ω-PRIMALITY IN ARITHMETIC LEAMER MONOIDS

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Abstract. Let Γ be a numerical semigroup. The Leamer monoid S_Γ^s, for $s \in \mathbb{N}\backslash\Gamma$, is the monoid consisting of arithmetic sequences of step size s contained in Γ. In this note, we give a formula for the ω-primality of elements in S_Γ^s when Γ is an numerical semigroup generated by a arithmetic sequence of positive integers.

1. Preliminaries

A numerical monoid S is an additive submonoid of the nonnegative integers $\mathbb{N}_0$ under regular addition such that $|\mathbb{N}_0 - S| < \infty$ ([11] is a good general reference on this subject). A great deal of literature has appeared over the past 15 years which studies the nonunique factorization properties of these monoids (for instance, see [4], [6], and [5] and the references therein). Among the factorization constants studied on these objects is the *ω-primality function* (referred to hereafter as the ω-function), which in some sense measures how far an element $x \in S$ is from being a prime element. A general survey of these results can be found in [16], while the papers [2], [3], and [9] all consider issues related to algorithms for computing specific values of the ω-function. Other papers that touch on this subject in more specific terms are [7], [8], [14], and [17]. In this paper, we pick up on the study begun in [12] of the factorization properties of Leamer monoids, which are constructed using numerical monoids. Leamer monoids first appeared in [10] and were used in that paper to study the Huneke-Wiegand conjecture from commutative algebra. In our current work, we address a particular case of Problem 5.4 in [12] and completely determine the behavior of the ω-function on a Leamer monoid generated by an arithmetic numerical monoid (i.e, a numerical monoid generated by an arithmetic sequence of integers). Our final results are summarized in Theorems 2.3 and 2.6. We find these results of interest for several reasons reasons:

- ω-function calculations can be extremely complex, and an intrictate algorithm for their computation has recently appeared in [9];
- the complete behavior of the ω-function on general commutative cancellative monoids is known in only a few cases (one of which is the numerical monoid $\langle a, b\rangle$ which is proved in [2] and summarized in [16]);
- the complete behavior of the ω-function on the underlying arithmetical numerical monoid (of the Leamer monoid we are considering) is itself unknown.

Before proceeding to our main result, we offer a series of definitions. We begin with a general definition of the ω-function itself.

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Definition 1.1. Let S be a commutative cancellative monoid. For any nonunit $x \in S$, define $\omega(x) = m$ if m is the smallest positive integer such that whenever x divides $x_1 \cdots x_t$, with $x_i \in S$, then there is a set $T \subset \{1, 2, \ldots, t\}$ of indices with $|T| \leq m$ such that x divides $\sum_{i \in T} x_i$. If no such m exists, then set $\omega(x) = \infty$.

When S is clear from the context, we simply write $\omega(n)$. A collection of basic facts concerning the ω-function can be found in [1, Section 2]. Needless to say, an element $x \in S$ is prime if and only if $\omega(x) = 1$. The definition of a Leamer monoid follows.

Definition 1.2. Let Γ be a numerical monoid and $s \in \mathbb{N} \setminus \Gamma$. Set

$$S_\Gamma^s = \{(0,0)\} \cup \{(x,n) : \{x, x+s, x+2x, \ldots, x+ns\} \subset \Gamma\} \subset \mathbb{N}^2.$$

Thus S_Γ^s is the collection of arithmetic sequences of step size s contained in Γ. Under regular addition on $\mathbb{N}^2$, S_Γ^s is a monoid known as a *Leamer monoid*.

As we will be working within $\mathbb{N}^2$ under addition, we remind the reader of the notion of divisibility in $\mathbb{N}^2$. If x and $y \in \mathbb{N}^2$, then we say that x *divides* y if there is a $z \in \mathbb{N}^2$ such that $x + z = y$.

We define the column at $x \in \Gamma$ to be the set $\{(x,n) \in S_\Gamma^s : n \geq 1\}$. We say that the column at x is infinite (resp. finite) if the cardinality of the column at x is infinite (resp. finite). For a finite column, the height of the column is $\max\{n : (x,n) \in S_\Gamma^s\}$ and we define x_f to be the first infinite column in S_Γ^s. The largest positive integer not in Γ is know as the Frobenius number and we denote this as $F(\Gamma)$. Since $S_\Gamma^s \subseteq \mathbb{N}^2$, we can graphically represent S_Γ^s, and we do so below in the case where $\Gamma = \langle 12, 13, 20 \rangle$ with $s = 1$. The red dots in the graph represent irreducible elements of S_Γ^s.

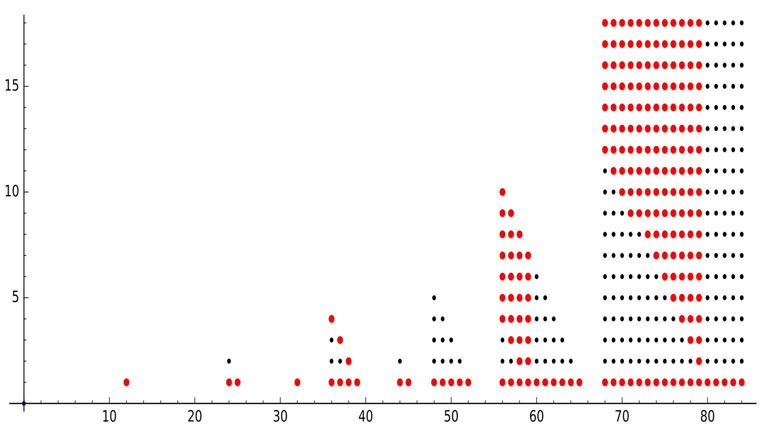

FIGURE 1. The Leamer monoid S_Γ^1 for $\Gamma = \langle 12, 13, 20 \rangle$

The following result from [12, Lemma 2.8] will give us some basic factorization properties of an arbitrary Leamer monoid. Note that $\mathcal{A}(S_\Gamma^s)$ is the set of irreducible elements (or atoms) of S_Γ^s.

Lemma 1.3. (a) For $n \gg 0$, $(x_f, n) \in \mathcal{A}(S_\Gamma^s)$.

(b) The column at every $x > F(\Gamma)$ is infinite.

Suppose that $\omega(n)$ is finite. To find this value, it is often helpful to consider the *bullets* for n. A product of irreducibles $x_1 \cdots x_k$ is said to be a bullet for n if n divides the product $x_1 x_2 \cdots x_k$ but does not divide any proper subproduct. If $\mathrm{bul}(x)$ represents the set of bullets of x, then the following proposition [16, Proposition 2.10] will be key in our coming calculations.

Proposition 1.4. If M is a commutative cancellative monoid and x a nonunit of M, then

$$\omega(x) = \sup\{r \mid x_1 \cdots x_r \in \mathrm{bul}(x) \text{ where each } x_i \text{ is irreducible in } M\}.$$

There has been fairly extensive study of the ω-function on numerical monoids in recent years. Of particular interest is the following result [15, Theorem 3.6], which describes the eventual behavior of the ω-function. If $S = \langle n_1, ..., n_k\rangle$ is a numerical monoid, then for n sufficiently large, $\omega(n)$ is quasilinear with period dividing n_1. In particular, there exists an explicit N_0 such that $\omega(n + n_1) = \omega(n) + 1$ for $n > N_0$. Hence, for sufficiently large n, $\omega(n) = \frac{n}{n_1} + a_0(n)$, where $a_0(n)$ has period dividing n_1.

For the remainder of our work, we focus on numerical monoids generated by arithmetic sequences (a good general reference on this topic is [13]). So let $S = \langle a, a + d, ..., a + kd\rangle$, where $\gcd(a, d)$=1 and $1 \le k < a$.

Lemma 1.5. [6, Lemmas 7 & 8]

1. Let n be a nonnegative integer. Then $n \in S$ if and only if $n = qa + jd$ with $q \in \mathbb{N}$ and $0 \le j \le kq$.
2. If $n = qa + jd$ with $q \in \mathbb{N}$ and $0 \le j \le kq$, then there is a factorization of n in S of length q.
3. Let n be an integer with $n = ua + vd = u'a + v'd$. Then there exists an integer λ such that $(u, v) - (u', v') = \lambda(d, -a)$.
4. If $n = qa + jd$ with $q \in \mathbb{N}$ and $0 \le j < a$, then q is the longest length of factorization of n in S.

We say that a Leamer monoid is *arithmetic* if Γ is an arithmetic numerical semigroup with $k \ge 2$ and s is the difference of the arithmetic sequence. If $\Gamma = \langle a, a + d, \cdots, a + kd\rangle$, then we will write $S^s_\Gamma = S^d_{a,k}$. We offer graphical representations of arithmetic Leamer monoids in Figures 2 and 3. Additionally, the following result tells us more about factorization properties of arithmetic Leamer monoids, which we will use to characterize the ω-function in such monoids.

Theorem 1.6. [12, Lemma 4.3 (a)] Fix an arithmetic Leamer monoid $S^d_{a,k}$, and let $x = ma + id$, where $m, i \in \mathbb{N}$ and $0 \le i < a$. Then $S^d_{a,k}$ has a finite column at x if and only if $m \le \lfloor \frac{a-2}{k} \rfloor$ and $0 \le i \le km - 1$. In this case, the column at x has height $km - i$.

Finally, we offer a lower bound on the ω-function in a general Leamer monoid. Note that we are only considering nonunit elements, i.e. $(x, n) \ne (0, 0)$, so $n \ge 1$ by the definition of a Leamer monoid.

Proposition 1.7. If $(x, n) \in S^s_\Gamma$, then (x, n) has a bullet of length $n + 1$. Hence, $\omega((x, n)) \ge n + 1$ and no element in a Leamer monoid is prime.

Proof. We wish to show that $(n + 1)(x + F(\Gamma), 1)$ is a bullet for (x, n). Since $nx + (n + 1)F(\Gamma) \ge F(\Gamma)$,

$$(n + 1)(x + F(\Gamma), 1) - (x, n) = (nx + (n + 1)F(\Gamma), 1) \in S^s_\Gamma$$

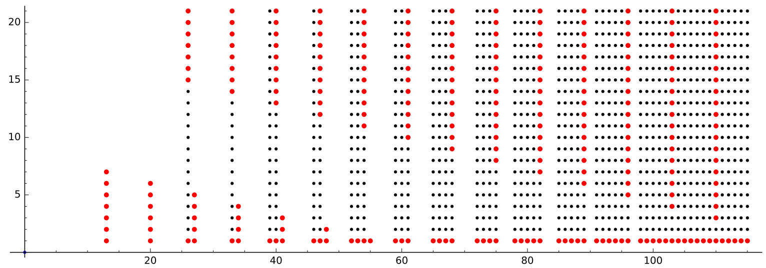

FIGURE 2. The Leamer monoid S_Γ^7 for $\Gamma = \langle 13, 20, 27, 34, 41, 48, 55, 62\rangle$

by Lemma 1.3(b). Additionally,

$$n(x + F(\Gamma), 1) - (x, n) = ((n-1)x + nF(\Gamma), 0) \notin S_\Gamma^s$$

since $(n-1)x + nF(\Gamma) > 0$. Thus, (x, n) divides $(n+1)(x + F(\Gamma), 1)$ but no proper subsum of it, so it is a bullet. The last statement clearly follows. □

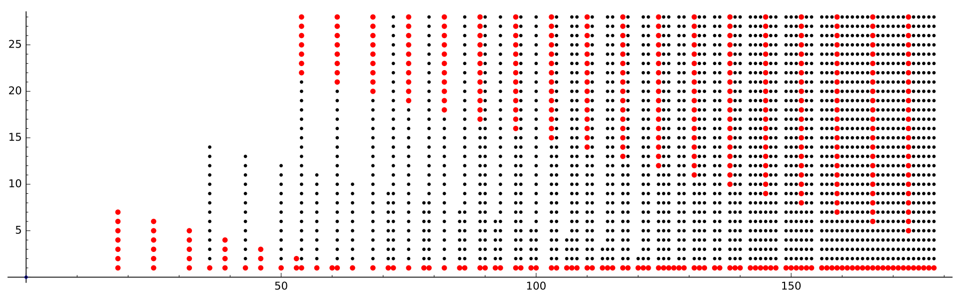

FIGURE 3. The Leamer monoid S_Γ^7 for $\Gamma = \langle 18, 25, 32, 39, 46, 53, 60, 67\rangle$

2. ω-values in arithmetic Leamer monoids

Throughout this section, let $S_{a,k}^d$ be an arithmetic Leamer monoid with $\gcd(a, d) = 1$ and $2 \le k \le d$. In [12], the authors study the factorization properties of arithmetic Leamer monoids. Now, we wish to extend use these results to find the ω-values of all elements in an arithmetic Leamer monoid. We will do so in Theorem 2.1 where we consider the case where (x, n) is not a multiple of (a, k), and then in Theorem 2.6 where consider the case where (x, n) is a multiple of (a, k).

2.1. (x, n) **is not a multiple of** (a, k)**.** We focus here on the case where $(x, n) \in S_{a,k}^d$ such that $(x, n) \ne p(a, k)$ for any $p \in \mathbb{N}$. By Lemma 1.5, we may choose the largest positive integer m such that $x = ma + id$ where $i \in \{0, \cdots, mk\}$. Additionally, let

$$w = \max\left(n + 1, m + \lfloor\frac{a-2}{k}\rfloor + 1 + \lfloor\frac{a+i-1}{a}\rfloor d\right). \tag{1}$$

Lemmas 2.2 and 2.3 will prove the following.

Theorem 2.1. If $(x,n) \in S^d_{a,k}$ such that $(x,n) \neq p(a,k)$ for any $p \in \mathbb{N}$, then $\omega((x,n)) = w$.

For notation purposes, we let $x \mod a$ represent the least residue of x modulo a.

Lemma 2.2. Let $(x,n) \in S^d_{a,k}$ such that $(x,n) \neq p(a,k)$ for any $p \in \mathbb{N}$ and suppose that $c \geq w$. Then (x,n) divides the sum of any c non-zero elements of $S^d_{a,k}$.

Proof. Let $y_0 = m + \lfloor \frac{a-2}{k} \rfloor + 1 + \lfloor \frac{a+i-1}{a} \rfloor d$, and let

$$\begin{aligned} x_0 &= y_0 a - (ma + id) \\ &= \left(m + \left\lfloor \frac{a-2}{k} \right\rfloor + 1 + \left\lfloor \frac{a+i-1}{a} \right\rfloor d \right) a - (ma + id) \\ &= \left(\left\lfloor \frac{a-2}{k} \right\rfloor + 1 \right) a + \left(\left\lfloor \frac{a+i-1}{a} \right\rfloor a - i \right) d. \\ &= \left(\left\lfloor \frac{a-2}{k} \right\rfloor + 1 \right) a + (a + i - 1 - ((a+i-1) \mod a) - i) d \\ &= \left(\left\lfloor \frac{a-2}{k} \right\rfloor + 1 \right) a + (a - 1 - ((i-1) \mod a)) d. \end{aligned}$$

Since $0 \leq a-1-((i-1) \mod a) < a$, there is an infinite column at x_0 by Theorem 1.6, so this also implies that there is an infinite column at $x_0 + sa + td$ for any $s, t \in \mathbb{N}$.

Now, for $1 \leq j \leq c$, let (x_j, n_j) be a non-zero element of $S^d_{a,k}$. Since $x_j \in \langle a, \cdots, a+kd \rangle$, there exists $q_j, i_j \in \mathbb{N}$ such that $x_j = q_j a + i_j d$. Therefore, $\sum_{j=1}^{c} q_j = y_0 + b$ for some $b \in \mathbb{N}$ since $c \geq y_0$ and each q_j is at least 1. As a result, we see that $\sum_{j=1}^{c} x_j - x = \sum_{j=1}^{c} (q_j a + i_j d) - (ma + id) = (y_0 + b)a - (ma + id) + \sum_{j=1}^{c} i_j d = x_0 + ba + \sum_{j=1}^{c} i_j d$ by the definition of x_0. So by our above discussion, there is an infinite column of $S^d_{a,k}$ at $\sum_{j=1}^{c} x_j - x$. Since $c \geq n+1$ and each $n_j \geq 1$, $\sum_{j=1}^{c} n_j - n \geq 1$. Therefore, this shows that $\sum_{j=1}^{c} (x_j, n_j) - (x,n) = \left(\sum_{j=1}^{c} x_j - x, \sum_{j=1}^{c} n_j - n \right)$ is in $S^d_{a,k}$, which completes the proof. □

If $w = n+1$, then by Proposition 1.7 there is a bullet for x of length $n+1$, and hence $\omega((x,n)) = n+1$. We consider the remaining case in the next lemma.

Lemma 2.3. Let $(x,n) \in S^d_{a,k}$ such that $(x,n) \neq p(a,k)$ for any $p \in \mathbb{N}$. If $w = m + \lfloor \frac{a-2}{k} \rfloor + 1 + \lfloor \frac{a+i-1}{a} \rfloor d$, then $w(a,k)$ is a bullet for (x,n).

Proof. Define x_0 and y_0 as they are defined in the proof of Lemma 2.2. Since $w = y_0$, (x,n) divides $y_0(a,k)$ by Lemma 2.2. Now, we wish to show that (x,n) does not divide $(y_0 - 1)(a,k)$.

First, note that

$$(y_0 - 1)a - x = x_0 - a = \left\lfloor \frac{a-2}{k} \right\rfloor a + (a - 1 - ((i-1) \mod a)) \, d,$$

so by Theorem 1.6, there is a finite column at $(y_0-1)a-x$ of height

$$\begin{aligned}&\lfloor\frac{a-2}{k}\rfloor k-(a-1-((i-1) \mod a))\\ =&\lfloor\frac{a-2}{k}\rfloor k-(a-2)-1+((i-1) \mod a)\\ =&(-(a-2) \mod k)+((i-1) \mod a)-1.\end{aligned}$$

Now, we wish to show that $(y_0-1)k-n$ is greater than this height. To do so, we will first show that $mk+\left\lfloor\frac{a+i-1}{a}\right\rfloor dk>n$.

First, suppose that there is an infinite column at $x=ma+id$. Then by Theorem 1.6, $m>\left\lfloor\frac{a-2}{k}\right\rfloor$. Also, since $w=y_0$, $y_0-1\geq n$. Therefore, since $k\geq 2$ by assumption,

$$\begin{aligned}mk+\left\lfloor\frac{a+i-1}{a}\right\rfloor dk&\geq 2m+\left\lfloor\frac{a+i-1}{a}\right\rfloor dk\\ &>m+\left\lfloor\frac{a-2}{k}\right\rfloor+\left\lfloor\frac{a+i-1}{a}\right\rfloor d=y_0-1\geq n.\end{aligned}$$

Now, suppose that there is a finite column at $x=ma+id$. Then the height of this column is $mk-i$ by Theorem 1.6, so $n\leq mk-i\leq mk+\left\lfloor\frac{a+i-1}{a}\right\rfloor dk$. Note that if equality holds, then the second inequality implies that $i=0$, so the first inequality then implies that $n=mk$. However, this would mean that $(x,n)=(ma,mk)=m(a,k)$, contradicting the fact that (x,n) is not a multiple of (a,k). Therefore, we obtain the desired result.

We have now shown that $mk+\left\lfloor\frac{a+i-1}{a}\right\rfloor dk>n$, so since $(a-2)\geq(i-1) \mod a-1$, $mk+(a-2)+\left\lfloor\frac{a+i-1}{a}\right\rfloor dk>(i-1) \mod a+n-1$. Rearranging this inequality, we see that

$$\begin{aligned}(y_0-1)k-n=mk+\left\lfloor\frac{a-2}{k}\right\rfloor k+\left\lfloor\frac{a+i-1}{a}\right\rfloor dk-n\\ >(i-1) \mod a-(a-2) \mod k-1,\end{aligned}$$

which is the desired result. Therefore, since $(y_0-1)k-n$ is greater than the height of the column at $(y_0-1)a-x$, $(y_0-1)(a,k)-(x,n)$ is not in $S^d_{a,k}$. □

2.2. (x,n) is a multiple of (a,k).

Throughout this subsection, let $(x,n)=m(a,k)$ for some $m\in\mathbb{N}^*$. We wish to show in the following lemmas that $\omega((x,n))=n+1$.

Lemma 2.4. Let $(x,n)\in S^d_{a,k}$ such that $(x,n)=m(a,k)$ for some $m\in\mathbb{N}$. If $c\geq mk+1$, then (x,n) divides the sum of any c non-zero elements of $S^d_{a,k}$.

Proof. For $1\leq j\leq c$, let $(x_j,n_j)\in S^d_{a,k}$ be a non-zero element. By Lemma 1.5, this means that there exists $q_j,i_j\in\mathbb{N}$ such that $x_j=q_ja+i_jd$ and $0\leq i_j\leq q_jk$. We now divide the claim into two cases.

First, suppose that some $q_l>\left\lfloor\frac{a-2}{k}\right\rfloor$. Then

$$\sum_{j=1}^{c}q_j-m=\sum_{j=1,j\neq l}^{c}q_j+q_l-m\geq mk-m+q_l\geq q_l\geq\left\lfloor\frac{a-2}{k}\right\rfloor+1.$$

Therefore,

$$\begin{aligned}\sum_{j=1}^{c} x_j - x &= \sum_{j=1}^{c}(q_j a + i_j d) - ma \\ &= \left(\sum_{j=1}^{c} q_j - m\right) a + \sum_{j=1}^{c} i_j d \\ &= \left(\left\lfloor \frac{a-2}{k} \right\rfloor + 1 + s\right) a + td\end{aligned}$$

for some $s, t \in \mathbb{N}$. By Theorem 1.6, there is an infinite column at $\left(\left\lfloor \frac{a-2}{k} \right\rfloor + 1\right) a$, so it follows that there is an infinite column at $\left(\left\lfloor \frac{a-2}{k} \right\rfloor + 1\right) a + sa + td$ for any $s, t \in \mathbb{N}$. Since there is an infinite column at $\sum_{j=1}^{c} x_j - x$ and since $\sum_{j=1}^{c} n_j - n \geq c - mk \geq 1$, (x, n) divides $\sum_{j=1}^{c} (x_j, n_j)$.

Now, suppose that $q_j \leq \left\lfloor \frac{a-2}{k} \right\rfloor$ for all $1 \leq j \leq c$. By Theorem 1.6, $1 \leq n_j \leq q_j k - i_j$, so $i_j \leq q_j k - n_j \leq q_j k - 1$. Therefore,

$$0 \leq \sum_{j=1}^{c} i_j \leq \sum_{j=1}^{c} q_j k - c \leq \sum_{j=1}^{c} q_j k - (mk+1) = \left(\sum_{j=1}^{c} q_j - m\right) k - 1,$$

so $\sum_{j=1}^{c} x_j - x = \left(\sum_{j=1}^{c} q_j - m\right) a + \sum_{j=1}^{c} i_j d$ is in $\langle a, a+d, \cdots, a+kd \rangle$ by Lemma 1.5. Additionally, by the same lemma, there exists unique $q', i' \in \mathbb{N}$ such that $\sum_{j=1}^{c} x_j - x = q'a + i'd$ and $0 \leq i' < a$. Let $\lambda \in \mathbb{Z}$ such that $(q', i') - \left(\sum_{j=1}^{c} q_j - m, \sum_{j=1}^{c} i_j\right) = \lambda(d, -a)$. Since $i' < a$ and each $i_j \geq 0$, $\lambda \geq 0$. Therefore, by Theorem 1.6, the height of column at $\sum_{j=1}^{c} x_j - x$ is

$$\begin{aligned}q'k - i' &= \left(\sum_{j=1}^{c} q_j - m + \lambda d\right) k - \left(\sum_{j=1}^{c} i_j - \lambda a\right) \\ &\geq \left(\sum_{j=1}^{c} q_j - m\right) k - \sum_{j=1}^{c} i_j \\ &\geq \sum_{j=1}^{c} n_j - mk,\end{aligned}$$

where the last inequality follows from the fact that $n_j \leq q_j k - i_j$. Therefore, since $c \geq mk + 1$, $1 \leq \sum_{j=1}^{c} n_j - n \leq q'k - i'$, so $\sum_{j=1}^{c} (x_j, n_j) - (x, n)$ is in $S_{a,k}^d$. □

Now, we wish to find an element of length $n + 1 = mk + 1$ such that no proper subsum of it is divisible by (x, n).

Lemma 2.5. Let $(x,n) \in S^d_{a,k}$ such that $(x,n) = m(a,k)$ for some $m \in \mathbb{N}$. Then $(n+1)(a+x,1)$ is divisible by (x,n), but no proper subsum is divisible by (x,n).

Proof. By Lemma 2.4, (x,n) divides $(n+1)(a+x,1)$. Therefore, we only need to show that it does not divide any proper subsum. Since $n(a+x,1) - (x,n) = (na + (n-1)x, 0)$, and $na + (n-1)x > 0$, the result follows. □

By the definition of the ω-function, the above lemmas imply the next result.

Theorem 2.6. If $(x,n) \in S^d_{a,k}$ such that $(x,n) = m(a,k)$ for some $m \in \mathbb{N}$, then $\omega((x,n)) = n+1$.

We close with a brief example.

Example 2.7. Setting $\Gamma = \langle 13, 20, 27, 34, 41, 48, 55, 62\rangle$, we return to the Leamer monoid of Figure 2. In the language of Theorems 2.1 and 2.6, we have that $a = 13$, $d = 7$, and $k = 7$. If $(x,n) \in S^7_{13,7}$, then from (1), we have that

$$w = \max\left(n+1, m+2+\left\lfloor \frac{12+i}{13} \right\rfloor 7\right).$$

Hence

$$\omega(x) = \begin{cases} \max\left(n+1, m+2+\left\lfloor \frac{12+i}{13} \right\rfloor 7\right) & \text{if} \quad (x,n) \neq p(13,7) \text{ for any } p \in \mathbb{N} \\ n+1 & \text{if} \quad (x,n) = p(13,7) \text{ for some } p \in \mathbb{N}. \end{cases}$$

Notice if x is relatively large with respect to n (i.e., $m > n$), then the last array reduces to

$$\omega(x) = \begin{cases} m+2+\left\lfloor \frac{12+i}{13} \right\rfloor 7 & \text{if} \quad (x,n) \neq p(13,7) \text{ for any } p \in \mathbb{N} \\ n+1 & \text{if} \quad (x,n) = p(13,7) \text{ for some } p \in \mathbb{N}. \end{cases}$$

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