GENERATING IDEALS IN SUBRINGS OF $K[[X]]$
VIA NUMERICAL SEMIGROUPS

SCOTT T. CHAPMAN

Abstract. Let $K$ be a field and $S$ be the numerical semigroup generated by the positive integers $n_1, \ldots, n_k$. We discuss issues involving ideal generation in the subring $K[[X^{n_1}, \ldots, X^{n_k}]] = K[[X;S]]$ of $K[[X]]$. Bounding the generators of a numerical semigroup, and its ideals, requires no more than elementary number theory; we take advantage of this and offer similar bounds on the ideals of $K[[X;S]]$. The closeness of this connection allows us to say much about how the generators of an ideal can be chosen. We will show that the first generator in a minimal generating set of an ideal $I$ of $K[[X;S]]$ can “almost” be chosen at random.

1. Introduction

In an introductory abstract algebra course, a central focus in ring theory is the introduction of polynomial rings. In fact, a major step in conceptualization in this course is the jump from the usual division algorithm in the ring of integers (or $\mathbb{Z}$) to a similar statement for the polynomial ring over a field (or $K[X]$). Once a student masters these ideas, the introduction of formal power series rings, not usually a mainstay of such a course, might be in order. Recall that if $K$ is a field, then the formal power series ring over $K$ is denoted by

$$K[[X]] = \left\{ \sum_{i=0}^{\infty} a_i X^i \mid a_i \in K \right\}$$

with addition and multiplication defined in the usual polynomial manner. For readers unfamiliar with power series rings, Brewer [1] is a good reference.

Why power series rings? While division works nicely via the division algorithm in $K[X]$ (where we assume $K$ to be a field) it works even better in $K[[X]]$, as the following easily proved result indicates.

Proposition 1.1. Let $f$ and $g$ be nonzero elements of $K[[X]]$. Then either $f|g$ or $g|f$.

This paper is based on an unpublished manuscript [4] which deals with similar problems in power series rings of the form $D[[X]]$ where $D$ is a general integral domain.

The author gratefully acknowledges support under an Academic Leave funded by Sam Houston State University.
So once we pass to power series rings, remainders are not necessary - in some sense, division here is perfect. This has extremely deep implications with regards to the ideal theory in $K[[X]]$. While the division algorithm in $K[X]$ implies that $K[X]$ is a principal ideal domain (or PID), Proposition 1.1 implies (with a little work) that $K[[X]]$ is a discrete valuation ring (or DVR). The ramifications of this with respect to the structure of the ideals in $K[[X]]$ is staggering (the interested reader can consult a variety of references to admire the ideal theoretic properties of the DVR $K[[X]]$ - we like the presentation in [11, Chapter V]).

In more general terms, consider this impressive list of easily verified properties of $K[[X]]$.

Proposition 1.2. Let $K$ be a field and $f = \sum_{i=0}^{\infty} a_i X^i \in K[[X]]$.

1. $f$ is a unit of $K[[X]]$ if and only if $a_0 \neq 0$.
2. The only irreducible elements of $K[[X]]$ are $uX$ where $u$ is a unit of $K[[X]]$.
3. If $f$ is a nonunit of $K[[X]]$, then there is a $k \in \mathbb{N}$ and $u$ a unit of $K[[X]]$ such that $f = uX^k$.
4. The ideal $(X) = \{ fX \mid f \in K[[X]] \}$ is the unique maximal ideal of $K[[X]]$.
5. If $I$ is a proper ideal of $K[[X]]$, then $I = (X^k) = \{ fX^k \mid f \in K[[X]] \}$ for some $k \in \mathbb{N}$.
6. If $I$ and $J$ are proper ideals of $K[[X]]$, then either $I \subseteq J$ or $J \subseteq I$.
7. $K[[X]]$ is a principal ideal domain (or PID) and thus a unique factorization domain (or UFD).

In this paper, we play off the above list, particularly (4) and (5), and consider a natural class of subrings of $K[[X]]$. We describe, using fairly elementary methods, exactly how their ideals are generated. To describe this class of subrings, we will require the notion of a numerical semigroup, which is merely a subsemigroup of $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ under addition. If $S$ is any such subsemigroup generated by $n_1, \ldots, n_k$, then set

$$K[[X; S]] = K[[X^{n_1}, \ldots, X^{n_k}]] = \{ \sum_{i=0}^{\infty} a_i X^i \mid a_i = 0 \text{ if } i \not\in S \}. $$

We will refer to $K[[X; S]]$ as a subring of $K[[X]]$ generated by monomials.

Our interest in these subrings lies in the close connection between the ideal theory of the semigroup $S$, and the ideal theory of $K[[X; S]]$. Bounding the generators of a numerical semigroup, and its ideals, requires no more than elementary number theory; we take advantage of this and offer similar bounds on the ideals of $K[[X; S]]$. The closeness of this connection allows us to not only bound the number of generators of an ideal in $K[[X; S]]$, but say much about how the generators can be chosen. We will show that the first generator in a minimal generating set of an ideal $I$ of $K[[X; S]]$ can “almost” be chosen at random.
The paper is constructed as follows. We open in Section 2 with a summary of facts concerning numerical semigroups and their ideals. Simple corollaries to these facts give lower bounds on the numbers of generators required on an ideal $I$ of $K[[X; S]]$. In Section 3, we use a variant of the division algorithm to give an upper bound on the number of generators of $I$. In Section 4, we consider questions related to how the first generator of $I$ can be chosen.

We now describe the terminology, notation, and background that will be required for the discussion in Section 4. A commutative ring $R$ is said to have the $n$-generator property if each finitely generated ideal of $R$ has a basis of $n$ elements. The study of questions involving the choice of generators of ideals in Noetherian rings is not new. Using $J$ to denote the Jacobson radical of a ring $R$ and $I$ any proper ideal of $R$, Heitman [6] and Heitman and Levy [7] define a commutative ring to have the 1-generator property if $R$ has the two-generator property and any element of $I - IJ$ can be arbitrarily chosen as one of two generators of $I$. (Elements of $IJ$ can be eliminated from consideration as generators by a simple application of Nakayama’s Lemma [9, Theorem 78].) A proper ideal $I$ of such a ring $R$ is said to be strongly two-generated if any nonzero element $\alpha$ of $I$ can be chosen as one of two generators of $I$. Strongly two-generated ideals are studied in detail by Lantz and Martin in [10]. Further, define $R$ to have the strong two-generator property if each proper ideal is strongly two-generated. Under the hypothesis that $R$ is semisimple (i.e., $J = (0)$), the $1 \frac{1}{2}$ and strong two-generator properties are equivalent. Clearly if $J \neq (0)$, then $R$ cannot be strongly two-generated, while $R$ strongly two-generated implies that $R$ is $1 \frac{1}{2}$-generated. It is a well-known fact that Dedekind domains possess the strong two-generator property and thus the $1 \frac{1}{2}$-generator property. In a similar manner, a nonzero element $\alpha$ of $R$ is called a strong $n$-generator of $R$ if it can be chosen as one of $n$ generators of every finitely generated ideal in which it is contained.

For our purposes, we will generalize the definitions above. Let $R$ be a ring with the $n$-generator property. We shall call $R$ ($n - \frac{1}{2}$)-generated if any nonzero element of $I - IJ$ can be chosen as one of $n$-generators of $I$, for any proper ideal $I$ of $R$. Further, if $\alpha$ is any nonzero element of $R$, then $\alpha$ is a strong $n$-generator if it can be chosen as one of $n$ generators of every finitely generated ideal in which it is contained. For the domains $K[[X; S]]$, the set of strong $n_1$-generators can be characterized (where $n_1$ is the smallest positive element in $S$) using the properties of the semigroup $S$. Further, we show that the $K[[X; S]]$ are $(n_1 - \frac{1}{2})$-generated, but since $J \neq (0)$, they are not "strongly $n$-generated."

It is important to note that these questions have been studied in a broader setting. In particular, since the domains $K[[X; S]]$ are one dimensional Noetherian local rings, our results are special cases of theorems from Herzog and Kunz [8] and Sally [13].

Before continuing, we set some notation. If $f \in K[[X; S]]$, then the smallest power of $X$ with a nonzero coefficient is called the order of $f$ and
denoted $\partial f$. The first nonzero coefficient in such an element $f$ will be called the *initial term of $f$*. For a proper ideal $I$ of $K[[X; S]]$, define the *order of $I$* to be $\partial I = \inf\{\partial f | f \in I\}$. We note that it is easy to argue that Proposition \ref{prop:1.2} (1) also holds for the rings $K[[X; S]]$, and hence $\partial I > 0$.

2. **Numerical Semigroups, Their Ideals, and a Lower Bound on the Number of Generators**

We begin with a brief summary of facts concerning a numerical semigroup $S$. Both \cite{5} and \cite{12} are good general references on this subject. Using elementary number theory, it is easy to show that there is a finite set of positive integers $n_1, \ldots, n_k$ such that if $s \in S$, then $s = x_1n_1 + \cdots + x_kn_k$ where each $x_i$ is a nonnegative integer. The elements $n_1, \ldots, n_k$ are called a *generating set* for $S$, and to denote this we use the notation

$$S = \langle n_1, \ldots, n_k \rangle = \{x_1n_1 + \cdots + x_kn_k \mid x_i \in \mathbb{N}_0\}.$$  

If the generators $n_1, \ldots, n_k$ are relatively prime, then $S$ is called *primitive*. We shall need three elementary facts concerning numerical semigroups; we offer brief explanations of each.

**Elementary Fact A.** *If $S = \langle n_1, \ldots, n_k \rangle$ is a numerical semigroup, then $S$ is isomorphic to a primitive numerical semigroup $S'$.*

*Proof.* Let $d = \gcd(n_1, \ldots, n_k)$ with $n_i = dn'_i$ for each $i$ and set $S' = \langle n'_1, \ldots, n'_k \rangle$. Define a map $\varphi : S \to S'$ by

$$\varphi(x_1n_1 + \cdots + x_kn_k) = x_1n'_1 + \cdots + x_kn'_k.$$  

That $\varphi$ is a semigroup isomorphism can be easily checked. \qed

**Elementary Fact B.** *A numerical semigroup $S$ has a unique minimal cardinality generating set. The elements of this minimal generating set are pairwise mutually incongruent modulo $n_1$, where $n_1$ is the minimal positive element of $S$. This minimal generating set is also contained in every generating set for $S$.*

*Proof.* Let $y_i = n_1 + i$ for $0 \leq i \leq n_1 - 1$ and set $S_{y_i} = \{s \mid s \in S$ and $s \equiv y_i \pmod{n_1}\}$. Clearly $\bigcup_{i=0}^{n_1-1} S_{y_i} = S$ and set $m_i = \min S_{y_i}$ for $0 \leq i \leq n_1 - 1$. Since each element $s$ in $S_{y_k}$ can be written as $s = m_i + kn_1$ for some $k$, we easily see that $S = \langle m_0, \ldots, m_{n_1-1} \rangle$. By deleting any $m_i$ with $m_i \in \langle m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_{n_1-1} \rangle$, we obtain a generating set whose minimality and uniqueness are straightforward. By construction, these generators are clearly pairwise incongruent modulo $n_1$. The last statement follows as the minimal generators cannot be written in a nontrivial manner as a sum of other elements in the semigroup. \qed

**Elementary Fact C.** *If $S = \langle n_1, \ldots, n_k \rangle$ is a primitive numerical semigroup, then there is a largest element $F(S) \not\in S$ with the property that any $s > F(S)$ is in $S$.***
Proof. Write

\begin{equation}
1 = x_1n_1 + \cdots + x_kn_k
\end{equation}

where each \( x_i \) is an integer. Let \( M > n_1 \max \{|x_1|, \ldots, |x_k|\} \) and set \( z = Mn_1 + \cdots + Mn_k \) in \( S \). By adding \((2.1)\) to \( z \) a total of \( n_1 \) times, we obtain that \( z, z + 1, \ldots, z + n_1 - 1 \) are in \( S \). By repeatedly adding \( n_1 \) to these elements, we obtain that all elements larger than \( z \) are in \( S \). If \( z' \) is the minimal such \( z \), then the hypothesized \( F(S) \) is \( z' - 1 \). \( \square \)

Due to Elementary Fact A we assume that \( S \) is primitive throughout the remainder of this work. The value \( F(S) \) is known as the Frobenius number of \( S \) and its computation remains a matter of current mathematical research. If \( S = \langle a, b \rangle \), then it is well-known that \( F(S) = ab - a - b \) (see [12, Section 1.3]), but for more than 2 generators, no general formula is known (see [12, Section 1.3] for more on Frobenius numbers).

A semigroup ideal of \( S \) is any nonempty subset \( D \) of \( S \) such that \( D \supseteq s + D = \{s + d|d \in D\} \) for each \( s \in S \). A subset \( \{d_1, \ldots, d_t\} \) of \( D \) is said to generate \( D \) as a semigroup ideal (denoted \( D = \ll d_1, \ldots, d_k \gg \)) if for any \( \alpha \in D \) there exists a representation

\begin{equation}
\alpha = \gamma_1n_1 + \gamma_2n_2 + \cdots + \gamma_kn_k + \delta_1d_1 + \delta_2d_2 + \cdots + \delta_td_t
\end{equation}

with \( \{\gamma_i\}_{i=1}^k \) and \( \{\delta_j\}_{j=1}^t \) nonnegative integers and \( \delta_j \neq 0 \) for some \( j \). (To avoid ambiguity, if \( n_i \in D \) for any \( i \) between 1 and \( k \), then we will require that \( n_i \) be listed as a generator of \( D \).) We need a further elementary fact before proceeding.

**Elementary Fact D.** Let \( S = \langle n_1, \ldots, n_k \rangle \) be a numerical semigroup and \( D \) a semigroup ideal of \( S \). Then \( D \) has a unique minimal cardinality generating set which is contained in every generating set for \( D \). Moreover, the elements of this minimal generating set are mutually incongruent modulo \( n_1 \).

The proof of Elementary Fact D is nearly identical to that of Elementary Fact B.

Let \( \mu(D) \) denote the cardinality of a minimal generating set of \( D \). Thus if

\[
\mu(S) = \sup \{\mu(D)|D \text{ is a semigroup ideal of } S\}
\]

then \( \mu(D) \leq n_1 \) and \( \mu(S) \leq n_1 \). We begin by pointing out that the last inequality is actually an equality.

**Proposition 2.1.** If \( S = \langle n_1, \ldots, n_k \rangle \) is a numerical semigroup, then the semigroup ideal

\[
D = \ll F(S) + 1, F(S) + 2, \ldots, F(S) + n_1 \gg
\]

requires \( n_1 \) generators. Thus \( \mu(D) = n_1 \) and it follows that \( \mu(S) = n_1 \).

**Proof.** We use Elementary Fact D and argue that none of the listed generators of \( D \) can be omitted. Note that \( n_1 \leq F(S) + 1 \). Suppose that we omit \( F(S) + i \). In the expansion of \( F(S) + i \) via \((2.2)\), \( \delta_j \neq 0 \) for some \( j \) with
0 \leq j \leq n_1 - 2$. Since $F(S) + i$ is not a multiple of any of the other listed generators, some other coefficient in (2.2) is nonzero, which forces the sum to be bigger than $F(S) + n_1 - 1$, a contradiction. Hence, $F(S) + i$ cannot be omitted. The last two assertions now follow. □

If an ideal $I$ of $K[[X;S]]$ is generated by $f_1, \ldots, f_t$, then we use the notation $I = (f_1, \ldots, f_t)$. Define analogously as above $\mu(I)$ to be the minimum number of generators required for $I$, and

$$\mu(K[[X;S]]) = \sup\{\mu(I) | I \text{ a nonzero ideal of } K[[X;S]]\}.$$ 

We finally define

$$D_I = \{\partial f | f \in I\}.$$ 

As $D_I$ is clearly closed under addition, it is a semigroup ideal of $S$ and moreover $D_I = \ll \partial(f_1), \ldots, \partial(f_t) \gg$. We immediately deduce the following.

**Proposition 2.2.** If $S$, $K$, and $I$ are as above, then

$$\mu(I) \geq \mu(D_I).$$

**Example 2.3.** The inequality in Proposition 2.2 may be strict. Let $D = \mathbb{R}$, $S = \langle 4, 5 \rangle$ and

$$I = (X^4 + X^5, X^9 + X^{12}, X^{13}).$$

Thus $D_I = \ll 4, 9 \gg$ which clearly can be reduced to $D_I = \ll 4 \gg$. By noting that

$$X^{13} = (X^4 + X^5)(-X^{12} + \sum_{i=0}^{\infty} (-1)^{i+1} X^{16+i}) + (X^9 + X^{12})(X^4 + X^8)$$

we can reduce the generating set for $I$ down to $I = (X^4 + X^5, X^9 + X^{12})$. Since $X^4 + X^5$ does not divide $X^9 + X^{12}$ over $K[[X;S]]$, $I$ is not principal, and clearly $\mu(I) = 2 > \mu(D_I)$.

Although not a necessary condition, requiring $I$ to be generated by monomials is sufficient to guarantee that $\mu(I) = \mu(D)$. Corollary 2.4 now follows directly from Propositions 2.1 and 2.2.

**Corollary 2.4.** Let $S$ and $F(S)$ be as in Proposition 2.1. The ideal

$$I = \left(X^{F(S)+1}, X^{F(S)+2}, \ldots, X^{F(S)+n_1}\right)$$

requires $n_1$ generators over $K[[X;S]]$. Thus $\mu(K[[X;S]]) \geq n_1$.

3. Generating Ideals in $K[[X;S]]$

We return for a moment to Proposition 1.1. If $S$ is a nontrivial numerical semigroup (i.e., it requires more than one generator), then Proposition 1.1 clearly fails in $K[[X;S]]$. As an example, if $S = \langle 2, 3 \rangle$, then neither $X^2|X^3$ nor $X^3|X^2$ in $K[[X;S]]$. As we show below, while the proposition fails, in some sense, it “almost” works.
Almost Division Algorithm for $K[[X; S]]$. Let $f$ and $g$ be elements of $K[[X; S]]$ with $\partial g \leq \partial f$. Then there exists $q$ and $r$ in $K[[X; S]]$ such that

$$f = qg + r$$

where $r = 0$ or $\partial g < \partial r \leq \partial g + \mathcal{F}(S)$.

Proof. Over $K[[X]]$ there exists $q_0$ such that

$$f = q_0g.$$ If $q_0 \in K[[X; S]]$ we are done for $r = 0$ satisfies our conclusion. Suppose $q_0 \not\in K[[X; S]]$. Then

$$q_0 = \alpha_0 + \alpha_1X + \ldots + \alpha_{\mathcal{F}(S)}X^{\mathcal{F}(S)} + \sum_{i=\mathcal{F}(S)+1}^{\infty} \alpha_iX^i$$

with at least one of the $\alpha_1, \ldots, \alpha_{\mathcal{F}(S)}$ not zero and the series $\sum_{i=\mathcal{F}(S)+1}^{\infty} a_iX^i$ an element of $K[[X; S]]$. Then

$$f = q_0g = (q_0 - [\alpha_1X + \ldots + \alpha_{\mathcal{F}(S)}X^{\mathcal{F}(S)}])g + [\alpha_1X + \ldots + \alpha_{\mathcal{F}(S)}X^{\mathcal{F}(S)}]g.$$ Setting $q = q_0 - [\alpha_1X + \ldots + \alpha_{\mathcal{F}(S)}X^{\mathcal{F}(S)}]$ and $r = [\alpha_1X + \ldots + \alpha_{\mathcal{F}(S)}X^{\mathcal{F}(S)}]g$ provides the result. \qed

We note that a similar almost division algorithm holds for many polynomial subrings of $K[X]$ (see [3] for details). With the correct conditions on $f$ and $g$, a result similar to Proposition 1.1 can be salvaged.

Corollary 3.1. Suppose $f, g \in K[[X; S]]$ with $\partial g = m$ and $\partial f > m + \mathcal{F}(S)$. Then there exists $q \in K[[X; S]]$ such that $f = qg$.

Proof. Since the division yields no terms in the quotient $q$ of power less than $X^{\mathcal{F}(S)+1}$, the result follows directly from the proof of the almost division algorithm. \qed

Using the machinery thus far developed, we produce an upper bound on $\mu(I)$.

Theorem 3.2. If $I$ is a proper ideal of $K[[X; S]]$ with $\partial I = m$, then $\mu(I) \leq n_1$.

Proof. In $D_I$ choose $d_0, \ldots, d_{n_1-1}$ so that $d_i$ is the smallest element of $D$ with $d_i \equiv i \pmod{n_1}$. Notice that $\ll d_0, \ldots, d_{n_1-1} \gg$ forms a generating set of $D_I$ over $S$ (which may not be minimal). For convenience, we set $\Delta_I = \{d_0, \ldots, d_{n_1-1}\}$ and $d^* = \min \Delta_I$. For each $d_i$, choose $f_i \in I$ such that $\partial f_i = d_i$. We claim that $I = (f_0, \ldots, f_{n_1-1})$ and thus have produced a generating set of $I$ of cardinality $n_1$. Since we can multiply generators by units and not affect $I$, assume without loss of generality that each $f_i$ has an initial term one.
Let \( g \in I \). Suppose that
\[
\partial g = \gamma_0 d_0 + \ldots + \gamma_{n_1-1} d_{n_1-1} + \alpha_1 n_1 + \alpha_k n_k
\]
and suppose further that \( t_1 \) is the smallest integer \( i \) for which \( \gamma_i \neq 0 \). Let
\[
g_1 = g - a_{i} \cdot f_{t_1} \cdot X^\partial g - t_1
\]
where \( a_{i} \partial g \) is the coefficient of \( X^\partial g \) in \( g \). Thus \( g_1 \in I \) and the coefficient of \( X^\partial g \) in \( g_1 \) is zero. Hence either \( \partial g_1 > \partial g \) or \( g_1 = 0 \). Now, if the later is true, \( g \in (f_{d_1}) \). So suppose \( \partial g_1 > \partial g \). Perform the same process on \( g_1 \) and obtain an element \( g_2 \in I \) with \( \partial g_2 > \partial g_1 \) or \( g_2 = 0 \). If \( g_2 = 0 \) then \( g_1 \in (f_{d_2}) \); consequently \( g \in (f_{d_1}, f_{d_2}) \). Repeat this process until it terminates (i.e., \( g_k = 0 \)) or \( \partial g_k > \mathcal{F}(S) + d^* \). In the latter case, by Corollary 3.2, \( g_k = f \cdot d^* \) for some \( f \in K[[X; S]] \). Hence, in either case \( g \in (f_1, \ldots, f_{n_1-1}) \) and \( I = (f_1, \ldots, f_{n_1-1}) \).

Corollary 2.4 and Theorem 3.2 immediately imply the following.

**Corollary 3.3.** Let \( S = \langle n_1, \ldots, n_k \rangle \) be as in Theorem 3.2 and \( I \) be an ideal of \( K[[X; S]] \).

1. \( \mu(K[[X; S]]) = n_1 \).
2. If \( f \in I \) and \( \partial(f) \in \Delta_I \), then \( f \) can be chosen as one of \( n_1 \) generators of \( I \).

As an added bonus to Theorem 3.2, the generators of an ideal \( I \) of \( K[[X; S]] \) can be chosen to be polynomials.

**Theorem 3.4.** Let \( I \) be a proper ideal of \( K[[X; S]] \) with \( \partial I = m \). The generating set selected for \( I \) in Theorem 3.2 can be chosen so that its elements are polynomials of degree no more than \( m + \mathcal{F}(S) \).

**Proof.** Suppose \( \{\beta_1, \ldots, \beta_k\} \) is a generating set for \( I \) as described in Theorem 3.2 with the elements listed in increasing order. By the proof of Theorem 3.2, the \( \beta_i \)'s are chosen such that \( \partial \beta_i \leq m + \mathcal{F}(S) \) for \( 1 \leq i \leq k \). If \( \beta_i \), \( 1 \leq i \leq k \), is not a polynomial consider
\[
\beta_i = \sum_{i=m}^{\mathcal{F}(S)+m} a_i X^i + \sum_{j=\mathcal{F}(S)+m+1}^{\infty} a_j X^j.
\]
Let \( H = \sum_{j=\mathcal{F}(S)+m+1}^{\infty} a_j X^j \). By Corollary 3.1 there exists \( q \in K[[X; S]] \) such that \( \beta_i q = H \). Thus \( H \in I \). Then
\[
\delta_i = \beta_i - H = \sum_{i=m}^{\mathcal{F}(S)+m} a_i X^i \in I
\]
and has the same order as \( \beta_i \). By Theorem 3.2, \( \delta_i \) may be chosen as a generator in place of \( \beta_i \). \( \square \)
In the case that \(S = \langle n, n+1, \ldots, 2n - 1 \rangle\), we can easily characterize the ideals of \(K[[X; S]]\) which requires \(n\) generators.

**Lemma 3.5.** Suppose \(S = \langle n, n+1, \ldots, 2n - 1 \rangle\) and that \(I\) is an ideal of \(K[[X; S]]\) which requires \(n\) generators. Then \(I = \langle X^m, X^{m+1}, \ldots, X^{m+n-1} \rangle\) for some positive integer \(m \geq n\).

**Proof:** Suppose \(\partial I = m\). By Theorem 3.4 there is a set of generators \(G\) of \(I\), each of degree no more than \(m + n - 1\). If \(I\) requires \(n\) generators, then there must be one of each degree \(i\) for \(m \leq i \leq m + n - 1\). Since, by Theorem 3.2, any element of a particular order can be chosen as a generator, repeated reduction of the elements of \(G\) will result in the claimed set of monomials. □

**Example 3.6.** We demonstrate the results of this section with an example. Let \(S = \langle 3, 5 \rangle\) and thus \(\mathcal{F}(S) = 7\). Consider the ideal \(I = \langle h, f, g, \rangle\) generated by

\[
h = X^5 + \sum_{i=13}^{\infty} 2X^i, \quad f = \sum_{i=3}^{\infty} i \cdot X^{2i}, \quad \text{and} \quad g = X^9 + X^{12}
\]

over \(Q[[X; S]]\). Clearly \(\Delta_I = \{5, 6\}\) and \(D_I = \langle 5, 6 \rangle\). Thus by Theorem 3.2 \(I = \langle f, h \rangle\). By Theorem 3.4 we can reduce the generators of \(I\) to polynomials of degree no more than 12. We can achieve this by the following. Let \(h = X^5 \cdot j\) with \(j\) a unit in \(Q[[X; S]]\). Then

\[
f = 3X^6 + 4X^8 + 5X^{10} + 6X^{12} + X^5 \cdot \left( \sum_{i=7}^{\infty} i \cdot X^{2i-5} \right) = 3X^6 + 4X^8 + 5X^{10} + 6X^{12} + h(\frac{1}{2} \sum_{i=7}^{\infty} i \cdot X^{2i-5})
\]

and thus \(\bar{f} = 3X^6 + 4X^8 + 5X^{10} + 6X^{12} \in I\). In a similar manner we can reduce \(h\) to \(\bar{h} = X^5\). By the appropriate manipulation with \(\bar{f}\) and \(\bar{h}\), the ideal can be further reduced to \(I = \langle X^5, X^6 \rangle\).

4. Choosing Generators of Ideals in \(K[[X; S]]\)

We close by considering more specific results concerning which elements of an ideal \(I\) of \(K[[X; S]]\) can be chosen as one of \(n_1\) generators of \(I\).

**Corollary 4.1.** Let \(S = \langle n_1, \ldots, n_k \rangle\) be a numerical monoid \(S\). Any power series of order \(n_i\), \(1 \leq i \leq k\), is a strong \(n_1\)-generator of \(K[[X; S]]\).

**Proof.** By Theorem 3.2 if \(I\) is a proper ideal of \(K[[X; S]]\) and \(\Delta_I = \{d_0, \ldots, d_{n_1-1}\}\), then any power series of order \(d_i\), \(0 \leq i \leq n_1 - 1\), can be chosen as a generator of \(I\). By definition, if \(n_i \in D\) then \(n_i\) is one of the \(d_i\’s\) and thus any power series \(f_{n_i}\) of order \(n_i\), \(1 \leq i \leq j\), can be chosen as a generator of \(I\). It follows that \(f_{n_i}\) is a strong \(n_1\)-generator. □

**Lemma 4.2.** Let \(S = \langle n_1, \ldots, n_k \rangle\) be as in the previous corollary and let

\[
M = \langle X^{n_1}, X^{n_2}, \ldots, X^{n_k} \rangle
\]
be the maximal ideal of $K[[X;S]]$. If $f$ is any nonunit of $K[[X;S]]$ with order $\neq n_i$, $1 \leq i \leq k$, then $f$ cannot be chosen as one of $n_1$-generators of $M$.

**Proof.** Again, we will use the fact that $S = \langle n_1, \ldots, n_k \rangle$ is a minimal set of generators. The generating set $\Delta_M$ is $\{n_1, \ldots, n_k\}$. Note that this is the minimal generating set, and by Elementary Fact [D] must be unique.

Suppose $f$ is an element of $K[[X;S]]$ as in the statement of the lemma. Since $M$ consists of all the nonunits of $K[[X;S]], f \in M$. Suppose $f$ could be chosen as one of $n_1$ generators of $M$. Then $\partial f$ would be a member of the minimal set of generators of $D_I$ over $S$. This contradicts the uniqueness property from Elementary Fact [D] mentioned above. \qed

We combine Corollary 4.1 and Lemma 4.2 to get the following.

**Corollary 4.3.** Let $S = \langle n_1, \ldots, n_k \rangle$ be a minimum set of generators for the monoid $S$. The set of strong $n_1$-generators of $K[[X;S]]$, denoted $\mathcal{L}$, is

$$\mathcal{L} = \{f \in K[[X;S]] | \partial f = n_i, 1 \leq i \leq k\}.$$ \[EQUATION 4.3\]

**Example 4.4.** The idea behind Lemma 4.2 can be illustrated with an example. Let $S = \langle 2, 3 \rangle$ and $M = (X^2, X^3)$ be as in Lemma 4.2. Suppose that $X^4$ could be chosen as a generator of $M$. Then there exists an element $g$ of $M$ such that $(X^4, g) = M$. Since $X^4$ contributes no term of order 2 or 3, $g$ must have order 2 or $X^2 \notin M$. Thus there must exist a linear combination of $X^4$ and $g$ such that

$$h_1 \cdot X^4 + h_2 \cdot g = X^3$$

with $h_1$ and $h_2 \in K[[X;S]]$. Now if $\partial h_2 = 0$, then $\partial(h_1 \cdot X^4 + h_2 \cdot g) = 2$. If $\partial h_2 \geq 2$ then $\partial(h_1 \cdot X^4 + h_2 \cdot g) \geq 4$. Thus no possible linear combination exists.

Note that if $D$ and $C$ are semigroup ideals of $S$, then we can define $D + C = \{\delta + \zeta | \delta \in D \text{ and } \zeta \in C\}$. We are now ready to show for any numerical monoid $S$ that $K[[X;S]]$ is $(n_1 - \frac{1}{2})$ generated.

**Theorem 4.5.** $K[[X;S]]$ has the $(n_1 - \frac{1}{2})$-generator property.

**Proof.** Let $I = \langle f_1, \ldots, f_{n_1} \rangle$ be an ideal of $K[[X;S]]$ which requires $n_1$ generators. We consider only ideals of this type since clearly any element of $I - IJ$ could be chosen as the first of $n_1$ generators of an ideal which requires less than $n_1$ generators.

Let $\Delta_I = \{d_1, \ldots, d_{n_1}\}$ and let $J = (X^{n_1}, \ldots, X^{n_k})$ be the maximal ideal of $K[[X;S]]$. Let $f$ be an element of $I$ with $\partial f \neq d_i$ for any $i$ from 1 to $n_1$. Then

$$f = \varepsilon_1 f_1 + \varepsilon_2 f_2 + \ldots + \varepsilon_{n_1} f_{n_1}$$

with $\varepsilon_i$ a nonunit or zero for all $1 \leq i \leq n_1$ (otherwise $\partial f = d_i$ for some $i$). Since each $\varepsilon_i$ is in $J$, $f \in IJ$ and $f \notin I - IJ$. Thus by Theorem 3.2 any element of $I - IJ$ can be chosen as one of $n_1$ generators of $I$ since $\{\partial g | g \in I - IJ\} = \{d_1, \ldots, d_{n_1}\}. \qed
The result of Theorem 4.5 fails for the associated semigroup rings $K[X; S]$ (see [5] for a general description of these rings). For example, an almost identical argument to the one used in the Example 4.4 shows that $X^4$ cannot be chosen as a generator of $(X^2, X^3)$ in $K[X^2, X^3]$. Since $J = (0)$ for this ring, it clearly does not have the $1 \frac{1}{2}$ generator property [2]. As can be seen, this argument easily generalizes to the element $X^{2n}$ in the ideals $(X^n, X^{n+1}, \ldots, X^{2n-1})$ in the semigroup rings $K[X; S]$ for $S = \langle n, n+1, \ldots, 2n-1 \rangle$.

REFERENCES


Department of Mathematics and Statistics, Sam Houston State University, Huntsville, TX 77341
E-mail address: scott.chapman@shsu.edu
URL: \url{www.shsu.edu/~stc008/}