THE CATENARY DEGREES OF ELEMENTS IN NUMERICAL MONOIDS GENERATED BY ARITHMETIC SEQUENCES

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Abstract. We compute the catenary degree of elements contained in numerical monoids generated by arithmetic sequences. We find that this can be done by describing each element in terms of the cardinality of its length set and of its set of factorizations. As a corollary, we find for such monoids that the catenary degree becomes fixed on large elements. This allows us to define and compute the *dissonance number*- the largest element with a catenary degree different from the fixed value. We determine the dissonance number in terms of the arithmetic sequence's starting point and its number of generators.

1. Introduction

The study of the arithmetic of integral domains and monoids which fail to satisfy the Fundamental Theorem of Arithmetic has become a popular area of research over the past twenty years (see the monograph [11] for a survey of this area as well as an extensive bibliography). Of the various combinatorial constants studied in this field, the *catenary degree* (cf. Section 2) has been the subject of many recent papers in the literature (for example, see [1], [3], [5], [6], [9], [10], [12], and [13]). If $c(S)$ denotes the catenary degree of the monoid S, then in [6] the authors show the following for a numerical monoid generated by an arithmetic sequence.

Theorem 1.1. [6, Theorem 14] *Let $S = \langle a, a+d, \ldots, a+kd\rangle$ where $a, d,$ and k are positive integers, $\gcd(a,d) = 1$, and $1 \le k \le a-1$. Then*

$$c(S) = \left\lceil \frac{a}{k} \right\rceil + d.$$

The papers cited previously consider (as does Theorem 1.1) the computation of the catenary degree of an entire monoid. In this paper, we explore a different avenue and consider the catenary degrees of individual elements of a monoid. We note that this is similar in spirit to previous research with respect to the elasticity of a monoid in [7]. We show in Theorem 3.1 that aside from the value obtained in Theorem 1.1, the elements of such an S can take on only two other catenary degrees (namely 0 and 2) and completely characterize which elements take on which values. If $c(s)$ denotes the catenary degree of an individual element $s \in S$, then as a by-product of Theorem 3.1, we show that the sequence $\{c(s)\}_{s\in S}$ eventually becomes constant at the value $c(S)$ given in Theorem 1.1. We set the dissonance of S, denoted $\mathrm{dis}(S)$, equal to the largest $s \in S$ with $c(s) \neq c(S)$. In Theorem 4.3, we determine the dissonance of all numerical monoids covered by Theorem 1.1. We begin our work with a brief introduction in Section 2. Section 3 contains a proof of Theorem 3.1 and Section 4 a proof of Theorem 4.3. Any undefined notation concerning numerical monoids can be found in [15] and undefined

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terms concerning factorization theory in [11]. Calculations done to support this work were completed using the GAP numerical semigroups package [8]. We believe this paper is but a first step, as computing the complete set of catenary degrees in various classes of integral domains and monoids promises to be a challenging question.

2. Definitions and Preliminaries

A *numerical monoid* S is a co-finite additive submonoid of $\mathbb{N}_0$. A set of positive integers $n_1, \ldots, n_k$ is said to generate S if $S = \langle n_1, \ldots, n_k \rangle = \{a_1 n_1 + \cdots + a_k n_k \mid a_1, \ldots, a_k \in \mathbb{N}_0\}$. It easily follows from elementary number theory that every numerical monoid is finitely generated and in fact has a unique set of generators of minimal length. If k is the cardinality of this minimal set of generators, then we call k the *embedding dimension* of S. Moreover, co-finite additivity forces $\gcd(n_1, \ldots, n_k) = 1$. Since $\mathbb{N}_0 \setminus S$ is finite, the largest number in the complement of S has special algebraic properties and is called the *Frobenius number* of S, denoted $\mathcal{F}(S)$.

Let $S = \langle n_1, \cdots, n_k \rangle$ be a numerical monoid. We assume that $n_1 < n_2 < \cdots < n_k$. Hence, the minimal generating set $\{n_1, \ldots, n_k\}$ of S constitutes the set of irreducibles of S in the normal sense. For $s \in S$, let $Z(s)$ be the *set of factorizations* of s. We denote an arbitrary element $z \in Z(s)$ with the k-tuple of natural numbers $(a_1, \ldots, a_k)$, which represents the factorization $(a_1)n_1 + (a_2)n_2 + \ldots + (a_k)n_k$. We say that the **length** of a factorization $z \in Z(s)$ is

$$|z| = a_1 + \cdots + a_k.$$

The *set of lengths* of an element, denoted $\mathcal{L}(s)$, is the set containing the numerical values of the length of each factorizations of s, that is,

$$\mathcal{L}(s) = \{|z| : z \in Z(s)\}.$$

The *delta set* of an element, denoted $\Delta(s)$, is the set containing the differences of lengths of consecutive elements of $\mathcal{L}(s)$. That is, if $\mathcal{L}(s) = \{m_1, \ldots, m_t\}$ with $m_1 < m_2 < \cdots < m_t$, then

$$\Delta(s) = \{m_{i+1} - m_i \mid 1 \leq i < t\}.$$

The *delta set of* S is then defined as

$$\Delta(S) = \bigcup_{s \in S, s > 0} \Delta(s).$$

Let $z = (a_1, \ldots, a_k)$ and $z' = (b_1, \ldots, b_k) \in Z(s)$. We say that the *greatest common divisor (GCD)* of z and z' is

$$\gcd(z, z') = (\min\{a_1, b_1\}, \ldots, \min\{a_k, b_k\}),$$

and we define the *distance* between z and z' as

$$d(z, z') = \max\{|z - \gcd(z, z')|, |z' - \gcd(z, z')|\},$$

where the subtraction is taken component-wise. By [11, Proposition 1.2.5], this distance function yields a well-defined metric.

Definition 2.1. *Given two factorizations z and z' of $s \in S$, an N-**chain** connecting them is a sequence of factorizations*

$$z = z_0, z_1, \ldots, z_{n-1}, z_n = z'$$

such that each $z_i \in Z(s)$ and $d(z_i, z_{i+1}) \le N$ for all $i < n$. For $s \in S$, we define the **catenary degree** *of s (denoted $c(s)$) to be the minimal N such that there is an N-chain between any two factorizations of s. We define the catenary degree of the whole monoid as*

$$c(S) = \sup\{c(s) \mid s \in S\}.$$

Remark 2.2. We note that computing catenary degrees for elements in an embeddding dimension two monoid is essentially trivial. If $S = \langle a, b\rangle$ with $\gcd(a, b) = 1$ and $a < b$, then moving from one factorization to another is merely an application of the rule

$$\underbrace{b + \cdots + b}_{a \text{ times}} = \underbrace{a + \cdots + a}_{b \text{ times}}$$

(see [15, Example 8.22]). Thus the catenary degree of an element in S is either 0 or b, and can be described as follows.

$$c(s) = \begin{cases} 0 & \text{if } s < ab \\ b & \text{if } s = ab \\ 0 & \text{if } ab < s < 2ab - a - b \text{ and } s - ab \notin \langle a, b\rangle \\ b & \text{if } ab < s < 2ab - a - b \text{ and } s - ab \in \langle a, b\rangle \\ 0 & \text{if } s = 2ab - a - b \\ b & \text{if } 2ab - a - b < s. \end{cases}$$

3. The Catenary Degree of Elements in Numerical Monoids Generated By an Arithmetic Sequence

Throughout the remainder of our work, $S = \langle a, a + d, \ldots, a + kd\rangle$ is a numerical monoid with $1 < k < a$ and $\gcd(a, d) = 1$. When presented in this form, we assume that $\{a, a + d, \ldots, a + kd\}$ is the minimal generating set for S. Notice that if $a = 2$, then $k < a$ implies that we are in the two generator case. Since this is addressed in Remark 2.2, we assume $a > 2$. For monoids generated by an arithmetic sequence, the Frobenius number is known to be $\mathcal{F}(S) = (\lfloor \frac{a-2}{k} \rfloor + 1)a + (d - 1)(a - 1) - 1$ (see [14]). Moreover, by [4], $\Delta(S) = \{d\}$. The remainder of this section contains a proof of the following Theorem.

Theorem 3.1. *Given $S = \langle a, a + d, \ldots, a + kd\rangle$, where $\gcd(a, d) = 1$, $1 < k < a$, and $s \in S$, then*

$$c(s) = \begin{cases} 0 & \text{if } |Z(s)| = 1, \\ 2 & \text{if } |Z(s)| > 1 \text{ and } |\mathcal{L}(s)| = 1, \\ \lceil \frac{a}{k} \rceil + d & \text{if } |\mathcal{L}(s)| > 1. \end{cases}$$

We begin developing the machinery needed to prove Theorem 3.1 with a distance two factorization lemma. We will eventually deduce that all factorizations produced in this manner can be connected by chains where each step has distance two.

Lemma 3.2. *Let $s \in S$ and take $z \in Z(s)$, $z = (\rho_0, \ldots, \rho_k)$. If $\rho_i \neq 0$ and $\rho_j \neq 0$ for some $i, j \in \{0, \ldots, k\}$, $i < j-2$, then, $z' = (\rho_0, \ldots, \rho_i - 1, \rho_{i+1} + 1, \ldots, \rho_{j-1} + 1, \rho_j - 1, \ldots, \rho_k) \in Z(s)$ and $|z| = |z'|$. In addition, $d(z, z') = 2$.*

Proof. Clearly, $|z| = |z'|$. Now notice that

$$\begin{aligned}(\rho_i - 1)(a + id) + (\rho_{i+1} + 1)(a + (i + 1)d) + (\rho_{j-1} + 1)(a + (j - 1)d) + (\rho_j - 1)(a + jd) \\ = \rho_i(a + id) + \rho_{i+1}(a + (i + 1)d) + \rho_{j-1}(a + (j - 1)d) + \rho_j(a + jd).\end{aligned}$$

Since the other factors besides $\rho_i, \rho_{i+1}, \rho_{j-1}, \rho_j$ in z are the same as the ones in z', we can say that $z' \in Z(s)$. Also, since

$$\gcd(z, z') = (\rho_0, \dots, \rho_i - 1, \rho_i, \dots, \rho_{j-1}, \rho_j - 1, \dots, \rho_k)$$

we have that

$$d(z, z') = \max\{|z - \gcd(z, z')|, |z' - \gcd(z, z')|\} = 2. \quad \square$$

Lemma 3.3. *Let $s \in S$ and take $z = (\rho_0, \dots, \rho_k) \in Z(s)$. If $\rho_i \neq 0$ and $\rho_j \neq 0$ for some $i, j \in \{0, \dots, k\}$, $i = j - 2$, then, $z' = (\rho_0, \dots, \rho_i - 1, \rho_{i+1} + 2, \rho_j - 1, \dots, \rho_k) \in Z(s)$ and $|z| = |z'|$. In addition, $d(z, z') = 2$.*

Proof. Clearly, $|z| = |z'|$. Recall that $j = i + 2$. Now notice that

$$(\rho_i - 1)(a + id) + (\rho_{i+1} + 2)(a + (i+1)d) + (\rho_{i+2} - 1)(a + (i+2)d)$$

$$= \rho_i(a + id) + \rho_{i+1}(a + (i+1)d) + \rho_{i+2}(a + (i+2)d).$$

Since the other factors besides $\rho_i, \rho_{i+1}, \rho_{i+2}$ in z are the same as the ones in z', we can say that $z' \in Z(s)$. Also, since

$$\gcd(z, z') = (\rho_0, \dots, \rho_i - 1, \rho_{i+1}, \rho_{i+2} - 1, \dots, \rho_k)$$

we have that

$$d(z, z') = \max\{|z - \gcd(z, z')|, |z' - \gcd(z, z')|\} = 2. \quad \square$$

Lemma 3.4. *Let $s \in S$ and take $z \in Z(s)$. Then, there exists $z' \in Z(s)$ such that $|z| = |z'|$ and z' has at most two nonzero entries at i and j such that $j = i + 1$ or $i = j$, where $j \in \{0, \dots, k\}$. In addition, there exists a 2-chain between z and z'.*

Proof. Let $z = (\rho_0, \dots, \rho_k)$. Take the smallest i such that $\rho_i \neq 0$. Similarly, take the maximum j such that $\rho_j \neq 0$. If $i = j$ or $i = j - 1$, then the proof is complete.

Suppose $i = j - 2$. By Lemma 3.3 we have that there exists $z_1 \in Z(s)$ with $|z_1| = |z|$ with the following structure

$$z_1 = (0, \dots, \rho_i - 1, \rho_{i+1} + 2, \rho_j - 1, \dots, 0).$$

Suppose $i < j - 2$. By Lemma 3.2 we have that there exists $z_1 \in Z(s)$ with $|z_1| = |z|$ with the following structure

$$z_1 = (0, \dots, \rho_i - 1, \rho_{i+1} + 1, \rho_{i+2}, \dots, \rho_{j-1} + 1, \rho_j - 1, \dots, 0).$$

Observe that by applying Lemma 3.2 or Lemma 3.3 $\min\{\rho_i, \rho_j\}$-times, we will obtain other factorizations with the same length as z in which either the i-coordinate or the j-coordinate is zero. Thus, if $i \neq j$ and $i \neq j - 1$, we can always create a new factorization with the same length as z in which the element's nonzero coordinates are indexed closer together. After finitely many iterations, it will be reduced to a factorization z' with the desired properties. In addition, in each application of Lemmas 3.2 and 3.3, $d(z, z_1) = 2$. After finitely many applications, we have created 2-chain from z to z'. $\square$

Lemma 3.5. *Let $s \in S$ and take $z, z' \in Z(s)$, with $|z| = |z'|$. If for some $i, j \in \{0, \dots, k - 1\}$ we have that $z = (0, \dots, \rho_i, \rho_{i+1}, \dots, 0)$ and $z' = (0, \dots, \beta_j, \beta_{j+1}, \dots, 0)$, where ρ_i and β_j are nonzero, then $z = z'$.*

Proof. If $p_{i+1} = \beta_{j+1} = 0$, then the result is trivial. So assume that at least one is nonzero.

Assume without loss of generality that $i \geq j$. Since $|z|= |z'|$,

$$\rho_i + \rho_{i+1} = \beta_j + \beta_{j+1}. \tag{1}$$

In addition,

$$\begin{aligned}\rho_i(a + id) + \rho_{i+1}(a + (i + 1)d) = \beta_j(a + jd) + \beta_{j+1}(a + (j + 1)d) \implies \\ (\rho_i + \rho_{i+1})a + (i\rho_i + i\rho_{i+1} + \rho_{i+1})d = (\beta_j + \beta_{j+1})a + (j\beta_j + j\beta_{j+1} + \beta_{j+1})d.\end{aligned} \tag{2}$$

By (1), we can eliminate the a's from (2), which results in:

$$\begin{aligned}(i\rho_i + i\rho_{i+1} + \rho_{i+1})d = (j\beta_j + j\beta_{j+1} + \beta_{j+1})d \implies \\ i\rho_i + i\rho_{i+1} + \rho_{i+1} = j\beta_j + j\beta_{j+1} + \beta_{j+1} \implies \\ i(\rho_i + \rho_{i+1}) + \rho_{i+1} = j(\beta_j + \beta_{j+1}) + \beta_{j+1}.\end{aligned} \tag{3}$$

By (1), we have that (3) can be transformed to

$$(i - j)(\rho_i + \rho_{i+1}) + \rho_{i+1} = \beta_{j+1}. \tag{4}$$

If $i = j$, then by (4), $\rho_{i+1} = \beta_{j+1}$, and so $\rho_i = \beta_i$. Moreover, $z = z'$.

Now if $i \neq j$, then $i - j \geq 1$. Let $i - j = m$. We have that

$$m(\rho_i + \rho_{i+1}) + \rho_{i+1} \quad = \quad \beta_{j+1}. \tag{5}$$

Substituting (5) into (1), we get

$$m\rho_i + m\rho_{i+1} + \rho_{i+1} + \beta_j \quad = \quad \rho_i + \rho_{i+1}.$$

Cancelling common terms, we get

$$m\rho_i + m\rho_{i+1} + \beta_j \quad = \quad \rho_i.$$

This is a contradiction as we assumed that $m > 0$ and $\beta_j \neq 0$. Therefore $i = j$ and thus $z = z'$. □

Lemma 3.6. *Let $s \in S$ and take $z, z' \in Z(s)$, with $|z|= |z'|$. Then, there exists a 2-chain from z to z'.*

Proof. By Lemma 3.4 we have that there exists $f, f' \in Z(s)$ such that $|f|= |z|$, $|f'|= |z'|$, both have at most two nonzero consecutive entries, and there exists a 2-chain from f to z as well as from f' to z'. In addition, by Lemma 3.5, we have that $f' = f$. Then, there exists a 2-chain from z to z'. □

Finally, we can state the much anticipated first theorem.

Theorem 3.7. *Let $s \in S$. Then, $c(s) = 2$ if and only if $|Z(s)|> 1$ and $|\mathcal{L}(s)|= 1$*

Proof. ($\Rightarrow$) The proof in this direction follows immediately from [11, Lemma 1.6.2].

($\Leftarrow$) Now suppose $|Z(s)|> 1$ and all the factorizations of s have the same length. Let us show that $c(s) = 2$. Take two arbitrary factorizations of s, say $z, z' \in Z(s)$. Notice that we can take two different factorizations because $|Z(s)|> 1$. By Lemma 3.6, there exists a 2-chain from z to z'. Clearly, $c(s) = 2$. □

To complement the last result, we consider what happens when the set of factorizations yields more than one length.

Theorem 3.8. *If $s \in S$ with $|\mathcal{L}(s)| > 1$, then $c(s) = c(S)$.*

Proof. Let $s \in S$ with $|\mathcal{L}(s)| > 1$. Consider an N-chain of minimal value N connecting two elements of different length. Such a chain must also have a link between elements of different length. Take $z, z' \in Z(s)$ such that $d(z, z') \leq N$ to be such a link. Denote $z = (a_0, a_1, a_2, \cdots, a_k)$ and $z' = (b_0, b_1, b_2, \cdots, b_k)$. Without loss of generality assume that $|z| > |z'|$. For simplicity, for each $0 \leq i \leq k$ set $n_i = a + id$. We can say that

$$s = \sum_{i=0}^{k} a_i n_i = \sum_{i=0}^{k} b_i n_i.$$

Let $y = z - gcd(z, z') = (y_0, y_1, \ldots, y_k)$ and $y' = z' - gcd(z, z') = (y'_0, y'_1, \ldots, y'_k)$. Observe that $d(z, z') = \max\{|y|, |y'|\} = |y|$. Also, $|y'| = |z' - gcd(z, z')|$.

Notice that since $|Z(s)| > 1$ and $\Delta(S) = \{d\}$ (see [4]), we have $|z| = |z'| + qd$ where $q \in \mathbb{N}$. Since y and y' are factorizations of the same element, we have $|y| = |y'| + qd$. So, we obtain

$$\sum_{i=0}^{k} y'_i n_k \geq \sum_{i=0}^{k} y'_i n_i = \sum_{i=0}^{k} y_i n_i \geq \sum_{i=0}^{k} y_i n_1$$

which implies that

$$\sum_{i=0}^{k} y'_i n_k \geq \sum_{i=0}^{k} y_i n_1 = n_1(\sum_{i=0}^{k} y'_i + qd).$$

Thus,

$$\sum_{i=0}^{k} y'_i n_k \geq n_1 \sum_{i=0}^{k} y'_i + n_1 qd$$

which implies that

$$\sum_{i=0}^{k} y'_i (n_k - n_1) \geq n_1 qd.$$

Now, $|y'| = \sum_{i=0}^{k} y'_i \geq \frac{n_1 qd}{kd}$ which implies that $|y'| \geq \lceil \frac{a}{k} q \rceil$ and so

$$|y| = \sum_{i=0}^{k} y_i \geq \left\lceil \frac{a}{k} \cdot q \right\rceil + q \cdot d \geq \left\lceil \frac{a}{k} \right\rceil + d = c(S).$$

Therefore, $d(z, z') = |y| \geq c(S)$. Since by definition $c(s) \leq c(S)$, the result follows. □

The piecewise representation of $c(s)$ as represented in Theorem 3.1 now follows.

4. The Dissonance Number

As in the previous section, we continue assuming that $S = \langle a, a+d, \ldots, a+kd \rangle$ with $1 < k < a$ and $\gcd(a, d) = 1$. Moreover, $\mathcal{F}(S) = (\lfloor \frac{a-2}{k} \rfloor + 1)a + (d-1)(a-1) - 1$ and $c(S) = \lceil \frac{a}{k} \rceil + d$. In our calculations below, residues (i.e., $a \mod b$) are always computed as least positive residues.

Proposition 4.1. *If $s \in S$ with $s > a \cdot c(S) + \mathcal{F}(S)$, then $c(s) = c(S)$. Thus the sequence $\{c(s)\}_{s \in S}$ is eventually constant.*

Proof. If $v > 0$ is an integer, then clearly $s = a \cdot c(S) + \mathcal{F}(S) + v$ is in S and has at least one factorization in terms of irreducibles in S. Now, by [6, Lemma 13], $a \cdot c(S)$ can be represented in terms of irreducibles in at least two different ways. One as $c(S)$ many copies of a and other as

$$a \cdot c(S) = \begin{cases} \frac{a}{k}(a+kd) & \text{if } k \mid a \\ \lfloor \frac{a}{k} \rfloor (a+kd) + (a + (a \mod k)d) & \text{otherwise.} \end{cases}$$

In either case, $\frac{a}{k}$ or $\lfloor \frac{a}{k} \rfloor + 1$ are strictly less than $c(S)$, so $|\mathcal{L}(s)| > 1$. Therefore, by Theorem 3.8 we can conclude that $c(s) = c(S)$. The second statement now follows. □

Based on Proposition 4.1, we make the following definition.

Definition 4.2. *If $s \in S$ is the biggest element in S such that $c(s) \neq c(S)$, then we call s the* **dissonance** *of S and we denote it by $\mathrm{dis}(S) = s$.*

From Proposition 4.1 it is clear that $\mathrm{dis}(S) \leq a \cdot c(S) + \mathcal{F}(S)$. In this section, we compute the dissonance as follows.

Theorem 4.3.

$$\mathrm{dis}(S) = \begin{cases} a \cdot c(S) + \mathcal{F}(S) & \text{if } 1 \leq k < 2 + [a-1 \bmod k] + [a-2 \bmod k] \\ a \cdot c(S) + \mathcal{F}(S) - a & \text{if } k \geq 2 + [a-1 \bmod k] + [a-2 \bmod k]. \end{cases}$$

Note that by Remark 2.2, when $k = 1$ and $S = \langle a, b \rangle$, then $\mathrm{dis}(S) = 2ab - a - b$, which matches the value in the formula above. Hence, we can assume throughout the remainder of our work that $k > 1$ and we can freely use the results of Section 3. In the next theorem, we begin to verify the second part of this equality.

Theorem 4.4. $\mathrm{dis}(S) < a \cdot c(S) + \mathcal{F}(S)$ *if and only if* $k \geq 2 + (a-1 \bmod k) + (a-2 \bmod k)$.

Proof. By Lemma 3.8, we know if an element $s \in S$ has $|\mathcal{L}(s)| > 1$, then $c(s) = c(S)$. Thus, we will look at when $a \cdot c(S) + \mathcal{F}(S)$ has $|\mathcal{L}(s)| > 1$. From Lemma 2.1 in [2], for any $s \in S$ there exist $c_1, c_2 \in \mathbb{N}$ and $0 \leq c_2 < a$ such that $s = c_1 a + c_2 d$. So let $s = a \cdot c(S) + \mathcal{F}(S)$, and simplify to

$$s = \left(\left\lceil \frac{a}{k} \right\rceil + \left\lfloor \frac{a-2}{k} \right\rfloor + d \right) a + (a-1)d.$$

By Theorem 2.2 in [2], we know $|\mathcal{L}(s)| > 1$ if and only if $\dfrac{c_2 - c_1 k}{a + kd} \leq -1$. Therefore,

$$(a-1) - k \left(\left\lceil \frac{a}{k} \right\rceil + \left\lfloor \frac{a-2}{k} \right\rfloor + d \right) \leq -a - kd \text{ implies that } 2a - 1 - k \left\lceil \frac{a}{k} \right\rceil - k \left\lfloor \frac{a-2}{k} \right\rfloor \leq 0.$$

Hence, $2a - 1 \leq k \left\lceil \frac{a}{k} \right\rceil + k \left\lfloor \frac{a-2}{k} \right\rfloor$ implies that

$$2a - 1 \leq k \left\lfloor \frac{a-1}{k} \right\rfloor + k + k \left\lfloor \frac{a-2}{k} \right\rfloor. \tag{6}$$

We know that,

$$\begin{aligned} k \left\lfloor \frac{a-1}{k} \right\rfloor + k + k \left\lfloor \frac{a-2}{k} \right\rfloor &= (a-1) - [a-1 \bmod k] + k + (a-2) - [a-2 \bmod k] \\ &= 2a - 3 - [a-1 \bmod k] - [a-2 \bmod k] + k. \end{aligned}$$

Therefore,

$$\begin{aligned} 2a-1 &\leq 2a-3-[a-1 \bmod k]-[a-2 \bmod k]+k \Rightarrow \\ k &\geq 2+[a-1 \bmod k]+[a-2 \bmod k]. \end{aligned}$$

Thus, if a, k satisfies the above inequality then $\text{dis}(S) < a\cdot c(S)+\mathcal{F}(\mathcal{S})$ because $|\mathcal{L}(a\cdot c(S)+\mathcal{F}(\mathcal{S}))| > 1$. Every statement in this proof is reversible, so the proof is complete. □

In the case that the dissonance number is not $a\cdot c(S)+\mathcal{F}(S)$, we next provide a lower bound for the dissonance number.

Corollary 4.5. *For S satisfying $\text{dis}(S) < a\cdot c(S)+\mathcal{F}(S)$, it follows that $c(a\cdot c(S)+\mathcal{F}(S)-a)\neq c(S)$.*

Proof. Assume that $|\mathcal{L}(a\cdot c(S)+\mathcal{F}(S)-a)| > 1$. We have that

$$\begin{aligned} s &= a\cdot c(S)+\mathcal{F}(S)-a = a\left(\left\lceil\frac{a}{k}\right\rceil+d\right)+a\left(\left\lfloor\frac{a-2}{k}\right\rfloor+1\right)+(d-1)(a-1)-1-a \\ &= a\left(\left\lceil\frac{a}{k}\right\rceil+\left\lfloor\frac{a-2}{k}\right\rfloor+d-1\right)+(a-1)d. \end{aligned}$$

By Theorem 2.2 in [2] and our assumption that $|\mathcal{L}(s)| > 1$, we have that

$$a-1-k(\left(\left\lceil\frac{a}{k}\right\rceil+\left\lfloor\frac{a-2}{k}\right\rfloor+d-1\right)\leq -a-kd$$

and hence

$$2a-1-k\left\lfloor\frac{a-1}{k}\right\rfloor-k-k\left\lfloor\frac{a-2}{k}\right\rfloor+k\leq 0. \tag{7}$$

We know,

$$\begin{aligned} k\left\lfloor\frac{a-1}{k}\right\rfloor+k\left\lfloor\frac{a-2}{k}\right\rfloor &= (a-1)-[a-1 \bmod k]+(a-2)-[a-2 \bmod k] \\ &= 2a-3-[a-1 \bmod k]-[a-2 \bmod k]. \end{aligned} \tag{8}$$

Now, substituting (8) into (7) and simplifying, we get

$$2+[a-1 \bmod k]+[a-2 \bmod k] \leq 0$$

which is a contradiction. Hence, we have proved that $a\cdot c(S)+\mathcal{F}(S)-a$ has factorizations of only one length and thus its catenary degree cannot be $c(S)$. □

This next lemma will provide us with a necessary condition for any of these 'in between' numbers to be the dissonance number.

Lemma 4.6. *If $c(a\cdot c(S)+\mathcal{F}(S)-v)=0$ or 2 for $0<v<a$, then $\left\lceil\frac{a-1}{k}\right\rceil=\left\lfloor\frac{a-1}{k}+\frac{1}{2}\right\rfloor$.*

Proof. Notice that $\mathcal{F}(S)\equiv -d \pmod a$. So,write $ma+nd=a\cdot c(S)+\mathcal{F}(S)-v$, where $0\leq n<a$. Clearly, $nd\equiv -d-v \pmod a$. Thus, $(n+1)d+v\equiv 0 \pmod a$. Suppose that $n=a-1$. Then we get $v\equiv 0 \pmod a$, which contradicts our bounds on v. Hence, $0\leq n\leq a-2$.

Since $d(n+1)+v\equiv 0 \pmod a$, there exists $l\in\mathbb{N}$ such that $la=d(n+1)+v\leq d(a-1)+(a-1)=(d+1)(a-1)$. Then, $l\leq\frac{(d+1)(a-1)}{a}$, and $l\in\mathbb{N}$, so $l\leq\left\lfloor\frac{(d+1)(a-1)}{a}\right\rfloor$.

We observe that $\left\lfloor \frac{(d+1)(a-1)}{a} \right\rfloor \leq d$. Reorganizing and then plugging in for $c(S)$ and for $\mathcal{F}(S)$, we see that

$$m = \frac{a \cdot c(S) + \mathcal{F} - v - nd}{a} = \frac{(\lceil \frac{a}{k} \rceil + d)a + [(\lfloor \frac{a-2}{k} \rfloor + 1)a + (d-1)(a-1) - 1] - v - nd}{a}$$
$$= \left\lceil \frac{a}{k} \right\rceil + 2d + \left\lfloor \frac{a-2}{k} \right\rfloor - \frac{d(n+1)+v}{a}.$$

Observe that $\left\lceil \frac{a}{k} \right\rceil + \left\lfloor \frac{a-2}{k} \right\rfloor = \left\lceil \frac{a-1}{k} \right\rceil + \left\lfloor \frac{a-1}{k} \right\rfloor$, and that $\frac{d(n+1)+v}{a} = l \leq d$ as above. So we can write

$$m = \left\lceil \frac{a-1}{k} \right\rceil + \left\lfloor \frac{a-1}{k} \right\rfloor + 2d - l \geq \left\lceil \frac{a-1}{k} \right\rceil + \left\lfloor \frac{a-1}{k} \right\rfloor + d.$$

If $c(a \cdot c(S) + \mathcal{F} - v) = 0$ or 2, then $\mathcal{L}(a \cdot c(S) + \mathcal{F} - v)$ contains one integer. Then by Theorem 2.2 in [2], we have that $\left\lceil \frac{n-mk}{a+kd} \right\rceil = 0$. Then $\frac{n-mk}{a+kd} > -1$. Rearranging, we get

$$m < \frac{a+kd+n}{k} \leq \frac{a+kd+(a-2)}{k} = \frac{2(a-1)}{k} + d.$$

So $m \leq \left\lfloor \frac{2(a-1)}{k} \right\rfloor + d = \left\lfloor \frac{a-1}{k} \right\rfloor + \left\lfloor \frac{a-1}{k} + \frac{1}{2} \right\rfloor + d$. Combining our two results so far, we get the inequality

$$\left\lceil \frac{a-1}{k} \right\rceil + \left\lfloor \frac{a-1}{k} \right\rfloor + d \leq m \leq \left\lfloor \frac{a-1}{k} \right\rfloor + \left\lfloor \frac{a-1}{k} + \frac{1}{2} \right\rfloor + d.$$

So $\lceil \frac{a-1}{k} \rceil \leq \lfloor \frac{a-1}{k} + \frac{1}{2} \rfloor$. Note that it is impossible for this inequality to be strict. We obtain that

$$\left\lceil \frac{a-1}{k} \right\rceil = \left\lfloor \frac{a-1}{k} + \frac{1}{2} \right\rfloor. \quad \square$$

Using Lemma 4.6 and its proof, we prove the next theorem by contradiction.

Theorem 4.7. *Suppose* $\mathrm{dis}(S) < a \cdot c(S) + \mathcal{F}(S)$, *and let* $s \in \{a \cdot c(S) + \mathcal{F}(S) - v \mid 0 < v < a\} \subseteq S$. *Then* $c(s) = c(S)$.

Proof. We use the notation of Lemma 4.6. From its proof, we know that

$$m \geq \left\lceil \frac{(a-1)}{k} \right\rceil + \left\lfloor \frac{(a-1}{k} \right\rfloor + d$$

as well as

$$m < \frac{2(a-1)}{k} + d = \frac{(2a-1)}{k} - \frac{1}{k} + d.$$

Now assume $\mathrm{dis}(S) < ac(S) + \mathcal{F}(S)$. By (6)

$$\frac{2(a-1)}{k} \leq \left\lfloor \frac{(a-1)}{k} \right\rfloor + \left\lfloor \frac{(a-2)}{k} \right\rfloor + 1.$$

Thus it follows immediately that

$$\left\lceil \frac{(a-1)}{k} \right\rceil < \left\lfloor \frac{(a-2)}{k} \right\rfloor + (1 - \frac{1}{k}).$$

This is a contradiction, because $\left\lceil \frac{(a-1)}{k} \right\rceil \geq \left\lfloor \frac{(a-2)}{k} \right\rfloor + 1$. Thus, the proof is complete. $\square$

Theorem 4.4 verifies the first part of the formula for $\operatorname{dis}(S)$ in Theorem 4.3. A combination of Corollary 4.5 and Theorem 4.7 verifies the second part, completing the proof of Theorem 4.3.

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