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# THE CATENARY AND TAME DEGREES ON A NUMERICAL MONOID ARE EVENTUALLY PERIODIC

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**Abstract**

Let $M$ be a commutative cancellative monoid. For $m$ a nonunit in $M$, the catenary degree of $m$, denoted $c(m)$, and the tame degree of $m$, denoted $t(m)$, are combinatorial constants that describe the relationships between differing irreducible factorizations of $m$. These constants have been studied carefully in the literature for various kinds of monoids, including Krull monoids and numerical monoids. In this paper, we show for a given numerical monoid $S$ that the sequences $\{c(s)\}_{s\in S}$ and $\{t(s)\}_{s\in S}$ are both eventually periodic. We show similar behavior for several functions related to the catenary degree which have recently appeared in the literature. These results nicely complement the known result that the sequence $\{\Delta(s)\}_{s\in S}$ of delta sets of $S$ also satisfies a similar periodicity condition.

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## 1. Introduction

Over the past 20 years, problems involving nonunique factorizations of elements in integral domains and commutative cancellative monoids have been widely popular in the mathematical literature (see [15] and its citation list). Much of this literature focuses on various combinatorial constants which describe in some sense how far these systems vary from the classical notion of unique factorization. While early work in this area focused on Krull domains and monoids (see [3, 4, 11, 13, 14, 16, 19]), many papers have recently considered these properties on numerical monoids (which are additive submonoids of the natural numbers). In particular, their elastic properties (see [8]), their delta sets (see [2, 5, 9]) and their catenary and tame degrees (see [1, 4, 6, 7, 12, 17, 18]) have been examined in detail. We take particular interest in the main result of [9], where, for a given numerical monoid $S$, the authors show that the sequence of delta sets $\{\Delta(s)\}_{s\in S}$ is eventually periodic. In this note, we prove an

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analogue of this result by showing that the similar sequences defined by the catenary degree, the tame degree and the various related forms of the catenary degree recently introduced in the literature (see [16]) are also eventually periodic. Our argument differs from the one offered in [9], as problems involving the catenary and tame degrees rely on the complete set of factorizations of an element, while those involving the delta sets are merely concerned with factorizations of differing lengths. We open in Section 2 with the necessary notation and definitions, and present our main result, with proofs, in Section 3.

## 2. Definitions and preliminaries

A *numerical monoid* $S$ is a cofinite additive submonoid of $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Both [20] and [21] are good general references on the subject. It is easy to show using elementary number theory that every numerical monoid has a unique minimal generating set. If these generators are $n_1, \ldots, n_k$ with $n_1 < n_2 < \cdots < n_k$, then we use the notation

$$S = \langle n_1, \ldots, n_k \rangle = \{a_1 n_1 + \cdots + a_k n_k \mid a_1, \ldots, a_k \in \mathbb{N}_0\}.$$

If $\gcd(n_1, \ldots, n_k) \neq 1$, then $\mathbb{N}_0 \setminus S$ is not finite, so we must have $\gcd(n_1, \ldots, n_k) = 1$. We call $k$ the *embedding dimension* of $s$. Since $\mathbb{N}_0 \setminus S$ is finite, there is a largest number in the complement of $S$, denoted $\mathcal{F}(S)$, and called the *Frobenius number* of $S$.

Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical monoid. For $s \in S$, let

$$Z(s) = \{(a_1, \ldots, a_k) \mid a_1 n_1 + \cdots + a_k n_k = s \text{ with each } a_i \in \mathbb{N}_0\}$$

be the *set of factorizations* of $s$ in $S$. We say that the *length* of $z = (a_1, \ldots, a_k) \in Z(s)$ is

$$|z| = a_1 + \cdots + a_k.$$

Set

$$\mathcal{L}(s) = \{|z| : z \in Z(s)\} = \{m_1, \ldots, m_l\},$$

where we assume that $m_1 < m_2 < \cdots < m_{l-1} < m_l$. The set $\mathcal{L}(s)$ is known as the *set of lengths* of $s$. The *delta set* of an element, denoted $\Delta(s)$, is the set containing the values of the difference of consecutive elements of $\mathcal{L}(s)$, that is,

$$\Delta(s) = \{m_{i+1} - m_i \mid 1 \leq i < l\}.$$

Let $z = (a_1, \ldots, a_k)$ and $z' = (b_1, \ldots, b_k) \in Z(s)$. We say that the greatest common divisor of $z$ and $z'$ is

$$\gcd(z, z') = (\min\{a_1, b_1\}, \ldots, \min\{a_k, b_k\}),$$

and we define the *distance* between $z$ and $z'$ as

$$d(z, z') = \max\{|z - \gcd(z, z')|, |z' - \gcd(z, z')|\}.$$

The distance function satisfies many of the usual properties of a metric; the interested reader can find these summarized in [15, Proposition 1.2.5]. If $Z' \subseteq Z(s)$, then set

$$d(z, Z') = \min\{d(z, z') \mid z' \in Z'\}.$$

A sequence

$$z = z_0, z_1, \dots, z_{n-1}, z_n = z'$$

of factorizations in $Z(s)$ is an *N-chain* if $d(z_i, z_{i+1}) \leq N$ for each $1 \leq i \leq n-1$. For $s \in S$, we define the *catenary degree* of $s$ (denoted $c(s)$) to be the minimal $N$ such that there is an $N$-chain between any two factorizations of $s$.

The *tame degree* of an element $t(s)$ is constructed as follows. For each $i \leq k$, let $Z^i(s) := \{(a_1, \dots, a_k) \in Z(s) \mid a_i \neq 0\}$. We further let

$$t_i(s) = \max_{z \in Z(s)} d(z, Z^i(s)) \quad \text{and} \quad t(s) = \max_{i \leq k} t_i(s).$$

Alternatively, we can say that $t(s)$ is the minimal number such that $d(z, Z^i(s)) \leq t(s)$ for all $z \in Z(s)$ and all $i \leq k$.

Three variations on the catenary degree have appeared in the literature (most recently in [16]; see also [19]). Their definitions are as follows.

(1) The *monotone catenary degree* of an element $c_{\mathrm{mon}}(s)$ is the minimal number such that for any $z, z' \in Z(s)$ with $|z| \leq |z'|$, there exists a $c_{\mathrm{mon}}(s)$-chain $z = z_1, z_2, \dots, z_k = z'$ with the added restriction that $|z_i| \leq |z_{i+1}|$.
(2) The *equivalent catenary degree* $c_{\mathrm{eq}}(s)$ of an element $s \in S$ is the minimal number such that given $z, z' \in Z(s)$ with $|z| = |z'|$, there exists a $c_{\mathrm{eq}}(s)$-chain $z = z_1, \dots, z_k = z'$ with the added restriction that $|z_i| = |z_{i+1}|$.
(3) We say that $a, b \in \mathcal{L}(s)$ (with $a < b$) are *adjacent* if $[a, b] \cap \mathcal{L}(s) = \{a, b\}$. Let $Z_l(s) = \{z \in Z(s) \mid |z| = l\}$. The *adjacent catenary degree* $c_{\mathrm{adj}}(s)$ of an element $s \in S$ is the minimal number such that $d(Z_a(s), Z_b(s)) \leq c_{\mathrm{adj}}(s)$ for all adjacent $a, b$.

We close this section by noting that computing done in connection with these results was run on the GAP numerical semigroups package [10]. Also, any undefined notation or definitions will be consistent with those used in the monograph [15].

## 3. Periodicity

Given a numerical monoid $S = \langle n_1, \dots, n_k \rangle$, we define $L(S) = \operatorname{lcm}\{n_1, \dots, n_k\}$. When there is no ambiguity, we shall simply write $L$. The remainder of this section will consist of a proof of our main result, which is as follows.

Theorem 3.1. *If $S = \langle n_1, \dots, n_k \rangle$ is a numerical monoid, then the sequences*

$$\{c(s)\}_{s \in S}, \quad \{t(s)\}_{s \in S}, \quad \{c_{\mathrm{mon}}(s)\}_{s \in S}, \quad \{c_{\mathrm{eq}}(s)\}_{s \in S}, \quad \text{and} \quad \{c_{\mathrm{adj}}(s)\}_{s \in S}$$

*are all eventually periodic with fundamental period a divisor of L.*

Let $S = \langle n_1, \ldots, n_k \rangle$ and suppose that $k = 2$. Using techniques from [6], one can readily verify that $t(s) = c(s) = n_2$ for large $s$ (see also [15, Example 3.1.6]). Moreover, it also follows for large $s$ that $c_{\text{adj}}(s) = c(s)$. Since for $k = 2$ we also have for all $s$ that $c_{\text{eq}}(s) = 0$ and $c_{\text{mon}}(s) = \max\{c_{\text{eq}}(s), c_{\text{adj}}(s)\} = c_{\text{adj}}(s) = c(s)$ (see [16, page 1003]), we can assume throughout the remainder of our paper that $k \geq 3$.

The proof of Theorem 3.1 will rely on the following basic sequencing lemma, whose proof is left to the reader.

Lemma 3.2. *Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical monoid and $f : S \to \mathbb{N}_0$ a function. If there exist positive integers $N$ and $M$ such that $s \in S$ and $s > M$ imply that $f(s - N) \geq f(s)$, then $\{f(s)\}_{s \in S}$ is eventually periodic with fundamental period a divisor of $N$.*

The following definition is critical to all of our remaining proofs.

Definition 3.3. Let $s$ be an element of a numerical monoid $S = \langle n_1, \ldots, n_k \rangle$ such that $s - L \in S$. For each $i$, with $1 \leq i \leq k$, define a map

$$\phi_i : Z(s - L) \to Z(s)$$

by

$$\phi_i : z \to z + \Big(0, \ldots, 0, \frac{L}{n_i}, 0, \ldots, 0\Big).$$

For each $i$, it is easy to verify that $\phi_i$ is *distance preserving* (that is, $d(z, z') = d(\phi_i(z), \phi_i(z'))$ for all $z, z' \in Z(s - L)$). In the next proposition, we describe the set $Z(s)$ in terms of the images under $\phi_i$ of $s - L$.

Proposition 3.4. *If $S = \langle n_1, \ldots, n_k \rangle$ and $s \in S$ are as in Definition 3.3 with $s \geq L(kn_k)$, then*

$$Z(s) = \bigcup_{i \leq k} \phi_i(Z(s - L)).$$

Proof. Let $(a_1, \ldots, a_k) \in Z(s)$. Then $\sum_{i=1}^{k} a_i n_i = s$. Observe that

$$kn_k \cdot \max_{i \leq k} a_i \geq \sum_{i=1}^{k} a_i n_i = s \geq L(kn_k).$$

If we denote $a_j = \max_{i \leq k} a_i$ and simplify, then $a_j \geq L > L/n_j$. So, we write $(a_1, \ldots, a_k) = (a_1, \ldots, a_j - (L/n_j), a_{j+1}, \ldots, a_k) + (0, \ldots, 0, L/n_j, 0, \ldots, 0)$. Hence, $(a_1, \ldots, a_k) = \phi_j(a_1, \ldots, a_j - (L/n_j), a_{j+1}, \ldots, a_k)$ and we conclude that $Z(s) = \bigcup_{i \leq k} \phi_i(Z(s - L))$, where the reverse inclusion is obvious. □

Proposition 3.4 leads to the following observations concerning the catenary and tame degrees of relatively large elements of a numerical monoid.

Theorem 3.5. *Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical monoid.*

(a) *If $s \in S$ with $s \geq \max\{L(kn_k), \mathcal{F}(S) + 2L + 1\}$, then $c(s - L) \geq c(s)$.*

(b) *If $s \in S$ with $s \geq \max\{L(kn_k), \mathcal{F}(S) + L + 1 + n_k\}$, then $t(s - L) \geq t(s)$.*

Proof. (a) Since $s \geq L(kn_k)$, we have $Z(s) = \bigcup_{i \leq k} \phi_i(Z(s-L))$ by Proposition 3.4. Let $j < l \leq k$. Then write $s = (s-2L) + (L/n_j)n_j + (L/n_l)n_l$. By hypothesis, we have $s - 2L \geq \mathcal{F}(S) + 1$. So, $Z(s-2L)$ is nonempty. Pick $(a_1, \ldots, a_k) \in Z(s-2L)$. We then observe that

$$\left(a_1, \ldots, a_j + \frac{L}{n_j}, a_{j+1}, \ldots, a_k\right) + \left(0, \ldots, \frac{L}{n_l}, 0, \ldots, 0\right) \in \phi_l(Z(s-L))$$

and

$$\left(a_1, \ldots, a_l + \frac{L}{n_l}, a_{l+1}, \ldots, a_k\right) + \left(0, \ldots, \frac{L}{n_j}, 0, \ldots, 0\right) \in \phi_j(Z(s-L))$$

represent the same factorization. Since $j, l$ were arbitrary, we know that the images $\phi_p(Z(s-L))$ have pairwise nontrivial intersection.

The $\phi_p$ are all distance-preserving maps, so they conserve catenary degree locally within their image. Pick $x, y \in Z(s)$. Then we have $x \in \phi_j(Z(s-L))$ for some $j \leq k$, and $y \in \phi_l(Z(s-L))$ for some $l \leq k$. Now we have two cases.

*Case 1.* $l = j$. There is nothing to do; there exists a path with sufficiently small catenary degree within $\phi_l(Z(s-L))$, by distance preservation.

*Case 2.* $l \neq j$. Then there exists some element $z \in \phi_l(Z(s-L)) \cap \phi_j(Z(s-L))$. We can move from $x$ to $z$ within $\phi_j(Z(s-L))$, and then from $z$ to $y$ within $\phi_l(Z(s-L))$. Each time we have sufficiently small catenary degree.

Thus, we have produced a $c(s-L)$-chain connecting $x$ and $y$. We conclude that $c(s-L) \geq c(s)$.

(b) Since $s \geq L(kn_k)$, we again have $Z(s) = \bigcup_{i \leq k} \phi_i(Z(s-L))$ by Proposition 3.4. Pick $z \in Z(s)$. By Proposition 3.4, we have $z = \phi_i(z') \in Z(s)$ for some $z' \in Z(s-L)$. For an arbitrary $j$, let $z'_j \in Z^j(s-L)$ be such that $d(z', z'_j) = d(z', Z^j(s-L))$. Let $z_j = \phi_i(z'_j) \in Z^j(s)$. Observe that

$$s - L - n_j \geq \mathcal{F}(S) + L + 1 + n_k - L - n_j \geq \mathcal{F}(S) + 1 + n_k - n_j \geq \mathcal{F}(S) + 1.$$

So, we have $Z^j(s-L) \neq \emptyset$, and thus $z'_j$ exists. We have

$$d(z, Z^j(s)) \leq d(z, z_j) = d(\phi_i(z'), \phi_i(z'_j)) = d(z', z'_j) \leq t(s-L).$$

Since $z$ and $j$ were arbitrary, let

$$d(z, Z^j(s)) = \max_{i \leq k} \max_{z'' \in Z(s)} d(z'', Z^i(s)) = t(s).$$

It follows that $t(s-L) \geq t(s)$. □

To approach periodicity for the related versions of the catenary degree, we will need some further results. Given a factorization $(a_1, \ldots, a_k)$ of $x$ with length $a$ and large values for all $a_i$, we will produce in Lemma 3.7 a new factorization of $x$ with length $a$. To begin with this process, pick some $i, j, k$ satisfying $1 \le i < j < l \le k$ and observe that

$$(\dots a_j - n_l, \dots, a_l + n_j, \dots) \tag{3.1}$$

is a factorization of $x$ with length $a + (n_j - n_l) = a - (n_l - n_j)$ and

$$(\dots a_i + n_j, \dots, a_j - n_i, \dots) \tag{3.2}$$

is a factorization of $x$ with length $a + (n_j - n_i)$. If we apply the exchange (3.1) $n_j - n_i$ times, and exchange (3.2) $n_l - n_j$ times, then we will produce a new factorization with length $a$. But, we need $a_j$ to be sufficiently large. For this reason, we need an additional definition.

Definition 3.6. Let $S = \langle n_1, \dots, n_k \rangle$ and assume that $k \ge 3$ throughout. Define

$$\omega(S) := \frac{L}{n_1} + \left\lceil \frac{L}{n_1 n_2} \right\rceil n_{k-1}(n_k - n_1).$$

When there is no ambiguity, this value will simply be denoted by $\omega$ and we call $\omega$ the *toppling number* of $S$.

Given Definition 3.6, we proceed with the previously promised lemma.

Lemma 3.7 (The toppling lemma). *Let $S$ be as in Definition 3.6 and suppose that $s \in S$. Let $z = (a_1, \dots, a_k) \in Z(s)$ with $a_j \ge \omega$ for some $j \ne 1, k$. For any $1 \le i, l \le k$ with $i, l \ne j$, there exists $z' \in Z(s)$, of the form*

$$\left(\dots, a_i + \left\lceil \frac{L}{n_1 n_2} \right\rceil [(n_l - n_j)n_j], \dots, a_j - \left\lceil \frac{L}{n_1 n_2} \right\rceil [n_j n_l - n_j n_i] \right.$$
$$\left. , \dots, a_l + \left\lceil \frac{L}{n_1 n_2} \right\rceil [(n_j - n_i)n_j], \dots \right),$$

*and $|z| = |z'|$.*

We refer to the process of changing $z$ into $z'$ in Lemma 3.7 as *toppling $a_j$ to $a_i$ and $a_l$*.

Proof. Observe that

$$\begin{aligned} |z'| &= |z| + \left\lceil \frac{L}{n_1 n_2} \right\rceil ((n_l - n_j)n_j - (n_j n_l - n_j n_i) + (n_j - n_i)n_j) \\ &= |z| + \left\lceil \frac{L}{n_1 n_2} \right\rceil (n_l n_j - n_j^2 - n_j n_l + n_j n_i + n_j^2 - n_i n_j) = |z| + \left\lceil \frac{L}{n_1 n_2} \right\rceil (0) = |z| \end{aligned}$$

and hence $|z| = |z'|$. Also,

$$\begin{aligned} \sum_{i=1}^{k} a'_i n_i &= \sum_{i=1}^{k} a_i n_i + \left\lceil \frac{L}{n_1 n_2} \right\rceil ((n_l - n_j)n_j n_i - (n_j n_l - n_j n_i)n_j + (n_j - n_i)n_j n_l) \\ &= \sum_{i=1}^{k} a_i n_i + \left\lceil \frac{L}{n_1 n_2} \right\rceil (n_l n_j n_i - n_j^2 n_i - n_j^2 n_l + n_j^2 n_i + n_j^2 n_l - n_i n_j n_l) \\ &= \sum_{i=1}^{k} a_i n_i + \left\lceil \frac{L}{n_1 n_2} \right\rceil (0) = \sum_{i=1}^{k} a_i n_i. \end{aligned}$$

So, $z$ and $z'$ are factorizations of the same element. Furthermore,

$$\begin{aligned} a_j - \left\lceil \frac{L}{n_1 n_2} \right\rceil [(n_j - n_i)n_l + (n_l - n_j)n_i] &\geq \omega - \left\lceil \frac{L}{n_1 n_2} \right\rceil n_j(n_l - n_i) \\ &\geq \omega - \left\lceil \frac{L}{n_1 n_2} \right\rceil n_{k-1}(n_k - n_1) \\ &= \frac{L}{n_1} + \left\lceil \frac{L}{n_1 n_2} \right\rceil n_{k-1}(n_k - n_1) - \left\lceil \frac{L}{n_1 n_2} \right\rceil n_{k-1}(n_k - n_1) = \frac{L}{n_1} > \frac{L}{n_j}. \end{aligned}$$

So, all of the coefficients are positive (that is, $z' \in Z(s)$). This completes the proof. □

Note that $z'$ as constructed in Lemma 3.7 is in the image of three maps. First, $z' \in \phi_j(Z(s - L))$ by the last calculation in the above proof. Moreover,

$$a_i + \left\lceil \frac{L}{n_1 n_2} \right\rceil [(n_l - n_j)n_j] \geq \left\lceil \frac{L}{n_1 n_2} \right\rceil [(1)n_2] \geq \frac{L}{n_1} \geq \frac{L}{n_i}$$

and so $z' \in \phi_i(Z(s - L))$. Similarly,

$$a_l + \left\lceil \frac{L}{n_1 n_2} \right\rceil [(n_j - n_i)n_j] \geq \left\lceil \frac{L}{n_1 n_2} \right\rceil [(1)n_2] \geq \frac{L}{n_1} \geq \frac{L}{n_l}$$

and so $z' \in \phi_l(Z(s - L))$. Thus,

$$z' \in \phi_j(Z(s - L)) \cap \phi_i(Z(s - L)) \cap \phi_l(Z(s - L)).$$

Lemma 3.7 now allows us to prove an analogue of Theorem 3.5 for the sequence $\{c_{\mathrm{eq}}(s)\}_{s \in S}$.

THEOREM 3.8. *Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical monoid and suppose that $s \in S$. If*

$$N = \left\lceil \frac{(k-1)\omega}{1 - (n_1/n_k)} \right\rceil$$

*and $s \geq Nkn_k$, then $c_{\mathrm{eq}}(s - L) \geq c_{\mathrm{eq}}(s)$.*

PROOF. Pick any two factorizations $z = (a_1, \ldots, a_k)$ and $z' = (b_1, \ldots, b_k)$ of $s$ of the same length. Observe for $a_j = \max_{i \leq k} a_i$ that

$$ka_j n_k \geq \sum_{i=1}^{k} a_i n_i = s \geq kNn_k$$

and so $a_j \geq N$. Hence, it is clear using the definitions that $a_j \geq N \geq \omega \geq L/n_j$. The same can be said for $b_l = \max_{i \leq k} b_i \geq L/n_l$.

*Case 1:* $j = l$. Then $z$ and $z'$ are factorizations in $\phi_{j=l}(Z(s - L))$, so there exists a $c_{\mathrm{eq}}(s - L)$-chain connecting them in $\phi_{j=l}(Z(s - L))$. Thus, $c_{\mathrm{eq}}(s) \leq c_{\mathrm{eq}}(s - L)$.

*Case 2:* $j \neq 1, k$ and $l \neq j$. Note that this case is symmetric to the case $l \neq 1, k$ and $j \neq l$. Observe that $z \in \phi_j(Z(s - L))$ and $z' \in \phi_l(Z(s - L))$.

*Case 2a:* $l < j$. Topple $z$ to some $z''$ by toppling $a_j$ to $a_l$ and $a_k$.

*Case 2b:* $l > j$. Topple $z$ to some $z''$ by toppling $a_j$ to $a_1$ and $a_l$.

Note that $z'' \in \phi_l(Z(s-L)) \cap \phi_j(Z(s-L))$. We can construct a $c_{\mathrm{eq}}(s-L)$-chain from $z$ to $z''$ in $\phi_j(Z(s-L))$, and then from $z''$ to $z'$ in $\phi_l(Z(s-L))$. Combining these chains, we have a $c_{\mathrm{eq}}(s-L)$ chain from $z$ to $z'$, so $c_{\mathrm{eq}}(s) \le c_{\mathrm{eq}}(s-L)$.

*Case 3: $a_1 \ge N$ and $b_k \ge N$ or $b_1 \ge N$ and $a_k \ge N$.* Without loss of generality, let $a_1 \ge N$ and $b_k \ge N$. We have

$$\sum_{i=1}^{k} a_i = \sum_{i=1}^{k} b_i \implies \sum_{i=1}^{k} a_i - b_k = \sum_{i=1}^{k-1} b_i$$
$$\implies \sum_{i=1}^{k} a_i - b_k \le (k-1)b_m \implies \frac{\sum_{i=1}^{k} a_i - b_k}{k-1} \le b_m,$$

where $b_m = \max_{i \ne k} b_i$. We also have

$$\sum_{i=1}^{k} a_i n_i = \sum_{i=1}^{k} b_i n_i \implies b_k = \frac{\sum_{i=1}^{k} a_i n_i - \sum_{i=1}^{k-1} b_i n_i}{n_k}$$
$$\implies b_k = \sum_{i=1}^{k} a_i \frac{n_i}{n_k} - \sum_{i=1}^{k-1} b_i \frac{n_i}{n_k}.$$

Combining these results,

$$\begin{aligned} b_m &\ge \frac{\sum_{i=1}^{k} a_i - (\sum_{i=1}^{k} a_i(n_i/n_k) - \sum_{i=1}^{k-1} b_i(n_i/n_k))}{k-1} \\ &\ge \frac{\sum_{i=1}^{k} a_i(1-(n_i/n_k)) + \sum_{i=1}^{k-1} b_i(n_i/n_k)}{k-1} \\ &\ge \frac{\sum_{i=2}^{k-1} a_i(1-(n_i/n_k)) + b_i(n_i/n_k)}{k-1} + \frac{a_1(1-(n_1/n_k)) + b_1(n_1/n_k)}{k-1} \\ &\ge \frac{a_1(1-(n_1/n_k))}{k-1} \ge \frac{N(1-(n_1/n_k))}{k-1} \ge \omega \end{aligned}$$

for some $1 \le m < k$. If $m = 1$, then both $a_1$ and $b_1 > \omega$, and we are in Case 1. If $m \ne 1$, then $b_m > \omega$ for some $m \ne 1, k$, and we are in Case 2. So, regardless of which pair of equal-length factorizations we choose, we can construct a $c_{\mathrm{eq}}(s)$-chain connecting them. We conclude that $c_{\mathrm{eq}}(s-L) \ge c_{\mathrm{eq}}(s)$, which completes the proof. □

We prove a version of Theorem 3.8 for the sequence $\{c_{\mathrm{adj}}(s)\}_{s\in S}$.

THEOREM 3.9. *Let $S = \langle n_1, \ldots, n_k\rangle$ be a numerical monoid and suppose that $s \in S$. Suppose further that*

$$N = \left\lceil \frac{(k-1)\omega + \Delta_{\max}}{1-(n_1/n_k)} \right\rceil,$$

*where $\Delta_{\max} = \max \Delta(S)$. If $s \ge kN$, then $c_{\mathrm{adj}}(s-L) \ge c_{\mathrm{adj}}(s)$.*

PROOF. We note that, by [5], $\Delta_{\max}$ is finite. Pick $a, b \in \mathcal{L}(s)$ that are adjacent. Then $a = b + \Delta$ for some $\Delta \in \Delta(s)$. It is sufficient to show that there exist $x, y \in Z(s)$ such that $|x| = a$, $|y| = b$ and $x, y \in \phi_i(Z(s - L))$ for some $i \leq k$. For this would imply that $Z_{a-(L/n_i)}(s - L)$ and $Z_{b-(L/n_i)}(s - L)$ are nonempty, so we can pick $p \in Z_{a-(L/n_i)}(s - L)$ and $q \in Z_{b-(L/n_i)}(s - L)$ such that $d(p, q) = d(Z_{a-(L/n_i)}(s - L), Z_{b-(L/n_i)}(s - L))$. Since $a - (L/n_i)$ and $b - (L/n_i)$ are adjacent, we get $d(Z_{a-(L/n_i)}(s - L), Z_{b-(L/n_i)}(s - L)) \leq c_{\mathrm{adj}}(s - L)$. Hence,

$$\begin{aligned} d(Z_a(s), Z_b(s)) &\leq d(\phi_i(p), \phi_i(q)) = d(p, q) \\ &= d(Z_{a-(L/n_i)}(s - L), Z_{b-(L/n_i)}(s - L)) \leq c_{\mathrm{adj}}(s - L). \end{aligned}$$

So, our goal is to show that there exist $x, y \in \phi_i(Z(s))$ with $x \in Z_a(s)$ and $y \in Z_b(s)$.

Pick $z_a = (a_1, \ldots, a_k) \in Z_a(s)$ and $z_b = (b_1, \ldots, b_k) \in Z_b(s)$. As before, $a_i \geq N$ and $b_j \geq N$ for some $i, j \leq k$. We break our argument into five cases.

*Case 1:* $i = j$. Then $z_a \in \phi_i(Z(s - L)) \cap Z_a(s)$ and $z_b \in \phi_i(Z(s - L)) \cap Z_b(s)$. This completes the argument for Case 1.

*Case 2:* $i \neq 1, k$. This case breaks into two subcases.

*Case 2a:* $j > i$. Topple $a_i$ to produce a factorization $(a'_1, \ldots, a'_k)$, where $a'_1 \geq L/n_1$, $a'_i \geq L/n_i$ and $a'_j \geq L/n_j$.

*Case 2b:* $j < i$. Topple $a_i$ to produce a factorization $(a'_1, \ldots, a'_k)$, where $a'_j \geq L/n_j$, $a'_i \geq L/n_i$ and $a'_k \geq L/n_k$.

We have $(a'_1, \ldots, a'_k) \in \phi_j(Z(s - L)) \cap Z_a(s)$. Since $z_b \in \phi_j(Z(s - L)) \cap Z_b(s)$, we are done with Case 2.

*Case 3:* $b_j \neq b_1, b_k$. This case also breaks into two subcases.

*Case 3a:* $i > j$. Topple $b_j$ to produce a factorization $(b'_1, \ldots, b'_k)$, where $b'_1 \geq L/n_1$, $b'_j \geq L/n_j$ and $b'_i \geq L/n_i$.

*Case 3b:* $i < j$. Topple $b_j$ to produce a factorization $(b'_1, \ldots, b'_k)$, where $b'_i \geq (L/n_i)$, $b'_j \geq L/n_j$ and $b'_k \geq L/n_k$.

We have $(b'_1, \ldots, b'_k) \in \phi_i(Z(s - L)) \cap Z_b(s)$. We also know that $(a_1, \ldots, a_k) \in \phi_i(Z(s - L)) \cap Z_a(s)$ by hypothesis. This completes Case 3.

*Case 4:* $i = k$ *and* $j = 1$. Set $a_m = \max_{i \neq k} a_i$. If $m = 1$, then $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ are both in the image of $\phi_1$ and we are done. Thus, we assume that $m > 1$. We have

$$\begin{aligned} \sum_{i=1}^{k} a_i = \sum_{i=1}^{k} b_i + \Delta \implies a_m(k-1) &\geq \sum_{i=1}^{k-1} a_i = \sum_{i=1}^{k} b_i - a_k + \Delta \\ \implies a_m &\geq \frac{\sum_{i=1}^{k} b_i - a_k + \Delta}{k-1}. \end{aligned}$$

We also get

$$\sum_{i=1}^{k} a_i n_i = \sum_{i=1}^{k} b_i n_i \quad \text{and} \quad a_k = \frac{\sum_{i=1}^{k} b_i n_i - \sum_{i=1}^{k-1} a_i n_i}{n_k} = \sum_{i=1}^{k} b_i \frac{n_i}{n_k} - \sum_{i=1}^{k-1} a_i \frac{n_i}{n_k}.$$

Combining the above two results, we get

$$\begin{aligned}
a_m &\geq \frac{\sum_{i=1}^{k} b_i - (\sum_{i=1}^{k} b_i(n_i/n_k) - \sum_{i=1}^{k-1} a_i(n_i/n_k)) + \Delta}{k-1} \\
\Longrightarrow\ a_m &\geq \frac{\sum_{i=1}^{k} b_i(1-(n_i/n_k)) + \sum_{i=1}^{k-1} a_i(n_i/n_k) + \Delta}{k-1} \\
&= \frac{\sum_{i=2}^{k-1} b_i(1-(n_i/n_k)) + a_i(n_i/n_k)}{k-1} + \frac{b_1(1-(n_1/n_k)) + a_1(n_1/n_k) + \Delta}{k-1} \\
&\geq \frac{b_1(1-(n_1/n_k)) + \Delta}{k-1} \\
&\geq \frac{N(1-(n_1/n_k)) + \Delta}{k-1} = \frac{\omega(k-1) + \Delta_{\max} + \Delta}{k-1} \geq \omega.
\end{aligned}$$

Topple $a_m$ to produce a factorization $(a'_1, \ldots, a'_k)$, with $a'_1 \geq L/n_1$, $a'_m \geq L/n_j$ and $a'_k \geq L/n_k$. We have $(a'_1, \ldots, a'_k) \in \phi_1(Z(s-L)) \cap Z_a(s)$. We also have $(b_1, \ldots, b_k) \in \phi_1(Z(s-L)) \cap Z_b(s)$ by hypothesis. This completes Case 4.

*Case 5: $i = 1$ and $j = k$.* We have

$$\sum_{i=1}^{k} a_i = \sum_{i=1}^{k} b_i + \Delta \Longrightarrow \sum_{i=1}^{k} a_i - b_k - \Delta = \sum_{i=1}^{k-1} b_i \leq (k-1)b_m \Longrightarrow b_m \geq \frac{\sum_{i=1}^{k} a_i - b_k - \Delta}{k-1},$$

where $b_m = \max_{i \neq k} b_i$. We also get

$$\sum_{i=1}^{k} a_i n_i = \sum_{i=1}^{k} b_i n_i \ \Longrightarrow\ b_k = \frac{\sum_{i=1}^{k} a_i n_i - \sum_{i=1}^{k-1} b_i n_i}{n_k} = \sum_{i=1}^{k} a_i \frac{n_i}{n_k} - \sum_{i=1}^{k-1} b_i \frac{n_i}{n_k}.$$

Combining the above two results, we get

$$\begin{aligned}
b_m &\geq \frac{\sum_{i=1}^{k} a_i - (\sum_{i=1}^{k} a_i(n_i/n_k) - \sum_{i=1}^{k-1} b_i(n_i/n_k)) - \Delta}{k-1} \\
\Longrightarrow\ b_m &\geq \frac{\sum_{i=1}^{k} a_i(1-(n_i/n_k)) + \sum_{i=1}^{k-1} b_i(n_i/n_k) - \Delta}{k-1} \\
&= \frac{\sum_{i=2}^{k-1} a_i(1-(n_i/n_k)) + b_i(n_i/n_k)}{k-1} + \frac{a_1(1-(n_1/n_k)) + b_1(n_1/n_k) - \Delta}{k-1} \\
&\geq \frac{a_1(1-(n_1/n_k)) - \Delta}{k-1} \geq \frac{N(1-(n_1/n_k)) - \Delta_{\max}}{k-1} \geq \omega.
\end{aligned}$$

Topple $b_m$ to produce a factorization $(b'_1, \ldots, b'_k)$, where $b'_1 \geq L/n_1$ and $b'_m \geq L/n_j$. Then $(b'_1, \ldots, b'_k) \in \phi_1(Z(s-L)) \cap Z_b(s)$. We also know that $(a_1, \ldots, a_k) \in \phi_1(Z(s-L)) \cap Z_a(s)$ by hypothesis. This completes Case 5.

We have covered all of the possible pairings of $i$ and $j$. We conclude that $d(Z_a(s), Z_b(s)) \le c_{\mathrm{adj}}(s - L)$. But, $a, b$ were arbitrary adjacent elements of $\mathcal{L}(s)$. It follows that $c_{\mathrm{adj}}(s) \le c_{\mathrm{adj}}(s - L)$. This completes the proof. □

We previously noted that $c_{\mathrm{mon}}(s) = \max\{c_{\mathrm{eq}}(s), c_{\mathrm{adj}}(s)\}$. From Theorems 3.8 and 3.9, we readily obtain the following result.

COROLLARY 3.10. *Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical monoid and suppose that $s \in S$. If*

$$N = \frac{(k-1)\omega + \Delta_{\max}}{1 - (n_1/n_k)}$$

*and $s \ge kN$, then $c_{\mathrm{mon}}(s - L) \ge c_{\mathrm{mon}}(s)$.*

Combining Theorems 3.5, 3.8 and 3.9 and Corollary 3.10 with Lemma 3.2 yields a proof of Theorem 3.1.

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