h) - g(x) \)
\[
\frac{g(x + h) - g(x)}{h}
\]

**Solutions.**

(a) \(g(0) = \sqrt{3}, \ g(-1) = 2, \ g(1) = \sqrt{2}, \ g(-5) = \sqrt{8}, \ g(-x) = \sqrt{3 + x}\)

(b) \(g(x + h) = \sqrt{3 - x - h}, \ g(x^2 - 3) = \sqrt{x^2} = |x|, \ -g(x) + 3 = -\sqrt{3 - x} + 3\)

(c) \(g(x^2 - 3) = \sqrt{3 - (x^2 - 3)} = \sqrt{x^2} = |x|\).

(d) \(g(x) + 3 = -\sqrt{3 - x} + 3\).

(e) \(g(x + h) - g(x) = \sqrt{3 - x - h} - \sqrt{3 - x}\).

(f) \(g(x + h) - g(x) = \frac{\sqrt{3 - x - h} - \sqrt{3 - x}}{h}\)\)

It turns out that this expression can be simplified (a little) by multiplying both numerator and denominator by the “conjugate” of \(\sqrt{3 - x - h} - \sqrt{3 - x}\), that is, multiplying numerator and denominator by \(\sqrt{3 - x - h} + \sqrt{3 - x}\):

\[
\frac{\sqrt{3 - x - h} - \sqrt{3 - x}}{h} \cdot \frac{\sqrt{3 - x - h} + \sqrt{3 - x}}{\sqrt{3 - x - h} + \sqrt{3 - x}} = \frac{(3 - x - h) - (3 - x)}{h(\sqrt{3 - x - h} + \sqrt{3 - x})} = \frac{-h}{h(\sqrt{3 - x - h} + \sqrt{3 - x})}
\]

This allows us to cancel out the \(h\) on the bottom.

\[\frac{1}{\sqrt{3 - x - h} + \sqrt{3 - x}}\]

This technique has value in calculus.

6. Find the domain of each of the following functions

(a) \(y(x) = \frac{1}{x - 9}\) has domain \((-\infty, 9) \cup (9, \infty)\).

(b) \(y(x) = \frac{x^2}{4 - x}\) has domain \((-\infty, 4) \cup (4, \infty)\).

(c) \(f(x) = \frac{x}{\sqrt{4 - x}}\) has domain \((-\infty, 4)\).

(d) \(g(x) = \frac{\sqrt{4 - x}}{x}\)

(e) \(f(x) = 4\)

(f) \(f(x) = 2^x\)

(g) \(h(x) = \frac{\sqrt{x - 4}}{x^2 - 5x + 6}\)

**Solutions.**

(a) \(y(x) = \frac{1}{x - 9}\) has domain \((-\infty, 9) \cup (9, \infty)\).

(b) \(y(x) = \frac{x^2}{4 - x}\) has domain \((-\infty, 4) \cup (4, \infty)\).

(c) \(f(x) = \frac{x}{\sqrt{4 - x}}\) has domain \((-\infty, 4)\).
(d) \( g(x) = \frac{\sqrt{4-x}}{x} \) has domain \((-\infty, 0) \cup (0, 4]\).

(e) \( f(x) = 4 \) has domain \((-\infty, \infty) = \mathbb{R}\).

(f) \( f(x) = 2^x \) has domain \((-\infty, \infty) = \mathbb{R}\).

(g) \( h(x) = \frac{\sqrt{x-4}}{x^2 - 5x + 6} \) has domain \((-\infty, 2) \cup (2, 3) \cup (3, \infty)\).