ω-PRIMALITY IN ARITHMETIC LEAMER MONOIDS

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Abstract. Let Γ be a numerical semigroup. The Leamer monoid S^s_Γ, for $s \in \mathbb{N}\backslash\Gamma$, is the monoid consisting of arithmetic sequences of step size s contained in Γ. In this note, we give a formula for the ω-primality of elements in S^s_Γ when Γ is an numerical semigroup generated by a arithmetic sequence of positive integers.

1. Preliminaries

A numerical monoid S is an additive submonoid of the nonnegative integers $\mathbb{N}_0$ under regular addition such that $|\mathbb{N}_0 - S| < \infty$ ([11] is a good general reference on this subject). A great deal of literature has appeared over the past 15 years which studies the nonunique factorization properties of these monoids (for instance, see [4], [6], and [5] and the references therein). Among the factorization constants studied on these objects is the *ω-primality function*, which in some sense measures how far an element $x \in S$ is from being a prime element. A general survey of these results can be found in [16], while the papers [2], [3], and [9] all consider issues related to algorithms for computing speciic values of the ω-function. Other papers that touch on this subject in more specific terms are [7], [8], [14], and [17]. In this paper, we pick up on the study begun in [12] of the factorization properties of Leamer monoids, which are constructed using numerical monoids. Leamer monoids first appeared in [10] and were used in that paper to study the Huenke-Wiegand conjecture from commutative algebra. In our current work, we address Problem 5.4 of [12] and completely determine the behavior of the ω function on a Leamer monoid generated by an arithmetic numerical monoid (i.e, a numerical monoid generated by an arithmetic sequence of integers). Our final results are summarized in Theorems 2.3 and 2.6. We find these results of interest, as the complete behavior of the ω-function on a numerical monoid is known in only a few cases (which are contained in [2]).

Before proceeding to our main result, we offer a series of definitions. We begin with a general definition of the ω-function itself.

Definition 1.1. Let S be a commutative cancellative monoid. For any nonunit $x \in S$, define $\omega(x) = m$ if m is the smallest positive integer such that whenever x divides $x_1 \cdots x_t$, with $x_i \in S$, then there is a set $T \subset \{1, 2, \ldots, t\}$ of indices with $|T| \leq m$ such that x divides $\sum_{i \in T} x_i$. If no such m exists, then set $\omega(x) = \infty$.

When S is clear from the context, we simply write $\omega(n)$. A collection of basic facts concerning the ω-function can be found in [1, Section 2]. Needless to say, an

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element $x \in S$ is prime if and only if $\omega(x) = 1$. The definition of a Leamer monoid follows.

Definition 1.2. Let Γ be a numerical monoid and $s \in \mathbb{N} \setminus \Gamma$. Set

$$S_\Gamma^s = \{(0,0)\} \cup \{(x,n) : \{x, x+s, x+2x, \ldots, x+ns\} \subset \Gamma\} \subset \mathbb{N}^2.$$

Thus S_Γ^s is the collection of arithmetic sequences of step size s contained in Γ. Under regular addition on $\mathbb{N}^2$, S_Γ^s is a monoid known as a *Leamer monoid*.

We define the column at $x \in \Gamma$ to be the set $\{(x,n) \in S_\Gamma^s : n \geq 1\}$. We say that the column at x is infinite (resp. finite) if the cardinality of the column at x is infinite (resp. finite). For a finite column, the height of the column is $\max\{n : (x,n) \in S_\Gamma^s\}$ and we define x_f to be the first infinite column in S_Γ^s. The largest positive integer not in Γ is know as the Frobenius number and we denote this as $F(\Gamma)$. Since $S_\Gamma^s \subseteq \mathbb{N}^2$, we can graphically represent S_Γ^s, and we do so below in the case where $\Gamma = \langle 12, 13, 20\rangle$ with $s = 1$. The red dots in the graph reprsent irredcible elements of S_Γ^s.

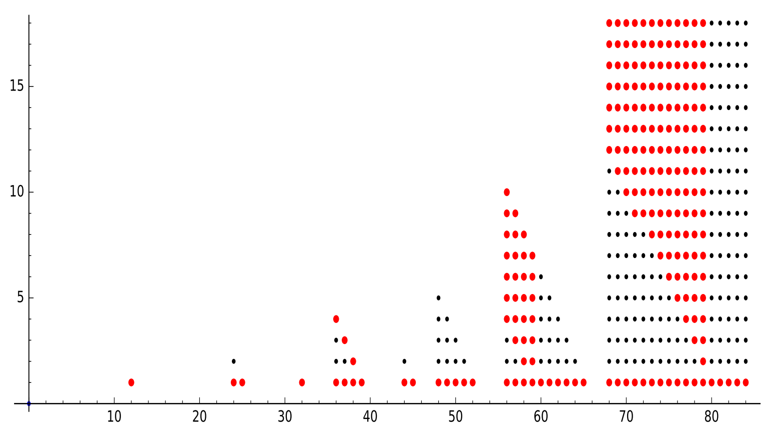

FIGURE 1. The Leamer monoid S_Γ^1 for $\Gamma = \langle 12, 13, 20\rangle$

The following result from [12, Lemma 2.8] will give us some basic factorization properties of an arbitrary Leamer monoid. Note that $\mathcal{A}(S_\Gamma^s)$ is the set of irreducible elements (or atoms) of S_Γ^s.

Lemma 1.3. (a) For $n \gg 0$, $(x_f, n) \in \mathcal{A}(S_\Gamma^s)$.
(b) The column at every $x > F(\Gamma)$ is infinite.

Suppose that $\omega(n)$ is finite. To find this value, it is often helpful to consider the *bullets* for n. A product of irreducibles $x_1 \cdots x_k$ is said to be a bullet for n if n divides the product $x_1 x_2 \cdots x_k$ but does not divide any proper subproduct. If $\text{bul}(x)$ represents the set of bullets of x, then the following proposition [16, Proposition 2.10] will be key in our coming calculations.

Proposition 1.4. If M is a commutative cancellative monoid and x a nonunit of M, then

$$\omega(x) = \sup\{r \mid x_1 \cdots x_r \in \text{bul}(x) \text{ where each } x_i \text{ is irreducible in } M\}.$$

There has been fairly extensive study of the ω-function on numerical monoids in recent years. Of particular interest is the following result [15, Theorem 3.6], which describes the eventual behavior of the ω-function. If $S = \langle n_1, ..., n_k\rangle$ is a numerical monoid, then for n sufficiently large, $\omega(n)$ is quasilinear with period dividing n_1. In particular, there exists an explicit N_0 such that $\omega(n + n_1) = \omega(n) + 1$ for $n > N_0$. Hence, for sufficiently large n, $\omega(n) = \frac{n}{n_1} + a_0(n)$, where $a_0(n)$ has period dividing n_1.

For the remainder of our work, we focus on numerical monoids generated by arithmetic sequences (a good general reference on this topic is [13]). So let $S = \langle a, a + d, ..., a + kd\rangle$, where gcd$(a, d)$=1 and $1 \le k < a$.

Lemma 1.5. [6, Lemmas 7 & 8]

(1) Let n be a nonnegative integer. Then $n \in S$ if and only if $n = qa + jd$ with $q \in \mathbb{N}$ and $0 \le j \le kq$.
(2) If $n = qa + jd$ with $q \in \mathbb{N}$ and $0 \le j \le kq$, then there is a factorization of n in S of length q.
(3) Let n be an integer with $n = ua + vd = u'a + v'd$. Then there exists an integer λ such that $(u, v) - (u', v') = \lambda(d, -a)$.
(4) If $n = qa + jd$ with $q \in \mathbb{N}$ and $0 \le j < a$, then q is the longest length of factorization of n in S.

We say that a Leamer monoid is *arithmetic* if Γ is an arithmetic numerical semigroup with $k \ge 2$ and s is the difference of the arithmetic sequence. If $\Gamma = \langle a, a + d, \cdots, a + kd\rangle$, then we will write $S^s_\Gamma = S^d_{a,k}$. We offer graphical representations of arithmetic Leamer monoids in Figures 2 and 3. Additionally, the following result tells us more about factorization properties of arithmetic Leamer monoids, which we will use to characterize the ω function in such monoids.

Theorem 1.6. [12, Lemma 4.3 (a)] Fix an arithmetic Leamer monoid $S^d_{a,k}$, and let $x = ma + id$, where $m, i \in \mathbb{N}$ and $0 \le i < a$. Then $S^d_{a,k}$ has a finite column at x if and only if $m \le \lfloor \frac{a-2}{k} \rfloor$ and $0 \le i \le km - 1$. In this case, the column at x has height $km - i$.

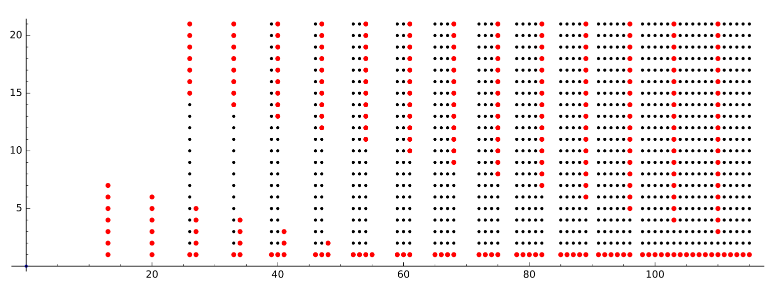

FIGURE 2. The Leamer monoid S^7_Γ for $\Gamma = \langle 13, 20, 27, 34, 41, 48, 55, 62\rangle$

Finally, we offer a lower bound on the ω function in a general Leamer monoid. Note that we are only considering non-unit elements, i.e. $(x, n) \neq (0, 0)$, so $n \ge 1$ by the definition of a Leamer monoid.

Proposition 1.7. If $(x, n) \in S_\Gamma^s$, then (x, n) has a bullet of length $n+1$. Hence, $\omega((x, n)) \geq n+1$ and no element in a Leamer monoid is prime.

Proof. We wish to show that $(n+1)(x+F(\Gamma), 1)$ is a bullet for (x, n). Since $nx + (n+1)F(\Gamma) \geq F(\Gamma)$,

$$(n+1)(x+F(\Gamma), 1) - (x, n) = (nx + (n+1)F(\Gamma), 1) \in S_\Gamma^s$$

by Lemma 1.3(b). Additionally,

$$n(x+F(\Gamma), 1) - (x, n) = ((n-1)x + nF(\Gamma), 0) \notin S_\Gamma^s$$

since $(n-1)x + nF(\Gamma) > 0$. Thus, (x, n) divides $(n+1)(x+F(\Gamma), 1)$ but no proper subsum of it, so it is a bullet. The last statment clearly follows. □

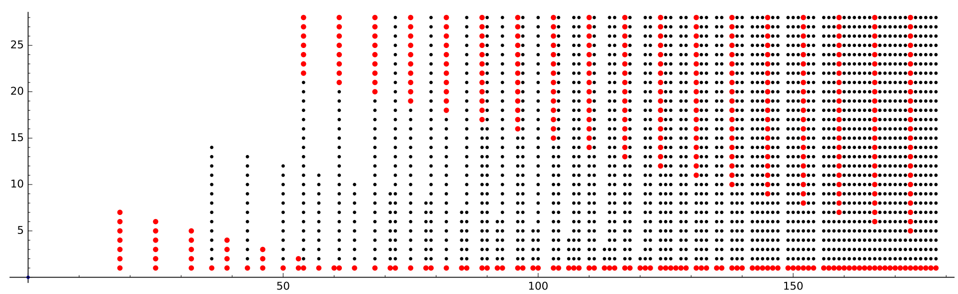

FIGURE 3. The Leamer monoid S_Γ^7 for $\Gamma = \langle 18, 25, 32, 39, 46, 53, 60, 67\rangle$

2. ω-VALUES IN ARITHMETIC LEAMER MONOIDS

Throughout this section, let $S_{a,k}^d$ be an arithmetic Leamer monoid with $\gcd(a, d) = 1$ and $2 \leq k \leq d$. In [12], the authors study the factorization properties of arithmetic Leamer monoids. Now, we wish to extend use these results to find the ω-values of all elements in an arithmetic Leamer monoid. To do so, we split the proof into two cases.

2.1. (x, n) is not a multiple of (a, k).

We focus here on the case where $(x, n) \in S_{a,k}^d$ such that $(x, n) \neq p(a, k)$ for any $p \in \mathbb{N}$. By Lemma 1.5, we may write $x = ma + is$ for $m \in \mathbb{N}$ and $i \in \{0, \cdots, mk\}$. Additionally, let

$$w = \max\left(n+1, m + \lfloor\frac{a-2}{k}\rfloor + 1 + \lfloor\frac{a+i-1}{a}\rfloor s\right).$$

We wish to show that $\omega((x, n)) = w$ in the following lemmas. For notation purposes, we let $x \mod a$ represent the least residue of x modulo a.

Lemma 2.1. Let $(x, n) \in S_{a,k}^d$ such that $(x, n) \neq p(a, k)$ for any $p \in \mathbb{N}$ and suppose that $c \geq w$. Then (x, n) divides the sum of any c non-zero elements of $S_{a,k}^d$.

Proof. Let $y_0 = m + \lfloor \frac{a-2}{k} \rfloor + 1 + \lfloor \frac{a+i-1}{a} \rfloor s$, and let

$$\begin{aligned} x_0 &= y_0 a - (ma + is) \\ &= \left(m + \left\lfloor \frac{a-2}{k} \right\rfloor + 1 + \left\lfloor \frac{a+i-1}{a} \right\rfloor s\right) a - (ma + is) \\ &= \left(\left\lfloor \frac{a-2}{k} \right\rfloor + 1\right) a + \left(\left\lfloor \frac{a+i-1}{a} \right\rfloor a - i\right) s. \\ &= \left(\left\lfloor \frac{a-2}{k} \right\rfloor + 1\right) a + (a+i-1-((a+i-1) \mod a) - i)s \\ &= \left(\left\lfloor \frac{a-2}{k} \right\rfloor + 1\right) a + (a-1-((i-1) \mod a))s. \end{aligned}$$

Since $0 \leq a-1-((i-1) \mod a) < a$, there is an infinite column at x_0 by Theorem 1.6, so this also implies that there is an infinite column at $x_0 + sa + td$ for any $s, t \in \mathbb{N}$.

Now, for $1 \leq j \leq c$, let (x_j, n_j) be a non-zero element of $S^d_{a,k}$. Since $x_j \in \langle a, \cdots, a+kd \rangle$, there exists $q_j, i_j \in \mathbb{N}$ such that $x_j = q_j a + i_j d$. Therefore, $\sum_{j=1}^{c} q_j = y_0 + b$ for some $b \in \mathbb{N}$ since $c \geq y_0$ and each q_j is at least 1. As a result, we see that $\sum_{j=1}^{c} x_j - x = \sum_{j=1}^{c} (q_j a + i_j d) - (ma + is) = (y_0 + b)a - (ma + is) + \sum_{j=1}^{c} i_j d = x_0 + ba + \sum_{j=1}^{c} i_j d$ by the definition of x_0. So by our above discussion, there is an infinite column of $S^d_{a,k}$ at $\sum_{j=1}^{c} x_j - x$. Since $c \geq n+1$ and each $n_j \geq 1$, $\sum_{j=1}^{c} n_j - n \geq 1$. Therefore, this shows that $\sum_{j=1}^{c} (x_j, n_j) - (x, n) = \left(\sum_{j=1}^{c} x_j - x, \sum_{j=1}^{c} n_j - n\right)$ is in $S^d_{a,k}$, which completes the proof. □

If $w = n+1$, then by Proposition 1.7 there is a bullet for x of length $n+1$, and hence $\omega((x,n)) = n+1$. We consider the remaining case in the next lemma.

Lemma 2.2. Let $(x, n) \in S^d_{a,k}$ such that $(x, n) \neq p(a, k)$ for any $p \in \mathbb{N}$. If $w = m + \lfloor \frac{a-2}{k} \rfloor + 1 + \lfloor \frac{a+i-1}{a} \rfloor s$, then $w(a, k)$ is a bullet for (x, n).

Proof. Define x_0 and y_0 as they are defined in the proof of Lemma 2.1. Since $w = y_0$, (x, n) divides $y_0(a, k)$ by Lemma 2.1. Now, we wish to show that (x, n) does not divide $(y_0 - 1)(a, k)$.

First, note that

$$(y_0 - 1)a - x = x_0 - a = \left\lfloor \frac{a-2}{k} \right\rfloor a + (a - 1 - ((i-1) \mod a))\, s,$$

so by Theorem 1.6, there is a finite column at $(y_0 - 1)a - x$ of height

$$\begin{aligned} &\lfloor \frac{a-2}{k} \rfloor k - (a - 1 - ((i-1) \mod a)) \\ =& \lfloor \frac{a-2}{k} \rfloor k - (a-2) - 1 + ((i-1) \mod a) \\ =& (-(a-2) \mod k) + ((i-1) \mod a) - 1. \end{aligned}$$

Now, we wish to show that $(y_0-1)k-n$ is greater than this height. To do so, we will first show that $mk+\lfloor\frac{a+i-1}{a}\rfloor sk>n$.

First, suppose that there is an infinite column at $x=ma+is$. Then by Theorem 1.6, $m>\lfloor\frac{a-2}{k}\rfloor$. Also, since $w=y_0$, $y_0-1\geq n$. Therefore, since $k\geq 2$ by assumption,

$$\begin{aligned}mk+\left\lfloor\frac{a+i-1}{a}\right\rfloor sk&\geq 2m+\left\lfloor\frac{a+i-1}{a}\right\rfloor sk\\&>m+\left\lfloor\frac{a-2}{k}\right\rfloor+\left\lfloor\frac{a+i-1}{a}\right\rfloor s=y_0-1\geq n.\end{aligned}$$

Now, suppose that there is a finite column at $x=ma+is$. Then the height of this column is $mk-i$ by Theorem 1.6, so $n\leq mk-i\leq mk+\lfloor\frac{a+i-1}{a}\rfloor sk$. Note that if equality holds, then the second inequality implies that $i=0$, so the first inequality then implies that $n=mk$. However, this would mean that $(x,n)=(ma,mk)=m(a,k)$, contradicting the fact that (x,n) is not a multiple of (a,k). Therefore, we obtain the desired result.

We have now shown that $mk+\lfloor\frac{a+i-1}{a}\rfloor sk>n$, so since $(a-2)\geq(i-1)$ mod $a-1$, $mk+(a-2)+\lfloor\frac{a+i-1}{a}\rfloor sk>(i-1) \mod a+n-1$. Rearranging this inequality, we see that

$$\begin{aligned}(y_0-1)k-n&=mk+\left\lfloor\frac{a-2}{k}\right\rfloor k+\left\lfloor\frac{a+i-1}{a}\right\rfloor sk-n\\&>(i-1)\mod a-(a-2)\mod k-1,\end{aligned}$$

which is the desired result. Therefore, since $(y_0-1)k-n$ is greater than the height of the column at $(y_0-1)a-x$, $(y_0-1)(a,k)-(x,n)$ is not in $S^d_{a,k}$. □

The last two lemmas prove the following result.

Theorem 2.3. If $(x,n)\in S^d_{a,k}$ such that $(x,n)\neq p(a,k)$ for any $p\in\mathbb{N}$, then $\omega((x,n))=\max\left(n+1,m+\lfloor\frac{a-2}{k}\rfloor+1+\lfloor\frac{a+i-1}{a}\rfloor s\right)$.

Now, we wish to complete our characterization of the ω-function in arithmetic Leamer monoids.

2.2. (x,n) **is a multiple of** (a,k)**.** Throughout this subsection, let $(x,n)=m(a,k)$ for some $m\in\mathbb{N}^*$. We wish to show in the following lemmas that $\omega((x,n))=n+1$.

Lemma 2.4. Let $(x,n)\in S^d_{a,k}$ such that $(x,n)\neq m(a,k)$ for some $m\in\mathbb{N}$. If $c\geq mk+1$, then (x,n) divides the sum of any c non-zero elements of $S^d_{a,k}$.

Proof. For $1\leq j\leq c$, let $(x_j,n_j)\in S^d_{a,k}$ be a non-zero element. By Lemma 1.5, this means that there exists $q_j,i_j\in\mathbb{N}$ such that $x_j=q_ja+i_jd$ and $0\leq i_j\leq q_jk$. We now divide the claim into two cases.

First, suppose that some $q_l>\lfloor\frac{a-2}{k}\rfloor$. Then

$$\sum_{j=1}^{c}q_j-m=\sum_{j=1,j\neq l}^{c}q_j+q_l-m\geq mk-m+q_l\geq q_l\geq\left\lfloor\frac{a-2}{k}\right\rfloor+1.$$

Therefore,

$$\begin{aligned}\sum_{j=1}^{c} x_j - x &= \sum_{j=1}^{c}(q_j a + i_j d) - ma \\ &= \left(\sum_{j=1}^{c} q_j - m\right) a + \sum_{j=1}^{c} i_j d \\ &= \left(\left\lfloor \frac{a-2}{k} \right\rfloor + 1 + s\right) a + td\end{aligned}$$

for some $s, t \in \mathbb{N}$. By Theorem 1.6, there is an infinite column at $\left(\left\lfloor \frac{a-2}{k} \right\rfloor + 1\right) a$, so it follows that there is an infinite column at $\left(\left\lfloor \frac{a-2}{k} \right\rfloor + 1\right) a + sa + td$ for any $s, t \in \mathbb{N}$. Since there is an infinite column at $\sum_{j=1}^{c} x_j - x$ and since $\sum_{j=1}^{c} n_j - n \geq c - mk \geq 1$, (x, n) divides $\sum_{j=1}^{c} (x_j, n_j)$.

Now, suppose that $q_j \leq \left\lfloor \frac{a-2}{k} \right\rfloor$ for all $1 \leq j \leq c$. By Theorem 1.6, $1 \leq n_j \leq q_j k - i_j$, so $i_j \leq q_j k - n_j \leq q_j k - 1$. Therefore,

$$0 \leq \sum_{j=1}^{c} i_j \leq \sum_{j=1}^{c} q_j k - c \leq \sum_{j=1}^{c} q_j k - (mk+1) = \left(\sum_{j=1}^{c} q_j - m\right) k - 1,$$

so $\sum_{j=1}^{c} x_j - x = \left(\sum_{j=1}^{c} q_j - m\right) a + \sum_{j=1}^{c} i_j d$ is in $\langle a, a+d, \cdots, a+kd \rangle$ by Lemma 1.5. Additionally, by the same lemma, there exists unique $q', i' \in \mathbb{N}$ such that $\sum_{j=1}^{c} x_j - x = q'a + i'd$ and $0 \leq i' < a$. Let $\lambda \in \mathbb{Z}$ such that $(q', i') - \left(\sum_{j=1}^{c} q_j - m, \sum_{j=1}^{c} i_j\right) = \lambda(d, -a)$. Since $i' < a$ and each $i_j \geq 0$, $\lambda \geq 0$. Therefore, by Theorem 1.6, the height of column at $\sum_{j=1}^{c} x_j - x$ is

$$\begin{aligned}q'k - i' &= \left(\sum_{j=1}^{c} q_j - m + \lambda d\right) k - \left(\sum_{j=1}^{c} i_j - \lambda a\right) \\ &\geq \left(\sum_{j=1}^{c} q_j - m\right) k - \sum_{j=1}^{c} i_j \\ &\geq \sum_{j=1}^{c} n_j - mk,\end{aligned}$$

where the last inequality follows from the fact that $n_j \leq q_j k - i_j$. Therefore, since $c \geq mk+1$, $1 \leq \sum_{j=1}^{c} n_j - n \leq q'k - i'$, so $\sum_{j=1}^{c} (x_j, n_j) - (x, n)$ is in $S_{a,k}^d$. □

Now, we wish to find an element of length $n + 1 = mk + 1$ such that no proper subsum of it is divisible by (x, n).

Lemma 2.5. Let $(x, n) \in S^d_{a,k}$ such that $(x, n) \neq m(a, k)$ for some $m \in \mathbb{N}$. Then $(n+1)(a+x, 1)$ is divisible by (x, n), but no proper subsum is divisible by (x, n).

Proof. By Lemma 2.4, (x, n) divides $(n+1)(a+x, 1)$. Therefore, we only need to show that it does not divide any proper subsum. Since $n(a+x, 1) - (x, n) = (na + (n-1)x, 0)$, and $na + (n-1)x > 0$, the result follows. □

By the definition of the ω-function, the above lemmas imply the next result.

Theorem 2.6. If $(x, n) \in S^d_{a,k}$ such that $(x, n) \neq m(a, k)$ for some $m \in \mathbb{N}$, then $\omega((x, n)) = n + 1$.

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