On Monoids and McNuggets

Scott Chapman

Sam Houston State University

November 16, 2017

References

This talk is based on two papers.

[1] *A Tale of Two Monoids: A Friendly Introduction to the Theory of Non-unique Factorization*, Mathematics Magazine **87**(2014), 163–173..

[2] *Factorization in the Chicken McNugget Monoid*, preprint (with Chris O'Neill).

A more general and technical version of this material can be found in:

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Notation

Let

$$\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

represent the integers

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

represent the natural numbers and

$$\mathbb{N}_0 = \{0, 1, 2, \ldots\}$$

represent the natural numbers adjoin 0.

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Day 1 in Number Theory

On day 1 in Number Theory you learn 2 things.

Definition

A positive integer $p > 1$ is prime if its only positive factors are 1 and p;

or

A positive integer $p > 1$ is *prime* if whenever $p \mid xy$ then either $p \mid x$ or $p \mid y$.

The Fundamental Theorem of Arithmetic

Every integer $x > 1$ can be factored uniquely up to order as a product of prime integers.

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The Fundamental Theorem of Arithmetic

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What does "up to order" mean?

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$$30 = 2 \cdot 3 \cdot 5 = 2 \cdot 5 \cdot 3 = 5 \cdot 2 \cdot 3 = 5 \cdot 3 \cdot 2 = 3 \cdot 2 \cdot 5 = 3 \cdot 5 \cdot 2$$

The Fundamental Theorem of Arithmetic

Given a positive integer $x > 1$ there is a unique list of prime integers $p_1, p_2, \ldots, p_k$ with $p_1 \le p_2 \le \cdots \le p_{k-1} \le p_k$ such that

$$x = p_1 \cdot p_2 \cdots p_k.$$

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Congruence Relations

In the first part of the talk, we will use the simple congruence relation on $\mathbb{Z}$ defined by

$$a \equiv b \pmod{n}$$

if and only if

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The Sequences

Let's consider two simple arithmetic sequences:

$$1,\ 5,\ 9,\ 13,\ 17,\ 21,\ 25,\ 29,\ 33,\ 37,\ 41,\ 45,\ 49\ \ldots$$

and

$$4,\ 10,\ 16,\ 22,\ 28,\ 34,\ 40,\ 46,\ 52,\ 58,\ 64,\ 70,\ 76,\ 82\ \ldots$$

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Background

$1, \ 5, \ 9, \ 13, \ 17, \ 21, \ 25, \ 29, \ 33, \ 37, \ 41, \ 45, \ 49 \ \ldots =$

$$\{1 + 4k \mid k \in \mathbb{N}_0\} =$$

$$1 + 4\mathbb{N}_0$$

is known as **The Hilbert Monoid**.

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Motivation

David Hilbert (1912)

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Hilbert's Argument

In $1 + 4\mathbb{N}_0$ we have

$$21 \cdot 33 = 9 \cdot 77$$

$$(3 \cdot 7) \cdot (3 \cdot 11) = (3 \cdot 3) \cdot (7 \cdot 11)$$

and clearly 9, 21, 33 and 77 cannot be factored in $1 + 4\mathbb{N}_0$. But notice that 9, 21, 33 and 77 are not **prime** in the usual sense of the definition in $\mathbb{Z}$.

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Don't Forget the Second Sequence

In

$$4,\ 10,\ 16,\ 22,\ 28,\ 34,\ 40,\ 46,\ 52,\ 58,\ 64,\ 70,\ 76,\ 82\ \ldots$$

$$= 4 + 6\mathbb{N}_0$$

we have

$$70 \cdot 22 = 154 \cdot 10$$

$$(2 \cdot 5 \cdot 7) \cdot (2 \cdot 11) = (2 \cdot 7 \cdot 11) \cdot (2 \cdot 5)$$

and clearly 70, 22, 154 and 10 cannot be factored in $4 + 6\mathbb{N}_0$. By appending 1 to this sequence we obtain a monoid $\mathbf{M} = 4 + 6\mathbb{N}_0 \cup \{1\}$ called **Meyerson's Monoid**.

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Basic Observation: Both of these examples are extremely elementary in nature.

Big Goal: To convince you that factorization of elements into irreducibles in the second example is far more complicated similar factorizations in the first.

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An Example in Almost Every Basic Abstract Algebra Textbook

Let $D = \mathbb{Z}[\sqrt{-5}]$.

In D

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$

represents a nonunique factorization into products of irreducibles in D.

To fully understand this, a student must understand *units* and *norms* in D. The previous examples avoid this problem.

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Basic Notation

Let M be a commutative cancellative monoid written multiplicatively with identity element 1 and associated group of units $M^\times$. Set $M^* = M \backslash M^\times$.

We use the usual conventions involving divisibility:

$$x \mid y \text{ in } M \Longleftrightarrow xz = y \text{ for some } z \in M.$$

If $x \mid y$ and $y \mid x$ in M, then x and y are *associates*.

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More Basic Notation

Call $x \in M^*$

(1) *prime* if whenever $x \mid yz$ for x, y, and z in M, then either $x \mid y$ or $x \mid z$.

(2) *irreducible (or an atom)* if whenever $x = yz$ for x, y, and z in M, then either $y \in M^\times$ or $z \in M^\times$.

As usual,

$$x \text{ prime in } M \implies x \text{ irreducible in } M$$

but not conversely.

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Yet More Notation

Set

$$\mathcal{A}(M) = \text{ the set of irreducibles of } M$$

If $M^* = \langle \mathcal{A}(M) \rangle$, then M is called *atomic*.

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A Note on Atomic Integral Domains

Notice that not all integral domains are atomic. Consider the domain

$$\mathbb{Z} + X\mathbb{Q}[X] = \{f(X) \in \mathbb{Q}[X] \mid f(0) \in \mathbb{Z}\}.$$

In this case, X cannot be factored as a product of irreducibles.

$$X = 2 \cdot (\frac{1}{2}X) = 2 \cdot 3 \cdot (\frac{1}{6}X) = \cdots$$

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How Things Factor in $1 + 4\mathbb{N}_0$

Lemma for Homework!

The element x is irreducible in $1 + 4\mathbb{N}_0$ if and only if x is either

1. *p where p is a prime and $p \equiv 1 \pmod 4$, or*
2. *$p_1 p_2$ where p_1 and p_2 are primes congruent to 3 modulo 4.*

Moreover, x is prime if and only if it is of type 1.

Corollary

Let $x \in 1 + 4\mathbb{N}_0$. If

$$x = \alpha_1 \cdots \alpha_s = \beta_1 \cdots \beta_t$$

for α_i and β_j in $\mathcal{A}(1 + 4\mathbb{N}_0)$, then $s = t$.

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An Example to Illustrate the Last Two Results

Let's Factor

$$141,851,281 = 4 \times (35,462,820) + 1 \in 1 + 4\mathbb{N}_0$$

Now

$$141,851,281 = 13 \times 17 \times 11 \times 23 \times 43 \times 59$$

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$$141{,}851{,}281 = \underbrace{13 \times 17}_{p \equiv 1 \pmod 4} \times \underbrace{11 \times 23 \times 43 \times 59}_{p \equiv 3 \pmod 4}$$

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Conclusion of $1 + 4\mathbb{N}_0$

In general, a monoid with this property, i.e.,

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for α_i and β_j in $\mathcal{A}(M)$, then $s = t$,

is called *half-factorial.*

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A general comment

Theorem

There is a map

$$\phi : \mathbb{Z}[\sqrt{-5}] \to 1 + 4\mathbb{N}_0$$

which preserves lengths of factorizations into products of irreducibles. Hence, $\mathbb{Z}[\sqrt{-5}]$ *is half-factorial.*

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$$\phi : \mathbb{Z}[\sqrt{-5}] \to 1 + 4\mathbb{N}_0$$

which preserves lengths of factorizations into products of irreducibles. Hence, $\mathbb{Z}[\sqrt{-5}]$ *is half-factorial.*

Factoring Things in $\mathbf{M}$

Notice that Meyerson's Monoid does not satisfy the half-factorial property:

$$10,000 = 10^4 = 250 \times 10 \times 4$$

Note that $250 = 2 \times 5^3 \equiv 4 \pmod 6$ is irreducible in $\mathbf{M}$.

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Let's consider a second example where we can more easily explore what is happening:

$$154^3 = 1372 \times 2662$$

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A Membership Criteria for **M**

The interplay between atoms exactly divisible by 2 and atoms exactly divisible by 4 is the key to determining how elements factor in **M**.

To determine the atoms of **M**, we begin with a key observation.

Lemma for Homework!

a) *If* $x \neq 1$ *is in* $\mathbb{N}$ *then* x *is in* **M** *if and only if* $x \equiv 0 \pmod{2}$ *and* $x \equiv 1 \pmod{3}$.

b) *If* $x \neq 1$ *is in* **M** *where* $x = 2^k w$ *with* w *odd and* $k \geq 3$, *then* x *is not an atom of* **M**.

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The Atoms of $\mathbf{M}$

Corollary

An element $x \in \mathbf{M}$ is irreducible if and only if it is of one of the following two types:

A) *atoms of the form $2r$ where r is an odd number congruent to $2 \pmod 3$. Such an atom must have at least one prime factor congruent to $2 \pmod 3$.*

B) *atoms of the form $4s$ where s is a product of odd primes all of which are congruent to $1 \pmod 3$.*

Moreover, $\mathbf{M}$ has no prime elements

The General Framework

Let M be an atomic monoid. Define for $x \in M^*$

$L(x) =$

the longest length of an irreducible factorization of x in M,

$l(x) =$

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and

$$\rho(x) = \frac{L(x)}{l(x)}$$

to be their quotient. $\rho(x)$ is called the *elasticity* of x.

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If there exists an $x \in M^*$ such that $\rho(M) = \rho(x) = \frac{L(x)}{l(x)}$, then we say that the elasticity of M is *accepted*.

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Elasticity in $\mathbf{M}$

Proposition (W. Meyerson, 2003)

- *If* $x \in \mathbf{M}$*, then* $1 \leq \rho(x) < 2$.
- $\rho(\mathbf{M}) = 2$

Sketch of Proof

Sketch of Proof: Let $x = 2^k \cdot w \in \mathbf{M} \neq 1$ and set

$$f(x) = k.$$

Notice that for $x, y \in \mathbf{M}$ we have

$$f(xy) = f(x) + f(y).$$

Suppose in $\mathbf{M}$ that $x = \alpha_1 \cdots \alpha_s = \beta_1 \cdots \beta_t$ where $s \geq t$ and each α_i and β_j is irreducible in $\mathbf{M}$.

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Sketch of Proof

We have

$$f(x) = f(\alpha_1) + \cdots + f(\alpha_s) = f(\beta_1) + \cdots + f(\beta_t)$$

and hence

$$s \cdot 1 \leq f(\alpha_1) + \cdots + f(\alpha_s) = f(\beta_1) + \cdots + f(\beta_t) \leq t \cdot 2.$$

This implies that $\frac{s}{t} \leq \frac{2}{1} = 2$, establishing the second part of the Theorem.

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Sketch of Proof

Now, consider the atoms $x = 2^2 \cdot 7$ and $y = 2 \cdot 7 \cdot 5$. For any positive integer c, set

$$t_c = (2 \cdot 7 \cdot 5)^{2c+1}.$$

Notice that it can also be written as

$$t_c = (2^2 \cdot 7)^c (2 \cdot 7^{c+1} \cdot 5^{2c+1}).$$

A straightforward computation shows that

$$\rho(x) \geq \frac{2c+1}{c+1}.$$

Since $\lim_{c \to \infty} \frac{2c+1}{c+1} = 2$, it follows that $\rho(\mathbf{M}) = 2$.

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Suppose $x \in \mathbf{M}$ has $\rho(x) = 2$.

Suppose further that $x = \alpha_1 \cdots \alpha_s = \beta_1 \cdots \beta_t$ where $s \geq t$.

Then $s = 2t$ and each β_i is an atom of type B and each α_j is an atom of type A.

By previous Lemma, when viewed as a product of type B atoms, x has no prime factor congruent to 2 modulo 3.

When viewed as a product of type A atoms, it has "many" prime factors congruent to 2 modulo 3.

This yields a contradiction.

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Conclusion

THE PUNCH LINE

M is an atomic monoid with elasticity 2 and the elasticity is **not accepted**.

McNuggets

Every day, 34 million Chicken McNuggets are sold worldwide. At most McDonalds locations in the United States today, Chicken McNuggets are sold in packs of 4, 6, 10, 20, 40, and 50 pieces. However, shortly after their introduction in 1979 they were sold in packs of 6, 9, and 20. The following problem spawned from the use of these latter three numbers.

The Chicken McNugget Problem

What numbers of Chicken McNuggets can be ordered using only packs with 6, 9, or 20 pieces?

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What are McNugget Numbers?

Positive integers satisfying the Chicken McNugget Problem are now known as *McNugget numbers*.

In particular, if n is a McNugget number, then there is an ordered triple (a, b, c) of nonnegative integers such that

$$6a + 9b + 20c = n. \tag{1}$$

We will call (a, b, c) a *McNugget expansion* of n. Since both $(3, 0, 0)$ and $(0, 2, 0)$ are McNugget expansions of 18, it is clear that McNugget expansions are not unique. This phenomenon will be the central focus of the remainder of this article.

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McNugget Table

#	(a, b, c)	#	(a, b, c)	#	(a, b, c)
0	(0, 0, 0)	17	NONE	34	NONE
1	NONE	**18**	(3, 0, 0) (0, 2, 0)	**35**	(1, 1, 1)
2	NONE	19	NONE	**36**	(0, 4, 0) (3, 2, 0) (6, 0, 0)
3	NONE	**20**	(0, 0, 1)	37	NONE
4	NONE	**21**	(2, 1, 0)	**38**	(0, 2, 1) (3, 0, 1)
5	NONE	22	NONE	**39**	(2, 3, 0) (5, 1, 0)
6	(1, 0, 0)	23	NONE	**40**	(0, 0, 2)
7	NONE	**24**	(4, 0, 0) (1, 2, 0)	**41**	(2, 1, 1)
8	NONE	25	NONE	**42**	(1, 4, 0) (4, 2, 0) (7, 0, 0)
9	(0, 1, 0)	**26**	(1, 0, 1)	43	NONE
10	NONE	**27**	(0, 3, 0) (3, 1, 0)	**44**	(1, 2, 1) (4, 0, 1)
11	NONE	28	NONE	**45**	(0, 5, 0) (3, 3, 0) (6, 1, 0)
12	(2, 0, 0)	**29**	(0, 1, 1)	**46**	(1, 0, 2)
13	NONE	**30**	(5, 0, 0) (2, 2, 0)	**47**	(0, 3, 1) (3, 1, 1)
14	NONE	31	NONE	**48**	(2, 4, 0) (5, 2, 0) (8, 0, 0)
15	(1, 1, 0)	**32**	(2, 0, 1)	**49**	(0, 1, 2)
16	NONE	**33**	(1, 3, 0) (4, 1, 0)	**50**	(2, 2, 1) (5, 0, 1)

43!

What happens with larger values? The Table has already verified that 44, 45, 46, 47, 48, and 49 are McNugget numbers. Hence, we have a sequence of 6 consecutive McNugget numbers, and by repeatedly adding 6 to these values, we obtain the following.

Proposition

Any $x > 43$ is a McNugget number.

Thus, 43 is the largest number of McNuggets that cannot be ordered with packs of 6, 9, and 20.

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A Diversion into Generalization

Given a set of k objects with predetermined values $n_1, n_2, \ldots, n_k$, what possible values of n can be had from combinations of these objects? Thus, if a value of n can be obtained, then there is an ordered k-tuple of nonnegative integers $(x_1, \ldots, x_k)$ which satisfy the linear diophatine equation

$$n = x_1 n_1 + x_2 n_2 + \cdots + x_k n_k. \tag{2}$$

We view this in a more algebraic manner. Given integers $n_1, \ldots, n_k > 0$, set

$$\langle n_1, \ldots, n_k \rangle = \{x_1 n_1 + \cdots + x_k n_k \mid x_1, \ldots, x_k \in \mathbb{N}_0\}.$$

Monoids of nonnegative integers under addition, like the one above, are known as *numerical monoids*, and $n_1, \ldots, n_k$ are called *generators*. We will call the numerical monoid $\langle 6, 9, 20 \rangle$ the *Chicken McNugget monoid*.

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A Question

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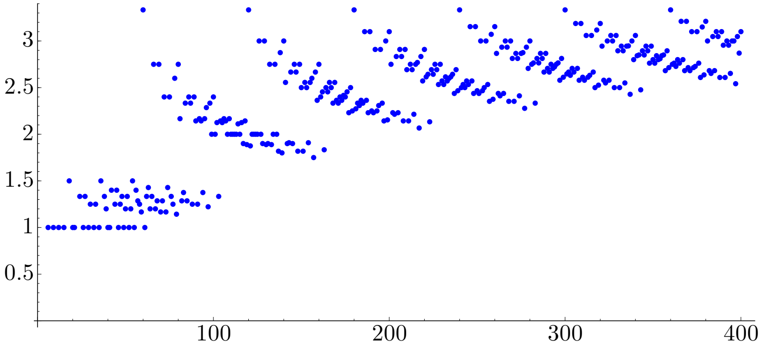

Figure: A plot depicting the elasticity function $\rho(n)$

A General Result

Proposition

Given $\langle n_1, \ldots, n_k \rangle$, *then* $\rho(\langle n_1, \ldots, n_k \rangle) = \frac{n_k}{n_1}$

Proof.

Let $n \in \langle n_1, \ldots, n_k \rangle$ and suppose $n = x_1 n_1 + \; + x_k a_k$. Then

$$\frac{n}{n_k} = \frac{n_1}{n_k} x_1 + \cdots + \frac{n_k}{n_k} x_k \le x_1 + \cdots + x_k \le \frac{n_1}{n_1} x_1 + \cdots + \frac{n_k}{n_1} x_k = \frac{n}{n_1}.$$

Thus $L(n) \le \frac{n}{n_1}$ and $l(n) \ge \frac{n}{n_k}$ for all $n \in \langle n_1, \ldots, n_k \rangle$, from which $\rho(\langle n_1, \ldots, n_k \rangle) \le \frac{n_k}{n_1}$. Also, $\rho(\langle n_1, \ldots, n_k \rangle) \ge \rho(n_1 n_k) = \frac{n_k}{n_1}$, so we have equality $\square$

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Thus . . .

Corollary

The elasticity of the Chicken McNugget Monoid is $\frac{20}{6} = \frac{10}{3}$.

Question

(HARDER!) Given $n \in \langle n_1, \ldots, n_k \rangle$, what are $L(n)$ and $l(n)$?

Thus ...

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$L(x)$

Theorem

For each x in the McNugget monoid, $L(x+6) = L(x) + 1$*. In particular, if we write* $x = 6q + r$ *for* $q, r \in \mathbb{N}$ *and* $r < 6$*, then*

$$L(x) = \begin{cases} q & \text{if } r = 0 \text{ or } 3 \\ q - 5 & \text{if } r = 1 \\ q - 2 & \text{if } r = 2 \text{ or } 5 \\ q - 4 & \text{if } r = 4 \end{cases}$$

for each x in the McNugget monoid.

$l(x)$

Theorem

For each x in the McNugget Monoid, $l(x+20) = l(x) + 1$. In particular, if we write $x = 20q + r$ for $q, r \in \mathbb{N}$ and $r < 20$, then

$$l(x) = \begin{cases} q & \text{if } r = 0 \\ q+1 & \text{if } r = 6, 9 \\ q+2 & \text{if } r = 1, 4, 7, 12, 15, 18 \\ q+3 & \text{if } r = 2, 5, 10, 13, 16 \\ q+4 & \text{if } r = 8, 11, 14, 19 \\ q+5 & \text{if } r = 3, 17 \end{cases}$$

for each x in the McNugget Monoid.