OTHER-HALF-FACTORIALITY IN COMMUTATIVE MONOIDS AND INTEGRAL DOMAINS

SCOTT T. CHAPMAN, JIM COYKENDALL, FELIX GOTTI, AND WILLIAM W. SMITH

Abstract. An atomic monoid M is called an other-half-factorial monoid if for each non-invertible element $x \in M$ no two distinct factorizations of x have the same length. The notion of other-half-factoriality was introduced by Coykendall and Smith in 2011 as a dual of the well-studied notion of half-factoriality. They proved that in the setting of integral domains, other-half-factoriality can be taken as an alternative definition of a UFD. However, being an other-half-factorial monoid is in general weaker than being a factorial monoid (i.e., a unique factorization monoid). Here we further investigate other-half-factoriality. First, we offer two characterizations of an other-half-factorial monoid. Then we study the connection between other-half-factoriality and purely long (resp., purely short) irreducibles, which are irreducible elements that appear in the longer (resp., shorter) part of any unbalanced factorization relation. Finally, we prove that an integral domain cannot contain both purely short and a purely long irreducibles, and we construct a Dedekind domain containing purely long (resp., purely short) irreducibles but not purely short (resp., purely long) irreducibles.

1. Introduction

A monoid M is called half-factorial if for all non-invertible $x \in M$, any two factorizations of x have the same length. In contrast to this, we say that M is other-half-factorial if for all non-invertible $x \in M$, any two distinct factorizations of x have different lengths. An integral domain is called half-factorial if its multiplicative monoid is half-factorial. Half-factorial monoids and domains have been investigated during the last six decades in connection with algebraic number theory and commutative algebra (see [8,9,15,18,23,37] and references there in). The term "half-factorial" was coined by A. Zaks in [37]. On the other hand, other-half-factorial monoids were introduced in 2011 by the second and the fourth authors [20]. As their main result, they proved that UFDs can be characterized as integral domains whose multiplicative monoids are other-half-factorial. Recently, other-half-factorial monoids have been classified in the class of torsion-free rank-1 monoids [31] and in the class of submonoids of finite-rank free monoids [30].

Here we offer a deeper investigation of other-half-factoriality in atomic monoids and integral domains as well as some connections between other-half-factoriality and the

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existence of certain extremal irreducible elements, which when introduced were called purely long and purely short irreducibles [20]. We say that a monoid satisfies the PLS property if it contains both purely short and purely long irreducibles. Every other-half-factorial monoid satisfies the PLS property, and here we determine classes of small-rank monoids where every monoid satisfying the PLS property is other-half-factorial. We will also establish that the multiplicative monoid of an atomic domain never satisfies the PLS property. As a result, we will rediscover that the multiplicative monoid of an integral domain is other-half-factorial if and only if the domain is a UFD, which was the main result in [20].

In Section 3, which is the first section of content, we offer two characterizations of other-half-factorial monoids. The first such characterization is given in terms of the integral independence of the set of irreducibles and the set of irreducibles somehow shifted. The second characterization states that a non-UFM monoid is other-half-factorial if and only if the kernel congruence of its factorization homomorphism is nontrivial and can be generated by a single factorization relation. This second characterization will allow us to recover [20, Proposition 2.9].

In Section 4 we delve into the study of purely long and purely short irreducibles. For an element x of a monoid M, a pair of factorizations (z_1, z_2) of x is called irredundant if they have no irreducibles in common and is called unbalanced if $|z_1| \neq |z_2|$. An irreducible a of M is called purely long (resp., purely short) provided that for any pair of irredundant and unbalanced factorizations of the same element, the longer (resp., shorter) factorization contains a. We prove that the set of purely long (and purely short) irreducibles of an atomic monoid is finite, and we use this result to decompose any atomic monoid as a direct sum of a half-factorial monoid and an other-half-factorial monoid.

Section 5 is devoted to the study of other-half-factoriality in connection with the PLS property on the class consisting of finite-rank atomic monoids. Observe that this class comprises all finitely generated monoids, all additive submonoids of $\mathbb{Z}^n$, and a large class of Krull monoids. We start by counting the number of non-associated irreducibles of a finite-rank other-half-factorial monoid. Then we show that for monoids of rank at most 2, being an other-half-factorial monoid is equivalent to satisfying the PLS property. We conclude the section by offering further characterizations of other-half-factoriality for rank-1 atomic monoids.

In Section 6 we investigate the existence of purely long and purely short irreducibles in the setting of integral domains, arriving to the surprising fact that an integral domain cannot simultaneously contain a purely long irreducible and a purely short irreducible. As a consequence, we rediscover the main result of [20], that the multiplicative monoid of an integral domain is other-half-factorial if and only if the domain is a UFD (a shorter proof of this result was later given in [1, Theorem 2.3]). We conclude exhibiting explicit examples of Dedekind domains containing purely long (resp., purely short) irreducibles, but not purely short (resp., purely long) irreducibles.

2. Fundamentals

2.1. **General Notation.** Throughout this paper, we let $\mathbb{N}$ denote the set of positive integers, and we set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $a, b \in \mathbb{Z}$ with $a \le b$, we let $[\![a, b]\!]$ be the discrete interval from a to b, that is, $[\![a, b]\!] = \{n \in \mathbb{Z} : a \le n \le b\}$. In addition, for $S \subseteq \mathbb{R}$ and $r \in \mathbb{R}$, we set $S_{\le r} := \{s \in S : s \le r\}$ and, with similar meaning, we use the symbols $S_{\ge r}$, $S_{<r}$, and $S_{>r}$. Unless we specify otherwise, when we label elements in a certain set by $s_i, s_{i+1} \dots, s_j$, we always assume that $i, j \in \mathbb{N}_0$ and that $i \le j$.

2.2. **Commutative Monoids.** We tacitly assume that each monoid (i.e. a semigroup with an identity element) we treat here is cancellative and commutative. As all monoids we shall be dealing with are commutative, we will use additive notation unless otherwise specified. If M is a monoid, then we let $M^\bullet$ denote the set $M \setminus \{0\}$, and $\mathscr{U}(M)$ denote the group consisting of all the units (i.e., invertible elements) of M. We say that M is *reduced* if $\mathscr{U}(M) = \{0\}$.

For the monoid M there exist an abelian group $\text{gp}(M)$ and a monoid homomorphism $\iota \colon M \to \text{gp}(M)$ such that any monoid homomorphism $M \to G$, where G is an abelian group, uniquely factors through ι. The group $\text{gp}(M)$, which is unique up to isomorphism, is called the *Grothendieck group*[1] of M. The monoid M is *torsion-free* if $nx = ny$ for some $n \in \mathbb{N}$ and $x, y \in M$ implies that $x = y$. A monoid is torsion-free if and only if its Grothendieck group is torsion-free (see [7, Section 2.A]). If M is torsion-free, then the *rank* of M, denoted by $\text{rank}(M)$, is the rank of the $\mathbb{Z}$-module $\text{gp}(M)$, that is, the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \text{gp}(M)$.

An equivalence relation ρ on the monoid M is called a *congruence* provided that it is compatible with the operation of M, that is, for all $x, y, z \in M$ the inclusion $(y, z) \in \rho$ implies that $(x + y, x + z) \in \rho$. The elements of a congruence are called *relations*. Let ρ be a congruence. Clearly, the set M/ρ of congruence classes (i.e., the equivalence classes) naturally turns into a commutative semigroup with identity (it may not be cancellative). The subset $\{(x, x) : x \in M\}$ of $M \times M$ is the smallest congruence of M, and is called the *trivial* (or *diagonal*) congruence. Every relation in the trivial congruence is called *diagonal*, while $(0, 0)$ is called the *trivial* relation. We say that $\sigma \subseteq M \times M$ generates the congruence ρ provided that ρ is the smallest (under inclusion) congruence on M containing σ. A congruence on M is *cyclic* if it can be generated by one element.

For $x, y \in M$, we say that y *divides* x *in* M and write $y \mid_M x$ provided that $x = y + y'$ for some $y' \in M$. If $x \mid_M y$ and $y \mid_M x$, then x and y are said to be *associated* elements (or *associates*) and, in this case, we write $x \sim y$. Being associates determines a congruence on M, and $M_{\text{red}} := M/\sim$ is called the *reduced monoid* of M. When M is reduced, we identify M_{red} with M. For $S \subseteq M$, we let $\langle S \rangle$ denote the smallest (under inclusion) submonoid of M containing S, and we say that S *generates* M if $M = \langle S \rangle$. An element $a \in M \setminus \mathscr{U}(M)$ is an *irreducible* (or an *atom*) if for each pair of elements $u, v \in M$

[1] The Grothendieck group of a monoid is often called the difference or the quotient group depending on whether the monoid is written additively or multiplicatively.

such that $a = u + v$ either $u \in \mathscr{U}(M)$ or $v \in \mathscr{U}(M)$. We let $\mathscr{A}(M)$ denote the set of irreducibles of M. The monoid M is called *atomic* if every element in $M \setminus \mathscr{U}(M)$ can be written as a sum of atoms. Clearly, M is atomic if and only if M_{red} is atomic. Each finitely generated monoid is atomic [25, Proposition 2.7.8].

2.3. **Factorizations.** The free commutative monoid on the set $\mathscr{A}(M_{\text{red}})$ is denoted by $\mathsf{Z}(M)$, and the elements of $\mathsf{Z}(M)$ are called *factorizations*. If $z \in \mathsf{Z}(M)$ consists of ℓ irreducibles of M_{red} (counting repetitions), then we call ℓ the *length* of z and write $|z| := \ell$. We say that $a \in \mathscr{A}(M)$ *appears* in z provided that $a + \mathscr{U}(M)$ is one of the ℓ irreducibles of z. The unique monoid homomorphism $\pi_M \colon \mathsf{Z}(M) \to M_{\text{red}}$ satisfying that $\pi(a) = a$ for all $a \in \mathscr{A}(M_{\text{red}})$ is called the *factorization homomorphism* of M. When there seems to be no risk of ambiguity, we write π instead of π_M. The kernel

$$\ker \pi := \{(z, z') \in \mathsf{Z}(M)^2 : \pi(z) = \pi(z')\}$$

of π is a congruence on $\mathsf{Z}(M)$, which we call the *factorization congruence* of M. In addition, we call an element $(z, z') \in \ker \pi$ a *factorization relation*. Let (z, z') be a factorization relation of M. We say that $a \in \mathscr{A}(M)$ *appears* in (z, z') if a appears in either z or z'. We call (z, z') *balanced* if $|z| = |z'|$ and *unbalanced* otherwise. Also, we say that (z, z') is *irredundant* provided that no irreducible of M appears in both z and z'. For each $x \in M$ we set

$$\mathsf{Z}(x) := \mathsf{Z}_M(x) := \pi^{-1}(x + \mathscr{U}(M)) \subseteq \mathsf{Z}(M).$$

Observe that $\mathsf{Z}(u) = \{0\}$ if and only if $u \in \mathscr{U}(M)$. In addition, note that M is atomic if and only if π is surjective, that is $\mathsf{Z}(x) \neq \emptyset$ for all $x \in M$. For each $x \in M$, we set

$$\mathsf{L}(x) := \mathsf{L}_M(x) := \{|z| : z \in \mathsf{Z}(x)\} \subset \mathbb{N}_0.$$

The monoid M is called a *UFM* (or a *unique factorization monoid*) if $|\mathsf{Z}(x)| = 1$ for all $x \in M$. On the other hand, M is called an *HFM* (or a *half-factorial monoid*) if $|\mathsf{L}(x)| = 1$ for all $x \in M$. Let R be an integral domain (i.e., a commutative ring with identity and without nonzero zero-divisors). We let $R^\bullet$ denote the multiplicative monoid $R \setminus \{0\}$ and, to simplify notation, we write π_R and $\mathsf{Z}(R)$ instead of $\pi_{R^\bullet}$ and $\mathsf{Z}(R^\bullet)$, respectively. In addition, for each $x \in R^\bullet$, we set $\mathsf{Z}_R(x) := \mathsf{Z}_{R^\bullet}(x)$ and $\mathsf{L}_R(x) := \mathsf{L}_{R^\bullet}(x)$. It is clear that R is atomic (resp., a UFD) if and only if the monoid $R^\bullet$ is atomic (resp., a UFM). We say that R is an *HFD* (or a *half-factorial domain*) provided that $R^\bullet$ is an HFM.

The notion of an HFM is therefore obtained from that of a UFM by keeping the existence and weakening the uniqueness of factorizations, i.e., replacing $|\mathsf{Z}(x)| = 1$ by $|\mathsf{L}(x)| = 1$ for every $x \in M$. In [20] the second and fourth authors proposed a dual way to weaken the unique factorization property and obtain a natural relaxed version of a UFM, which they called an other-half-factorial monoid.

Definition 2.1. Let M be an atomic monoid. We say that M is an OHFM (or an *other-half-factorial monoid*) if for all $x \in M$ and $z_1, z_2 \in \mathsf{Z}(x)$ the equality $|z_1| = |z_2|$ implies that $z_1 = z_2$.

Notice that a monoid is an OHFM if and only if its reduced monoid is an OHFM. It is clear that every UFM is an OHFM. We say that an OHFM is *proper* if it is not a UFM. The study of other-half-factoriality will be our primary focus of attention here. It has been proved in [20] that the multiplicative monoid of an integral domain is an OHFM if and only if the domain is a UFD, i.e., the multiplicative monoid of an integral domain cannot be a proper OHFM. We will obtain several additional fundamental results as a consequence of our investigation.

3. Characterizations of OHFMs

The main purpose of this section is to provide characterizations of an OHFM in terms of the integral dependence of its set of irreducibles and also in terms of its factorization congruence. First, we introduce a simple lemma which will streamline some of our future computations. The notion of integral independence plays a central role in our first characterization of an OHFM. Let M be a monoid, and let S be a subset of M. The set S is called *integrally independent* in M provided that for any distinct $s_1, \dots, s_n \in S$ and any $c_1, \dots, c_n \in \mathbb{Z}$ the equality $\sum_{i=1}^n c_i s_i = 0$ in $\mathrm{gp}(M)$ implies that $c_i = 0$ for every $i \in [\![1, n]\!]$. Let us proceed to characterize OHFMs.

Theorem 3.1. *Let M be an atomic monoid that is not a UFM. Then the following statements are equivalent.*

- *(a) The monoid M is an OHFM.*
- *(b) There exists $a \in \mathscr{A}(M_{\text{red}})$ such that $\mathscr{A}(M_{\text{red}}) \setminus \{a\}$ and $a - \mathscr{A}(M_{\text{red}}) \setminus \{a\}$ are integrally independent sets in $\mathrm{gp}(M_{\text{red}})$.*
- *(c) The congruence $\ker \pi$ is nontrivial and cyclic.*

Proof. Since M is an OHFM if and only if M_{red} is an OHFM and since the factorization homomorphisms of both M and M_{red} are the same, there is no loss in assuming that M is a reduced monoid. Accordingly, we identify M_{red} with M.

(a) $\Rightarrow$ (b): Assume that M is an OHFM. Observe that the set $\mathscr{A}(M)$ cannot be integrally independent as, otherwise, M would be a UFM. Then there exists $a \in \mathscr{A}(M)$ and $m \in \mathbb{N}$ such that

$$ma = \sum_{i=1}^{k} m_i a_i \tag{3.1}$$

for some $a_1, \dots, a_k \in \mathscr{A}(M) \setminus \{a\}$ and $m_1, \dots, m_k \in \mathbb{Z}$. Let us verify that $\mathscr{A}(M) \setminus \{a\}$ is an integrally independent set in $\mathrm{gp}(M)$. Suppose, for the sake of a contradiction, that this is not the case. Then there exist $b \in \mathscr{A}(M) \setminus \{a\}$ and $n \in \mathbb{N}$ satisfying

$$nb = \sum_{i=1}^{\ell} n_i b_i \tag{3.2}$$

for some $b_1, \dots, b_\ell \in \mathscr{A}(M) \setminus \{a, b\}$ and $n_1, \dots, n_\ell \in \mathbb{Z}$. Take $c_i = \frac{1}{2}(|m_i| - m_i)$ and $c'_i = \frac{1}{2}(|m_i| + m_i)$ for every $i \in [\![1, k]\!]$, and also take $d_i = \frac{1}{2}(|n_i| - n_i)$ and $d'_i = \frac{1}{2}(|n_i| + n_i)$ for every $i \in [\![1, \ell]\!]$. Then set

$$z := ma + \sum_{i=1}^{k} c_i a_i, \quad z' := \sum_{i=1}^{k} c'_i a_i, \quad w := nb + \sum_{j=1}^{\ell} d_j b_j, \quad \text{and} \quad w' := \sum_{j=1}^{\ell} d'_j b_j.$$

It follows from (3.1) and (3.2) that both (z, z') and (w, w') are irredundant factorization relations of M. Because (z, z') and (w, w') are irredundant and nontrivial, the other-half-factoriality of M guarantees that they are both unbalanced. Assume, without loss of generality, that $|z| > |z'|$ and $|w| < |w'|$. Clearly, $((|w'| - |w|)z, (|w'| - |w|)z')$ and $((|z| - |z'|)w, (|z| - |z'|)w')$ are both factorization relations of M. By adding them, one can produce a new balanced factorization relation with exactly one of its two component factorizations involving the irreducible a. However, this contradicts that M is an OHFM. Thus, $\mathscr{A}(M) \setminus \{a\}$ is integrally independent in $\text{gp}(M)$.

Let $a \in \mathscr{A}(M)$ be as in the previous paragraph. We proceed to argue that the set $a - \mathscr{A}(M) \setminus \{a\}$ is also integrally independent in $\text{gp}(M)$. Take this time $b_1, \dots, b_\ell \in \mathscr{A}(M) \setminus \{a\}$ and $n_1, \dots, n_\ell \in \mathbb{Z}$ such that $\sum_{i=1}^{\ell} n_i(b_i - a) = 0$. Then set $d_i = \frac{1}{2}(|n_i| - n_i)$ and $d'_i = \frac{1}{2}(|n_i| + n_i)$ for every $i \in [\![1, \ell]\!]$, and consider the factorizations

$$z_1 := \sum_{i=1}^{\ell} d_i b_i + \Big(\sum_{i=1}^{\ell} d'_i\Big)a \quad \text{and} \quad z_2 := \sum_{i=1}^{\ell} d'_i b_i + \Big(\sum_{i=1}^{\ell} d_i\Big)a.$$

The equality $\sum_{i=1}^{\ell} n_i b_i = \big(\sum_{i=1}^{\ell} n_i\big)a$ ensures that (z_1, z_2) is a balanced factorization relation. Since M is an OHFM, $z_1 = z_2$ and therefore $n_i = d_i - d'_i = 0$ for every $i \in [\![1, \ell]\!]$. As a consequence, we can conclude that $a - \mathscr{A}(M) \setminus \{a\}$ is an integrally independent set in $\text{gp}(M)$.

(b) $\Rightarrow$ (c): Suppose that there exists $a \in \mathscr{A}(M)$ such that both $\mathscr{A}(M) \setminus \{a\}$ and $a - \mathscr{A}(M) \setminus \{a\}$ are integrally independent sets in $\text{gp}(M)$. Let S be the subgroup of $\text{gp}(M)$ generated by $\mathscr{A}(M) \setminus \{a\}$. We have seen before that $\mathscr{A}(M)$ is an integrally dependent set. As a result, the annihilator $\text{Ann}(a+S)$ of $a+S$ in the $\mathbb{Z}$-module $\text{gp}(M)/S$ is not trivial. Since $\text{Ann}(a+S)$ is an additive subgroup of $\mathbb{Z}$, there exists $m \in \mathbb{N}$ such that $\text{Ann}(a + S) = m\mathbb{Z}$. Then there is an irredundant factorization relation $(w_1, w_2) \in \ker \pi$ such that exactly m copies of a appear in w_1 and no copies of a appear in w_2.

Let us verify that (w_1, w_2) is unbalanced. Suppose, by way of contradiction, that $|w_1| = |w_2|$. Note that $\pi(w_1) - \pi(w_2) = 0$ in $\text{gp}(M)$ ensures the existence of $a_0, \dots, a_k \in \mathscr{A}(M)$ (with $a_0 = a$) and $m_0, \dots, m_k \in \mathbb{Z}$ (with $m_0 = m$) such that $\sum_{i=0}^{k} m_i a_i = 0$. As $|w_1| = |w_2|$, the equality $\sum_{i=0}^{k} m_i = 0$ holds. As a consequence, one finds that

$$\sum_{i=1}^{k} m_i(a - a_i) = a\sum_{i=0}^{k} m_i - \sum_{i=0}^{k} m_i a_i = 0.$$

This, along with the fact that $m_i \neq 0$ for some $i \in [\![1,k]\!]$, contradicts that $a - \mathscr{A}(M) \setminus \{a\}$ is an integrally independent set. Hence $|w_1| \neq |w_2|$, and so (w_1, w_2) is unbalanced.

We still need to show that (w_1, w_2) generates the congruence $\ker \pi$. Towards this end, take a nontrivial irredundant factorization relation $(z_1, z_2) \in \ker \pi$. As $\mathscr{A}(M) \setminus \{a\}$ is integrally independent, a must appear in (z_1, z_2). Assume without loss that exactly n copies of a appear in z_1 for some $n \in \mathbb{N}$. Then the equality $\pi(z_1) = \pi(z_2)$ ensures that $n \in \operatorname{Ann}(a+S)$, and so $n = km$ for some $k \in \mathbb{N}$. Then after canceling na in both sides of $\pi(w_1^k z_2) = \pi(w_2^k z_1)$, we obtain two integral combinations of irreducibles in $\mathscr{A}(M) \setminus \{a\}$, whose corresponding coefficients must be equal. Thus, $(z_1, z_2) = (w_1, w_2)^k$.

(c) $\Rightarrow$ (a): Suppose that $\ker \pi$ is a cyclic congruence generated by an unbalanced irredundant factorization relation (w_1, w_2). Let $*$ denote the monoid operation of the congruence $\ker \pi$. Take $(z, z') \in \ker \pi$ such that $z \neq z'$. Since (w_1, w_2) generates $\ker \pi$, there exist $n \in \mathbb{N}$ and $z_0, \dots, z_n \in \mathsf{Z}(M)$ with $z_0 = z$ and $z_n = z'$ such that for every $i \in [\![1, n]\!]$ the equality

$$(z_{i-1}, z_i) = (w_1, w_2) * (d_i, d_i) \tag{3.3}$$

holds for some $d_i \in \mathsf{Z}(M)$. After multiplying all the identities in (3.3) (for every $i \in [\![1, n]\!]$), one finds that $(z, z') * (z_1 \cdots z_{n-1}, z_1 \cdots z_{n-1}) = (w_1^n, w_2^n) * (d, d)$, where $d = d_1 \cdots d_n$. Since $z_1 \cdots z_{n-1}$ divides both $w_1^n d$ and $w_2^n d$ in the free monoid $\mathsf{Z}(M)$ and $\gcd(w_1^n, w_2^n) = 1$, there exists $z'' \in \mathsf{Z}(M)$ such that $z_1 \cdots z_{n-1} z'' = d$. As a result, $(z, z') = (z'' w_1^n, z'' w_2^n)$ and so (z, z') is an unbalanced factorization relation. Hence M is an OHFM. □

Let M be an atomic monoid. Following [20] we call a factorization relation (z_1, z_2) *master* if any irredundant and unbalanced factorization relation of M has the form (z_1^n, z_2^n) or (z_2^n, z_1^n) for some $n \in \mathbb{N}$. A master factorization relation must be unbalanced unless M is an HFM. When M is a proper OHFM we have seen that $\ker \pi$ is a nontrivial cyclic congruence, and it is clear that any generator of $\ker \pi$ is a master factorization relation. Thus, as a consequence of Theorem 3.1, we obtain the following corollary, which was first established in the proof of the main theorem of [20].

Corollary 3.2. *Let M be an atomic monoid. Then M is a proper OHFM if and only if it admits an unbalanced master factorization relation.*

The numerical monoids that are proper OHFMs have been characterized in [20] as those having embedding dimension 2. This was generalized in [31, Proposition 4.3], which states that the additive submonoids of $\mathbb{Q}_{\geq 2}$ that are OHFMs are those generated by two elements. In general, every monoid that can be generated by two elements is an OHFM, as the following result indicates.

Corollary 3.3. *Let M be a monoid generated by two elements. Then $\ker \pi$ is cyclic, and M is an OHFM.*

Proof. As M is finitely generated, it is atomic. We can assume without loss that M is reduced. If M is a UFM, then there is nothing to show. Therefore assume that M is not a UFM. Then there exists a generating set A of M with $|A| = 2$. Because M is not a UFM, $\mathscr{A}(M) = A$. As both sets $A \setminus \{a\}$ and $a - A \setminus \{a\}$ are singletons, the corollary follows from Theorem 3.1. $\square$

When a monoid cannot be generated by two elements, its factorization congruence may not be cyclic (even if the monoid is finitely generated). The next example sheds some light upon this observation.

Example 3.4. For $n \in \mathbb{N}_{\ge 3}$, consider the additive submonoid $M = \{0\} \cup \mathbb{N}_{\ge n}$ of $\mathbb{N}_0$. It can be readily verified that M is atomic and $\mathscr{A}(M) = [\![n, 2n-1]\!]$. Since $2(n+1) = n + (n+2)$, it follows that M is not an OHFM. Then Theorem 3.1 guarantees that the factorization congruence of M is not cyclic.

4. Pure Irreducibles: The PLS Property

4.1. **Pure Irreducibles.** In this section, we study the notions of purely long and purely short irreducibles (as introduced in [20]) in connection with other-half-factoriality. Let M be an atomic monoid, and let (z_1, z_2) be an unbalanced factorization relation of M. Then we call the factorization of bigger (resp., smaller) length between z_1 and z_2 the *longer* (resp., *shorter*) factorization of (z_1, z_2).

Definition 4.1. Let M be a monoid, and take $a \in \mathscr{A}(M_{\text{red}})$. We say that a is *purely long* (resp., *purely short*) if a is not prime and for all irredundant and unbalanced factorization relations (z_1, z_2) of M, the fact that a appears in z_1 implies that $|z_1| > |z_2|$ (resp., $|z_1| < |z_2|$).

Remark 4.2. As by definition a purely long (or short) irreducible is not prime, it must appear in at least one nontrivial irredundant factorization relation of M.

We let $\mathscr{L}(M)$ (resp., $\mathscr{S}(M)$) denote the set comprising all purely long (resp., purely short) irreducibles of M_{red}. When M is a proper OHFM, it follows from Corollary 3.2 that both $\mathscr{L}(M)$ and $\mathscr{S}(M)$ are nonempty sets. More precisely, if $z_1, z_2 \in \mathsf{Z}(M)$ satisfy that $|z_1| < |z_2|$ and (z_1, z_2) is an irredundant factorization relation generating the factorization congruence of an OHFM M, then $\mathscr{L}(M)$ (resp., $\mathscr{S}(M)$) consists of all irreducibles that appear in z_2 (resp., z_1).

We call any element of $\mathscr{L}(M) \cup \mathscr{S}(M)$ a *pure* irreducible. As a consequence of the following proposition we will obtain that every atomic monoid contains only finitely many pure irreducibles.

Proposition 4.3. *For an atomic monoid M, let a be a purely short/long irreducible, and let (w_1, w_2) be an irredundant factorization relation. Then a appears in (w_1, w_2) if and only if (w_1, w_2) is unbalanced.*

Proof. To argue the direct implication suppose, by way of contradiction, that (w_1, w_2) is balanced. We also assume, without loss of generality, that a appears in w_2. Suppose first that $a \in \mathscr{L}(M)$, and take an irredundant factorization relation (z_1, z_2) such that $|z_1| > |z_2|$ and a appears in z_1. Then we can take $n \in \mathbb{N}$ large enough such that the number of copies of a that appear in $w_1^n z_1$ is strictly smaller than the number of copies of a that appear in $w_2^n z_2$. Therefore $(w_1^n z_1, w_2^n z_2)$ yields, after cancellations, an irredundant and unbalanced factorization relation whose shorter factorization involves a. However, this contradicts that a is purely long. Supposing that $a \in \mathscr{S}(M)$, one can similarly arrive to another contradiction.

For the reverse implication, assume that (w_1, w_2) is unbalanced with $|w_1| < |w_2|$. Suppose first that $a \in \mathscr{L}(M)$. Take an irredundant factorization relation (z_1, z_2) such that a appears in (z_1, z_2). There is no loss in assuming that a appears in z_1 and, therefore, that $|z_1| > |z_2|$. Then there exists $n \in \mathbb{N}$ such that $|w_1^n z_1| < |w_2^n z_2|$. Since a appears in the shorter factorization of $(w_1^n z_1, w_2^n z_2)$, the fact that a is a purely long irreducible guarantees that a also appears in the longer factorization of $(w_1^n z_1, w_2^n z_2)$. Hence a appears in w_2. For $a \in \mathscr{S}(M)$ the proof is similar. □

Corollary 4.4. *For an atomic monoid M, both sets $\mathscr{L}(M)$ and $\mathscr{S}(M)$ are finite.*

Proof. If M is an HFM, then both sets $\mathscr{L}(M)$ and $\mathscr{S}(M)$ are empty. Otherwise, there must exist an unbalanced factorization relation (z_1, z_2). It follows now from Proposition 4.3 that every pure irreducible of M appears in (z_1, z_2). Hence both sets $\mathscr{L}(M)$ and $\mathscr{S}(M)$ must be finite. □

Clearly, atomic monoids having both purely long and purely short irreducibles are natural generalizations of OHFMs, and they will play an important role in the remainder of this paper.

Definition 4.5. If an atomic monoid M contains both purely long and purely short irreducibles, then we say that M has the *PLS property* or that M is a *PLSM*.

For future reference, we highlight the following immediate corollary of Theorem 3.1.

Corollary 4.6. *Every proper OHFM is a PLSM.*

The converse of Corollary 4.6 does not hold even for finitely generated monoids. For any subset S of $\mathbb{R}^d$, we let $\mathsf{cone}(S)$ and $\mathsf{aff}(S)$ denote the cone and the affine space generated by S, respectively.

Example 4.7. Consider the submonoid $M = \langle a_i : i \in [\![1,5]\!] \rangle$ of $(\mathbb{N}_0^3, +)$, where $a_1 = (0,1,1)$, $a_2 = (0,2,1)$, $a_3 = (1,2,3)$, $a_4 = (2,2,2)$, and $a_5 = (3,2,1)$. Clearly, M is atomic and it is not hard to check that $\mathscr{A}(M) = \{a_i \mid i \in [\![1,5]\!]\}$. Let H be the hyperplane described by the equation $y = 2$. Since $a_1 \notin H$ and $a_i \in H$ for every $i \in [\![2,5]\!]$, the irreducible a_1 is purely long. Because $\mathsf{cone}(a_1, a_2)$ and $\mathsf{aff}(a_3, a_4, a_5)$ only intersect in the origin, a_1 and a_2 cannot be in the same part of any irredundant factorization relation of M. Thus, if a_2 appears in an irredundant factorization relation involving a_1, then it must appear in its shorter part. In addition, note that because $a_2 \notin \mathsf{aff}(a_3, a_4, a_5)$, there is no irredundant factorization relation of M involving a_2 but not a_1. Hence $a_2 \in \mathscr{L}(M)$, and so M is a PLSM. However, it follows from [30, Section 5] that M is not an OHFM.

For an atomic monoid M, none of the conditions $\mathscr{L}(M) = \emptyset$ and $\mathscr{S}(M) = \emptyset$ implies the other one. The following example sheds some light upon this observation.

Example 4.8. For the set $A = \{(0,3), (1,2), (2,1), (3,0)\}$, consider the submonoid M of $(\mathbb{N}_0^2, +)$ generated by A. It is clear that M is atomic, and one can readily check that $\mathscr{A}(M) = A$. Since all the irreducibles of M lie in the line determined by the equation $x + y = 3$, it follows from [30, Corollary 5.5] that M is an HFM.

Now consider the submonoid M_1 of $(\mathbb{N}_0^2, +)$ generated by the set $A_1 = A \cup \{(1,1)\}$. It is easy to verify that M_1 is atomic with $\mathscr{A}(M_1) = A_1$. Moreover, since the irreducibles of M_1 are not colinear, it follows from [30, Corollary 5.5] that M_1 is not an HFM. Therefore there exists an irredundant factorization relation (z_1, z_2) with $|z_1| \neq |z_2|$. Since M is an HFM, $(1,1)$ must appear in (z_1, z_2); say that $(1,1)$ appears in z_1. After projecting on the line determined by the equation $y = x$, one can easily see that $|z_1| > |z_2|$. As a result, $(1,1)$ is purely long. Note that the irreducibles in A are neither purely long nor purely short because they are precisely the irreducibles of M, which is an HFM. Hence M_1 contains a purely long irreducible but no purely short irreducibles.

Lastly, considering the submonoid M_2 of $(\mathbb{N}_0^2, +)$ generated by the set $A \cup \{(2,2)\}$ and proceeding as we did with M_1, one finds that $(2,2)$ is the only purely short irreducible in M_2, and also that M_2 contains no purely long irreducibles.

We know that HFMs contain neither purely long nor purely short irreducibles. However, there are monoids that are not HFMs and still contain neither purely long nor purely short irreducibles.

Example 4.9. Let M and A be as in Example 4.8, and let M_3 be the submonoid of $(\mathbb{N}_0^2, +)$ generated by the set $A_3 = A \cup \{(0,2), (1,1), (2,0)\}$. It is not hard to verify that M_3 is an atomic monoid with $\mathscr{A}(M_3) = A \cup \{(0,2), (1,1), (2,0)\}$. Since the equalities $2(1,1) = (0,2) + (2,0)$ and $(1,2) + (2,1) = (0,3) + (3,0)$ give rise to two irredundant and balanced factorizations involving each irreducible of M_3, both sets $\mathscr{L}(M_3)$ and $\mathscr{S}(M_3)$ must be empty. Because of this, M_3 cannot be an OHFM, which is confirmed by [30, Theorem 5.10]. In addition, as the points in A_3 are not colinear, it follows from [30, Corollary 5.5] that M_3 is not an HFM.

4.2. **Direct Sum Decomposition of PLSM.** Let M be a monoid, and let M_1 and M_2 be two submonoids of M. The monoid M is the *direct sum* of M_1 and M_2 provided that $M = M_1 + M_2$ and $M_1 \cap M_2 = \{0\}$; in this case, it is customary to write $M = M_1 \oplus M_2$. Let us remark that $M = M_1 \oplus M_2$ does not guarantee the uniqueness of the representation of an element of M as a sum of an element of M_1 and an element of M_2.

Theorem 4.10. *Let M be a PLSM. Then there exist submonoids H and O of M_{red} such that $M_{\text{red}} = H \oplus O$, where H is an HFM and O is a finitely generated proper OHFM.*

Proof. Let O be the submonoid of M_{red} generated by the set $\mathscr{L}(M) \cup \mathscr{S}(M)$. It is clear that O is an atomic monoid with $\mathscr{A}(O) = \mathscr{L}(M) \cup \mathscr{S}(M)$. Moreover, note that $\mathscr{L}(O) = \mathscr{L}(M)$ and $\mathscr{S}(O) = \mathscr{S}(M)$. By Corollary 4.4, the monoid O is finitely generated. To verify that O is an OHFM, let (z_1, z_2) be a nontrivial irredundant factorization relation in $\ker \pi_O$. Since at least one irreducible in $\mathscr{L}(M) \cup \mathscr{S}(M)$ appears in the relation (z_1, z_2), the latter must be unbalanced by Proposition 4.3. As a consequence, O is a proper OHFM.

Now let H be the submonoid of M_{red} generated by $\mathscr{A}(M) \setminus (\mathscr{L}(M) \cup \mathscr{S}(M))$. It follows immediately that H is atomic with $\mathscr{A}(H) = \mathscr{A}(M) \setminus (\mathscr{L}(M) \cup \mathscr{S}(M))$. To see that H is an HFM, it suffices to observe that since $\ker \pi_H \subseteq \ker \pi_M$, any irredundant factorization relation of $\ker \pi_H$ must be balanced by Proposition 4.3.

Because $\mathscr{A}(M_{\text{red}}) = \mathscr{A}(H) \cup \mathscr{A}(O)$, we find that $M_{\text{red}} = H + O$. To argue that this is indeed a direct sum, suppose that $x \in H \cap O$. As both H and O are atomic monoids, one can take $z_1 \in \mathsf{Z}_H(x)$ and $z_2 \in \mathsf{Z}_O(x)$. Therefore $(z_1, z_2) \in \ker \pi_M$. Since $\mathscr{L}(M) \neq \emptyset$ and $\mathscr{S}(M) \neq \emptyset$, if a pure irreducible appeared in z_2, then a pure irreducible would appear in z_1. As z_1 consists of non-pure irreducibles, z_2 must be the factorization with no irreducibles, whence $x = 0$. As a result, $H \cap O = \{0\}$, which implies that $M_{\text{red}} = H \oplus O$. □

The converse of Theorem 4.10 does not hold in general, as the following example indicates.

Example 4.11. Consider the additive submonoid M of $(\mathbb{N}_0^2, +)$ generated by the set of lattice points $\{(1,1), (0,3), (1,2), (2,1), (3,0)\}$. We have already seen in the second paragraph of Example 4.8 that $\mathscr{L}(M) = \{(1,1)\}$ and $\mathscr{S}(M) = \emptyset$. Therefore M is not a PLSM. The submonoid $H = \langle (1,2), (0,3) \rangle$ of M is clearly a UFM and, in particular, an HFM. On the other hand, one can see that the submonoid $O = \langle (1,1), (2,1), (3,0) \rangle$ of M is a proper OHFM by applying Theorem 3.1 with $a = (1,1)$. It follows immediately that $M = H \oplus O$ even though M is not a PLSM.

We conclude this section with the following proposition.

Proposition 4.12. *Let M be a PLSM. Then there exists an unbalanced factorization relation $(w_1, w_2) \in \ker \pi$ such that every factorization relation of $\ker \pi$ has the form $(w_1^n h_1, w_2^n h_2)$ for some $n \in \mathbb{N}_0$ and some balanced factorization relation $(h_1, h_2) \in \ker \pi$.*

Proof. Take $a \in \mathscr{L}(M)$. Set $A = \mathscr{A}(M) \setminus \{a\}$, and let S be the subgroup of $\text{gp}(M)$ generated by A. Since a appears in an irredundant and unbalanced factorization relation of M, there exists $m \in \mathbb{N}$ such that $\text{Ann}(a + S) = m\mathbb{Z}$, where $\text{Ann}(a + S)$ is the annihilator of $a + S$ in the $\mathbb{Z}$-module $\text{gp}(M)/S$. As $ma \in S$, there is an irredundant factorization relation (w_1, w_2) of M such that exactly m copies of a appear in w_1. It follows from Proposition 4.3 that $|w_1| > |w_2|$. Suppose now that (z_1, z_2) is an irredundant factorization relation of M with $|z_1| > |z_2|$, and let $k \in \mathbb{N}$ be the number of copies of a appearing in z_1. Notice that $k \in \text{Ann}(a + S)$, and therefore $k = nm$ for some $n \in \mathbb{N}$. Then $(w_1^n z_2, w_2^n z_1) \in \ker \pi$ yields, after cancellations, a factorization relation that does not involve a. Thus, such a factorization must be balanced by Proposition 4.3 and cannot involve any pure irreducible. So the number of copies of each irreducible b in $\mathscr{L}(M)$ (resp., $\mathscr{S}(M)$) that appear in z_1 (resp., z_2) equals n times the number of copies of b that appear in w_1 (resp., w_2). Hence $(z_1, z_2) = (w_1^n h_1, w_2^n h_2)$, where $h_1, h_2 \in \mathsf{Z}(M)$ involve no pure irreducibles. Clearly, $(h_1, h_2) \in \ker \pi$, and Proposition 4.3 guarantees that $|h_1| = |h_2|$. □

5. Finite-Rank Monoids

In this section we continue studying the OHF property and the PLS property, but we restrict our attention to the class of finite-rank monoids.

5.1. **Number of Irreducibles.** If M is a reduced finite-rank UFM, then it follows from [25, Proposition 1.2.3(2)] that $|\mathscr{A}(M)| = \mathsf{rank}(M)$. In parallel with this, the cardinality of $\mathscr{A}(M)$ in a finite-rank proper OHFM M can be determined.

Proposition 5.1. *Let M be a finite-rank proper OHFM. Then $|\mathscr{A}(M_{\text{red}})| = \mathsf{rank}(M)+1$.*

Proof. As $\text{gp}(M_{\text{red}}) \cong \text{gp}(M)/\mathscr{U}(M)$, the monoid M_{red} has finite rank. Hence one can replace M by M_{red} and assume that M is reduced. Set $r = \mathsf{rank}(M)$ and then embed M into the $\mathbb{Q}$-vector space $V := \mathbb{Q} \otimes_{\mathbb{Z}} \text{gp}(M) \cong \mathbb{Q}^r$ via $M \hookrightarrow \text{gp}(M) \to \mathbb{Q} \otimes_{\mathbb{Z}} \text{gp}(M)$, where the injectivity of the second map follows from the flatness of the $\mathbb{Z}$-module $\mathbb{Q}$. So we can think of M as an additive submonoid of the finite-dimensional vector space $\mathbb{Q}^r$. By Theorem 3.1, there exists $a \in \mathscr{A}(M)$ such that $\mathscr{A}(M) \setminus \{a\}$ and $a - \mathscr{A}(M) \setminus \{a\}$ are integrally independent sets in $\text{gp}(M)$. In particular, the sets $\mathscr{A}(M) \setminus \{a\}$ and $a - \mathscr{A}(M) \setminus \{a\}$ are linearly independent inside the vector space V. Because M is atomic, $\text{gp}(M)$ can be generated by $\mathscr{A}(M)$ as a $\mathbb{Z}$-module and, therefore, $\mathscr{A}(M)$ is a generating set of V. Since M is a proper OHFM, the monoid M is not a UFM and, consequently, $\mathscr{A}(M)$ is a linearly dependent set of V. This along with the fact that $\mathscr{A}(M) \setminus \{a\}$ is linearly independent in V implies that $\mathscr{A}(M) \setminus \{a\}$ is a basis for V. Hence $|\mathscr{A}(M)| = |\mathscr{A}(M) \setminus \{a\}| + 1 = r + 1$. □

Corollary 5.2. *Every finite-rank OHFM is finitely generated.*

The condition of having finite rank in Corollary 5.2 is not superfluous. For instance, consider the additive monoid $M = \langle 2, 3 \rangle \oplus \mathbb{N}_0^\infty$, where $\mathbb{N}_0^\infty$ is the direct sum of countably many copies of $\mathbb{N}_0$. Since $\langle 2, 3 \rangle$ is a proper OHFM and $\mathbb{N}_0^\infty$ is a UFM, the monoid M is a proper OHFM. However, M is not finitely generated because $\mathsf{rank}(M) = \infty$. The converse of Proposition 5.1 does not hold in general, as the following example shows.

Example 5.3. For every $r \in \mathbb{N}$, consider the submonoid M_r of $(\mathbb{N}_0^r, +)$ that is generated by the set $S = \{v_0, re_j : j \in [\![1, r]\!]\}$, where $v_0 := \{e_1 + \cdots + e_r\}$. It is not hard to verify that $\mathscr{A}(M_r) = S$, and so $|\mathscr{A}(M_r)| = r + 1$. Notice that each point in S lies in the hyperplane of $\mathbb{R}^r$ determined by the equation $x_1 + \cdots + x_r = r$. Hence it follows from [30, Corollary 5.5] that M_r is a proper HFM. Therefore M_r cannot be an OHFM.

As we have emphasized in Corollary 4.6, every proper OHFM is a PLSM. We proceed to show that being an OHFM is equivalent to being a PSLM in the class of torsion-free monoids with rank at most 2.

Theorem 5.4. *For a torsion-free monoid M with $\mathsf{rank}(M) \le 2$, the following statements are equivalent.*

(a) The monoid M is a proper OHFM.
(b) The monoid M is a PLSM.
(c) The congruence $\ker \pi$ can be generated by an unbalanced factorization relation.

Proof. (a) $\Leftrightarrow$ (c): This is part of Theorem 3.1.

(a) $\Rightarrow$ (b): This is Corollary 4.6.

(b) $\Rightarrow$ (a): Assume that M is a PLSM, and suppose for the sake of a contradiction that M is not a proper OHFM. Since M is finitely generated, it is atomic. As M is not a UFM, $|\mathscr{A}(M)| \ge 2$. We split the rest of the proof into three cases.

CASE 1: $|\mathscr{A}(M)| = 2$. In this case, the factorization congruence $\ker \pi$ is cyclic by Corollary 3.3, and the existence of purely long/short irreducibles implies that any generator of $\ker \pi$ must be unbalanced, contradicting that M is not a proper OHFM.

CASE 2: $|\mathscr{A}(M)| = 3$. Take $a_1, a_2, a_3 \in M$ such that $\mathscr{A}(M) = \{a_1, a_2, a_3\}$. Assume without loss that $a_1 \in \mathscr{L}(M)$ and $a_2 \in \mathscr{S}(M)$. Now take an irredundant and balanced factorization relation $(z_1, z_2) \in \ker \pi$. Since a_1 and a_2 are pure irreducibles, none of them can appear in (z_1, z_2). Therefore only copies of the irreducible a_3 appear in both z_1 and z_2. This implies that $z_1 = z_2$. As (z_1, z_2) was taking to be irredundant, it must be trivial. Hence M is a proper OHFM, a contradiction.

CASE 3: $|\mathscr{A}(M)| \ge 4$. Take $a_0 \in \mathscr{L}(M)$ and $a_3 \in \mathscr{S}(M)$, and then take $a_1, a_2 \in \mathscr{A}(M) \setminus \{a_0, a_3\}$ such that $a_1 \ne a_2$. Since a_0 is a purely long irreducible, the submonoid $M' := \langle a_1, a_2, a_3 \rangle$ of M must be an HFM. Now take a nontrivial factorization relation $(z_1, z_2) \in \ker \pi_{M'}$. As a_3 is a purely short irreducible, it does not appear in (z_1, z_2). Therefore either (z_1, z_2) or (z_2, z_1) equals (ma_1, ma_2) for some $n \in \mathbb{N}$. Now the fact that M is torsion-free, along with the equality $ma_1 = ma_2$, guarantees that $a_1 = a_2$, which is a contradiction. $\square$

Corollary 5.5. *If a torsion-free monoid M is generated by at most three elements, then it is a proper OHFM if and only if it is a PLSM.*

Proof. There is no loss in assuming that M is reduced. Clearly, $|\mathscr{A}(M)| \le 3$. Consider the $\mathbb{Q}$-space $V = \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{gp}(M)$, and identify M with its isomorphic copy inside V provided by the embedding $M \hookrightarrow \mathrm{gp}(M) \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{gp}(M)$. As M is atomic, $\mathscr{A}(M)$ is a spanning set of V, whence $\dim V \le 3$. If $\dim V = 3$, then $\mathscr{A}(M)$ is linearly independent over $\mathbb{Q}$, in which case M is the free monoid on $\mathscr{A}(M)$. In this case, M is neither a proper OHFM nor a PLSM. On the other hand, if $\dim V \le 2$, then $\mathsf{rank}(M) \le 2$ and we are done via Theorem 5.4. □

However, for a finitely generated monoid containing four or more irreducibles, the PLS property may not imply the OHF property. This has been illustrated in Example 4.7. In the same example, we have seen that the condition of having rank at most 2 is required in Theorem 5.4. On the other hand, the following example indicates that the condition of being torsion-free is also required in the statement of Theorem 5.4.

Example 5.6. Fix $n \in \mathbb{N}$ such that $n \ge 4$, and consider the submonoid $M := \langle a_k : k \in [\![1, n]\!] \rangle$ of the additive group $\mathbb{Z}_{n-2} \times \mathbb{Z}^2$, where $a_1 = (0, 0, 2)$, $a_2 = (0, 0, 3)$, and $a_k = (k - 3, 1, 0)$ for every $k \in [\![3, n]\!]$. Since M is finitely generated, it must be atomic. In addition, it can be readily verified that $\mathscr{A}(M) = \{a_k : k \in [\![1, n]\!]\}$. Now suppose that (z_1, z_2) is an irredundant and unbalanced factorization relation in $\ker \pi$, and assume that $|z_1| < |z_2|$. Since the second component of both a_1 and a_2 is 0 and the second component of $a_3, \dots, a_n$ is 1, the numbers of irreducibles in $\{a_3, \dots, a_n\}$ that appear in z_1 and in z_2 must coincide. A similar observation based on third components shows that a_1 appears in z_2 but not in z_1 and also that a_2 appears in z_1 but not in z_2. Hence $a_1 \in \mathscr{L}(M)$ and $a_2 \in \mathscr{S}(M)$, which implies that M is a PLSM. Checking that M is not an OHFM amounts to observing that the equality $(n-2)a_3 = (0, n-2, 0) = (n-2)a_4$ yields an irredundant and balanced nontrivial factorization relation of M.

Now we turn to characterize the PSLMs in the class consisting of all torsion-free rank-1 monoids, which have been recently studied under the name *Puiseux monoids*. Puiseux monoids have been studied in connection with commutative algebra [19], commutative factorization theory [11], and noncommutative factorization theory [6]. An updated survey on the atomic structure of Puiseux monoids is given in [12]. Notice that a Puiseux monoid is reduced unless it is a group (see [22, Section 24] and [28, Theorem 2.9]).

Proposition 5.7. *Let M be an atomic Puiseux monoid. Then the following statements are equivalent.*

(a) *The monoid M is a proper OHFM.*
(b) *The monoid M is a PLSM.*
(c) *Both inclusions* $\inf \mathscr{A}(M) \in \mathscr{L}(M)$ *and* $\sup \mathscr{A}(M) \in \mathscr{S}(M)$ *hold.*
(d) *At least one of the inclusions* $\inf \mathscr{A}(M) \in \mathscr{L}(M)$ *or* $\sup \mathscr{A}(M) \in \mathscr{S}(M)$ *holds.*
(e) *The equality $|\mathscr{A}(M)| = 2$ holds.*

If any of the conditions above holds, then $\mathscr{L}(M)$ and $\mathscr{S}(M)$ are singletons: $\mathscr{L}(M) = \{\inf \mathscr{A}(M)\}$ *and* $\mathscr{S}(M) = \{\sup \mathscr{A}(M)\}$.

Proof. (a) ⇒ (b): This is Corollary 4.6.

(b) ⇒ (c): Suppose that M is a PLSM, and take $a_\ell \in \mathscr{L}(M)$ and $a_s \in \mathscr{S}(M)$. Now take $a \in M$ such that $a \neq a_\ell$. Clearly, $n := \mathsf{n}(a)\mathsf{n}(a_\ell) \in M$ and, moreover, $z_1 := \mathsf{n}(a)\mathsf{d}(a_\ell)a_\ell$ and $z_2 := \mathsf{n}(a_\ell)\mathsf{d}(a)a$ are two factorizations in $\mathsf{Z}(n)$. Since the factorization relation (z_1, z_2) is irredundant and a_ℓ appears in z_1, one finds that $|z_1| > |z_2|$. Therefore $\mathsf{n}(a)\mathsf{d}(a_\ell) > \mathsf{n}(a_\ell)\mathsf{d}(a)$, which means that $a > a_\ell$. Then we conclude that $\inf \mathscr{A}(M) = a_\ell \in \mathscr{L}(M)$. The equality $\sup \mathscr{A}(M) = a_s$ can be argued similarly, from which one obtains that $\sup \mathscr{A}(M) \in \mathscr{S}(M)$.

(c) ⇒ (d): This is obvious.

(d) ⇒ (e): Assume now that $\inf \mathscr{A}(M) \in \mathscr{L}(M)$, and take $a_\ell \in \mathscr{L}(M)$. Since M is an atomic Puiseux monoid that is not a UFM, it follows that $|\mathscr{A}(M)| \geq 2$. Suppose, by way of contradiction, that $|\mathscr{A}(M)| \geq 3$, and take irreducibles $a_1, a_2 \in \mathscr{A}(M) \setminus \{a_\ell\}$ such that $a_1 \neq a_2$. Consider the element $n := \mathsf{n}(a_1)\mathsf{n}(a_2) \in M$. It is clear that both $z_1 := \mathsf{n}(a_2)\mathsf{d}(a_1)a_1$ and $z_2 := \mathsf{n}(a_1)\mathsf{d}(a_2)a_2$ are factorizations in $\mathsf{Z}(n)$, and they have different lengths because $a_1 \neq a_2$. However, the fact that a_ℓ does not appear in either z_1 or z_2 contradicts that $a_\ell \in \mathscr{L}(M)$. As a result, $|\mathscr{A}(M)| = 2$. One can similarly obtain $|\mathscr{A}(M)| = 2$ assuming that $\sup \mathscr{A}(M) \in \mathscr{S}(M)$.

(e) ⇒ (a): If $|\mathscr{A}(M)| = 2$, it follows from Corollary 3.3 that M is an OHFM. Taking a_1 and a_2 to be the two irreducibles of M, one finds that $\mathsf{n}(a_2)\mathsf{d}(a_1)a_1$ and $\mathsf{n}(a_1)\mathsf{d}(a_2)a_2$ are two different factorizations of $\mathsf{n}(a_1)\mathsf{n}(a_2) \in M$, and so M is not a UFM. Hence M must be a proper OHFM. □

Corollary 5.8. *Let N be a numerical monoid. Then $\mathscr{L}(N) \cup \mathscr{S}(N)$ is nonempty if and only if N has embedding dimension 2, in which case $\mathscr{L}(N) = \{\min \mathscr{A}(N)\}$ and $\mathscr{S}(N) = \{\max \mathscr{A}(N)\}$.*

6. Pure Irreducibles in Integral Domains

We proceed to study the existence of purely long and purely short irreducibles in the context of integral domains. Throughout this section we set $\mathscr{L}(R) := \mathscr{L}(R^\bullet)$ and $\mathscr{S}(R) := \mathscr{S}(R^\bullet)$ for any integral domain R. In addition, if R is a Dedekind domain, we let $\mathrm{Cl}(R)$ denote the divisor class group of R.

6.1. Integral Domains Do Not Satisfy the PLS Property.

In Section 5 we have proved that in the class of torsion-free monoids with rank at most 2, being a proper OHFM and being a PLSM are equivalent notions. It was proved in [20] that an integral domain has the OHF property only if it is a UFD, and having the OHF property implies having both purely long and purely short irreducibles. This begs the tantalizing question as to whether there is an atomic integral domain with both purely long and a purely short irreducibles. The next theorem will answer such a question.

Theorem 6.1. *Let R be an integral domain. Then either $\mathscr{L}(R) = \emptyset$ or $\mathscr{S}(R) = \emptyset$.*

Proof. Suppose, by way of contradiction, that R is an atomic domain such that both $\mathscr{L}(R)$ nor $\mathscr{S}(R)$ are nonempty sets. We recall that by Corollary 4.4 both $\mathscr{L}(R)$ and $\mathscr{S}(R)$ are finite sets. For convenience, we will write

$$\mathscr{L}(R) = \{\pi_1, \pi_2, \cdots, \pi_\ell\} \quad \text{and} \quad \mathscr{S}(R) = \{\xi_1, \xi_2, \dots, \xi_s\},$$

where $\ell = |\mathscr{L}(R)|$ and $s = |\mathscr{S}(R)|$. Take $(z, z') \in \ker \pi_R$ to be an irredundant and unbalanced factorization relation satisfying that $|z| > |z'|$. It follows from Proposition 4.3 that each π_i is appears in z and each ξ_j appears in z'. Therefore there exist factorizations $a, b \in \mathsf{Z}(R^\bullet)$ such that

$$z = \pi_1^{a_1}\pi_2^{a_2}\cdots\pi_\ell^{a_\ell} a \quad \text{and } z' = \xi_1^{b_1}\xi_2^{b_2}\cdots\xi_s^{b_s} b$$

for some $a_1, \dots, a_\ell, b_1, \dots, b_s \in \mathbb{N}$ and $a, b \in R$ such that none of the π_i's appears in a and none of the ξ_j's appears in b. In addition, as the factorization is irredundant, none of the π_i's appears in b and none of the ξ_j's appears in a. We now derive contradictions in the following three cases.

CASE 1: $\ell, s \ge 2$. Consider the element

$$x := \pi_2^{a_2}\pi_3^{a_3}\cdots\pi_\ell^{a_\ell}(\pi_1^{a_1} - \xi_1^{b_1})a.$$

We claim that $x \neq 0$. Because R contains no nonzero zero-divisors, verifying that $x \neq 0$ amounts to showing that $\pi_1^{a_1} - \xi_1^{b_1} \neq 0$. Indeed, this must be the case as, otherwise, $\ell \ge 2$ and $\pi_1, \pi_2 \in \mathscr{L}(R)$ would force π_2 to appear on the left side of the factorization relation $(\pi_1^{a_1}, \xi_1^{b_1})$. Hence both $\pi_1^{a_1} - \xi_1^{b_1}$ and x belong to $R^\bullet$. This, along with the fact that ξ_1 divides x, guarantees the existence of $w_0 \in \mathsf{Z}_R(\pi_1^{a_1} - \xi_1^{b_1})$ and $w_1 \in \mathsf{Z}(R^\bullet)$ such that $w := \pi_2^{a_2}\pi_3^{a_3}\cdots\pi_\ell^{a_\ell} w_0 a \in \mathsf{Z}_R(x)$ and $\xi_1 w_1 \in \mathsf{Z}_R(x)$. Now consider the factorization relation $(w, \xi_1 w_1) \in \ker \pi_R$.

CASE 1.1: ξ_1 does not appear in w. Since $\xi_1 \in \mathscr{S}(R)$, it follows that $|w| > |\xi_1 w_1|$. Then $\pi_1 \in \mathscr{L}(R)$ implies that π_1 must appear in w. Thus, π_1 divides $\pi_1^{a_1} - \xi_1^{b_1}$, which implies that π_1 divides $\xi_1^{b_1}$. As a result, there is a factorization relation $(\pi_1 w_1', \xi_1^{b_1}) \in \ker \pi_R$ for some $w_1' \in \mathsf{Z}(R^\bullet)$. Clearly, $(\pi_1 w_1', \xi_1^{b_1})$ is a non-diagonal factorization relation. Since $\xi_1 \in \mathscr{S}(R)$, it follows that $|\pi_1 w_1'| > |\xi_1^{b_1}|$. Therefore $(\pi_1 w_1', \xi_1^{b_1})$ is an unbalanced factorization relation in which ξ_2 does not appear. However, this contradicts that $\xi_2 \in \mathscr{S}(R)$.

CASE 1.2: ξ_1 appears in w. Because ξ_1 does not appear in a, we see that ξ_1 must appear in w_0. This implies that ξ_1 divides $\pi_1^{a_1} - \xi_1^{b_1}$ and, therefore, ξ_1 must divide $\pi_1^{a_1}$. Now we can follow an argument completely analogous to that we just used in CASE 1.1 to obtain the desired contradiction.

CASE 2: $\{\ell, s\} = \{1, k\}$ for some $k \in \mathbb{N}$. First, we assume that $\ell = 1$. Recall that $(z, z') = (\pi_1^{a_1} a, \xi_1^{b_1}\xi_2^{b_2}\dots\xi_s^{b_s} b)$. In this case, we also impose the condition that the exponent a_1 is the minimum number of copies of π_1 that can appear in any irredundant and unbalanced factorizations relation in $\ker \pi_R$. Using notation similar to that of CASE 1, we now set

$$x := \pi_1^{a_1-1}(\pi_1 - \xi_1^{b_1})a.$$

Notice that $x \neq 0$ as otherwise $\pi_1 = \xi_1^{b_1}$, which is clearly impossible. Since ξ_1 divides x, one can take $w_2 \in \mathsf{Z}_R(\pi_1 - \xi_1^{b_1})$ and $w_2' \in \mathsf{Z}(R)$ such that $(\pi_1^{a_1-1} w_2 a, \xi_1 w_2') \in \ker \pi_R$. It is clear that ξ_1 does not divide $\pi_1 - \xi_1^{b_1}$. As a result, ξ_1 does not appear in $\pi_1^{a_1-1} w_2 a$, which implies that $|\pi_1^{a_1-1} w_2 a| > |\xi_1 w_2'|$. By the minimality of a_1, we see that π_1 must appear in w_2. Thus, π_1 divides $\pi_1 - \xi_1^{b_1}$ and hence π_1 must also divide $\xi_1^{b_1}$. In this case, $(\pi_1 w_2'', \xi_1^{b_1}) \in \ker \pi_R$ for some $w_2'' \in \mathsf{Z}(R)$, which is a contradiction as ξ_2 does not appear in $\xi_1^{b_1}$. The case when $\ell > 1$ and $s = 1$ follows similarly. □

As a consequence of Theorem 6.1, we rediscover the main result of [20].

Corollary 6.2. *[20, Theorem 2.10] Let R be an integral domain. Then $R^\bullet$ is an OHFM if and only if R is a UFD.*

Proof. Clearly, if R is a UFD, then $R^\bullet$ is a UFM and, therefore, an OHFM. For the reverse implication, suppose that $R^\bullet$ is an OHFM. By Theorem 6.1, either $\mathscr{L}(R)$ is empty or $\mathscr{S}(R)$ is empty. Therefore $R^\bullet$ is not a proper OHFM, and so it is a UFM. Hence R is a UFD. □

6.2. **Examples of Dedekind Domains.** For a finite-rank monoid M, we have already seen that none of the conditions $\mathscr{L}(M) = \emptyset$ and $\mathscr{S}(M) = \emptyset$ implies the other one. We conclude the paper offering examples of Dedekind domains to illustrate that a similar statement holds in the context of atomic integral domains.

The celebrated Claborn's class group realization theorem [21, Theorem 7] states that for every abelian group G there exists a Dedekind domain D such that $\mathrm{Cl}(D) \cong G$. The following refinement of this result, due to Gilmer et al., will be crucial in our constructions.

Theorem 6.3. *[29, Theorem 8] Let G be a countably generated abelian group generated by $B \cup C$ with $B \cap C = \emptyset$ such that $B^* \cup C$ generates G as a monoid for each cofinite subset B^* of B. Then there exists a Dedekind domain D with class group G satisfying the following conditions:*

(1) the set $B \cup C$ consists of the classes of G containing maximal ideals;

(2) the C consists of the classes of G containing infinitely many maximal ideals;

(3) the set B consists of the classes of G containing finitely many maximal ideals.

Moreover, the number of maximal ideals in each class contained in B can be specified arbitrarily.

Further refinements of Claborn's theorem in the direction of Theorem 6.3 were given by Grams [33], Michel and Steffan [34], and Skula [35]. We are in a position now to exhibit a Dedekind domain with a purely short (resp., long) irreducible but no purely long (resp., short) irreducibles.

Example 6.4. In this example we will produce a Dedekind domain in which there is a purely short irreducible but no purely long irreducibles. To this end, consider a Dedekind domain D with class group $\mathrm{Cl}(D) \cong \mathbb{Z}/3\mathbb{Z}$. Since $\mathrm{Cl}(D)$ is finite and the class of $\mathrm{Cl}(D)$ corresponding to $2+3\mathbb{Z}$ generates $\mathrm{Cl}(D)$ as a monoid, one can invoke Theorem 6.3 to assume that the non-principal prime ideals of D distribute within $\mathrm{Cl}(D)$ in the following way. There is a unique prime ideal P in the generating class of $\mathrm{Cl}(D)$ corresponding to $1+3\mathbb{Z}$ and infinitely many prime ideals in the generating class corresponding to $2+3\mathbb{Z}$. Observe that we can separate any non-prime irreducible principal ideal I of D into the following three types, according to their (unique) factorizations into prime ideals:

$$(1)\;\; I = P^3, \qquad (2)\;\; I = PQ, \qquad \text{or} \qquad (3)\;\; I = Q_1Q_2Q_3,$$

where Q, Q_1, Q_2, and Q_3 are prime ideals in the class of $\mathrm{Cl}(D)$ corresponding to $2+3\mathbb{Z}$. By construction, there is only one irreducible principal ideal of type (1), namely, P^3.

Let a be an irreducible such that $(a) = P^3$. We will verify that $a \in \mathscr{S}(D)$. To do so, suppose that a appears in an irredundant factorization relation $(z_1, z_2) \in \ker \pi_D$, where

$$z_1 := (P^3)^k \prod_{i=1}^{m} (PQ_i) \prod_{j=1}^{n} (Q_{j,1}Q_{j,2}Q_{j,3}) \quad \text{and} \quad z_2 := \prod_{i=1}^{m'} (PQ'_i) \prod_{j=1}^{n'} (Q'_{j,1}Q'_{j,2}Q'_{j,3})$$

for some $k \in \mathbb{N}$, $m, n, m', n' \in \mathbb{N}_0$, and ideals $Q_i, Q'_i, Q_{j,i}$, and $Q'_{j,i}$ in the class corresponding to $2+3\mathbb{Z}$. We have abused the notation here by writing ideal factorizations of z_1 and z_2, but each ideal in parentheses is principal and irreducibly generated. As D is Dedekind, comparing primes in the ideal factorizations of z_1 and z_2, one obtains that $3k+m+n = m'+n'$. As a result, $|z_1| = k+m+n = m'+n'-2k = |z'|-2k < |z_2|$. Therefore $a \in \mathscr{S}(D)$, as desired.

Let us proceed to argue that D contains no purely long irreducibles. By the previous paragraph, $\mathscr{L}(D)$ contains no irreducibles generating ideals of type (1). Take $a_2, a_3 \in \mathscr{A}(D)$ such that the ideals (a_2) and (a_3) are of type (2) and type (3), respectively. Then take prime ideals $Q_1, \dots, Q_5$ in the ideal class of $\mathrm{Cl}(D)$ corresponding to $2+3\mathbb{Z}$ such that the equalities $(a_2) = PQ_1$ and $(a_3) = Q_2Q_3Q_4$ hold, and $Q_5 \notin \{Q_1, Q_2, Q_3, Q_4\}$. Now consider the ideal factorizations

$$\begin{aligned} z_1 &:= (P^3)(Q_2Q_3Q_4), \\ z_2 &:= (PQ_2)(PQ_3)(PQ_4), \\ z_3 &:= (P^3)(PQ_1)(Q_3Q_4Q_5)(Q_5^3), \text{and} \\ z_4 &:= (PQ_5)^4(Q_1Q_3Q_4). \end{aligned}$$

Notice that $(z_1, z_2) \in \ker \pi_D$ is irredundant and satisfies that $|z_1| < |z_2|$. Because $Q_2Q_3Q_4$ appears in z_1, it follows that $a_3 \notin \mathscr{L}(D)$. On the other hand, $(z_3, z_4) \in \ker \pi_D$ is also irredundant, and it satisfies that $|z_3| < |z_4|$. Because PQ_1 appears in z_3, we obtain that $a_2 \notin \mathscr{L}(D)$. As a result, no irreducible generating an ideal of type (2) or type (3) is purely long, whence $\mathscr{L}(D) = \emptyset$.

To complement Example 6.4, we conclude the paper constructing a Dedekind domain having a purely long irreducible but no purely short irreducibles.

Example 6.5. Let D be a Dedekind domain with $\mathrm{Cl}(D) \cong \mathbb{Z}$. Since $\mathrm{Cl}(D)$ is countable and the set $\{\pm 1\}$ generates $\mathrm{Cl}(D)$ as a monoid, we can assume in light of Theorem 6.3 that the non-principal prime ideals of D distribute within $\mathrm{Cl}(D)$ as follows. There is a unique prime ideal, which we denote by P, in the class of $\mathrm{Cl}(D)$ corresponding to $-2 \in \mathbb{Z}$; there is a unique prime ideal, which we denote by Q, in the class of $\mathrm{Cl}(D)$ corresponding to $2 \in \mathbb{Z}$; and there are infinitely many prime ideals in each of the classes corresponding to -1 and 1. We denote the prime ideals in the class corresponding to -1 by (annotated) N and the prime ideals in the class corresponding to 1 by (annotated) M. Notice that we can separate non-prime irreducible principal ideals I of D into the following four types, according to their (unique) factorizations into prime ideals:

$$(1)\ \ I = PQ, \qquad (2)\ \ I = PN_1N_2, \qquad (3)\ \ I = QM_1M_2, \qquad \text{or} \qquad (4)\ \ I = NM,$$

where N, N_1, N_2 belong to the class of $\mathrm{Cl}(D)$ corresponding to -1 and M, M_1, M_2 belong to the class of $\mathrm{Cl}(D)$ corresponding to 1.

Take $a \in \mathscr{A}(D)$ such that $(a) = PQ$, and let us prove that $a \in \mathscr{L}(D)$. To do this, suppose that (a) appears in an irredundant ideal factorization relation $(z_1, z_2) \in \ker \pi_D$, and write

$$z_1 := (PQ)^k \prod_{i=1}^{m} (PN_{i,1}N_{i,2}) \prod_{j=1}^{n} (QM_{j,1}M_{j,2}) \prod_{k=1}^{t} (N_kM_k)$$

and

$$z_2 := \prod_{i=1}^{m'} (PN'_{i,1}N'_{i,2}) \prod_{j=1}^{n'} (QM'_{j,1}M'_{j,2}) \prod_{k=1}^{t'} (N'_kM'_k)$$

for some $k, m, n, t, m', n', t' \in \mathbb{N}_0$. Since D is Dedekind, after comparing the numbers of copies of the prime ideals P and Q that appear in z_1 and z_2, one obtains that $m - m' = n - n' = -k$. In addition, after comparing the numbers of copies of prime ideals in the class corresponding to -1 that appear in z_1 and z_2, one obtains that $t - t' = -2(m - m') = 2k$. As a result,

$$|z_1| = k + m + n + t = (m' + n' + t') + k + (m - m') + (n - n') + (t - t') = |z'| + k > |z_2|.$$

Therefore we can conclude that $a \in \mathscr{L}(D)$, as desired.

Finally, let us verify that D contains no purely short irreducibles. Since any irreducible generator of PQ is purely long, D contains no purely short irreducibles of type (1). Take $a_2 \in \mathscr{A}(D)$ such that (a_2) has type (2), and then take prime ideals N_1, N_2 in the class of $\mathrm{Cl}(D)$ corresponding to -1 such that $(a_2) = PN_1N_2$. In addition, take distinct prime ideals N_3 and N_4 in the class of $\mathrm{Cl}(D)$ corresponding to -1 such that $N_3, N_4 \notin \{N_1, N_2\}$. Finally, take distinct prime ideals M_1 and M_2 in the class of $\mathrm{Cl}(D)$ corresponding to 1. Consider the ideal factorization relation $(z_1, z_2) \in \ker \pi_D$, where

$$z_1 := (PQ)(PN_1N_2)(M_1N_3)(M_2N_4) \qquad \text{and} \qquad z_2 := (PN_1N_3)(PN_2N_4)(QM_1M_2).$$

Observe that $(z_1, z_2) \in \ker \pi_D$ is an irredundant factorization relation satisfying $|z_1| > |z_2|$. Since PN_1N_2 appears in z_1, it follows that $a_2 \notin \mathscr{S}(D)$. As a result, no irreducible generating an ideal of type (2) can be purely short. In a similar manner, one can verify that no irreducible generating an ideal of type (3) is purely short. Now let a_4 be an irreducible of D such that (a_4) is a principal ideal of type (4). Take N and M in the classes of $\mathrm{Cl}(D)$ corresponding to -1 and 1, respectively, such that $(a_4) = NM$, and then consider the factorizations

$$z_3 := (PQ)(NM)^2 \quad \text{and} \quad z_4 := (PN^2)(QM^2).$$

Notice that $(z_3, z_4) \in \ker \pi_D$ is an irredundant factorization relation satisfying that $|z_3| > |z_4|$. Since NM appears in z_3, it follows that $a_4 \notin \mathscr{S}(D)$. Therefore none of the irreducibles generating ideals of type (4) is purely short. As a consequence, $\mathscr{S}(D) = \emptyset$.

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Department of Mathematics and Statistics, Sam Houston State University, Huntsville, TX 77341
Email address: scott.chapman@shsu.edu

Department of Mathematical Sciences, Clemson University, Clemson, SC 29634
Email address: jcoyken@clemson.edu

Department of Mathematics, MIT, Cambridge, MA 02139
Email address: fgotti@mit.edu

Department of Mathematics, The University of North Carolina, Chapel Hill, NC 27599
Email address: wwsmith@email.unc.edu