On the Catenary Degrees of Elements in Numerical Monoids Generated by an Arithmetic Sequence

Scott Chapman

Sam Houston State University

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The Student Authors

This talk is based on the work completed under my direction at the 2013 PURE REU at the University of Hawaii at Hilo by the following students.

Marly Corrales, University of Southern California

Andrew Miller, Amherst College

Chris Miller, The University of Wisconsin at Madison

Dhir Patel, Rutgers University

Prologue

This talk is based the papers:

[1] S.T. Chapman, M. Corrales, A. Miller, C. Miller, and D. Patel, The Catenary and Tame Degrees on a Numerical Monoid are Eventually Periodic, *J. Aust. Math. Soc.* **97**(2014), 289–300.

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More information and background on this area can be found in:

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The Notation of Factorization Theory

Throughout, we assume that M is a commutative cancellative monoid. Unless otherwise noted, we write the operation of M multiplicatively and hence represent its identity element by 1_M.

We use the standard notation of divisibility theory; if x and y are in M and there exists c in M with $cx = y$, then $x \mid y$.

Denote by

$$M^\times = \{u \in M \mid uv = 1_M \text{ for some } v \in M\}$$

the set of units of M. The *irreducibles* (or *atoms*) of M are denoted $\mathcal{A}(M)$, where

$$\mathcal{A}(M) = \{x \in M \backslash M^\times \mid x = rs \text{ with } r, s \in M \text{ implies } r \in M^\times \text{ or } s \in M^\times\}.$$

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$$\mathcal{A}(M) = \{x \in M \backslash M^{\times} \mid x = rs \text{ with } r, s \in M \text{ implies } r \in M^{\times} \text{ or } s \in M^{\times}\}.$$

Notation

The monoid M is *atomic* if every element of $M \backslash M^{\times} = M^{\bullet}$ posses a factorization into elements of $\mathcal{A}(M)$.

Two elements x and y in $\mathcal{A}(M)$ are called *associates* if there exists a unit $u \in M^{\times}$ such that $x = uy$. If x and y are associates, then we write $x \simeq y$.

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Even More Notation

Consider factorizations of the form

$$x = x_1 \cdots x_k = y_1 \cdots y_t$$

which may not be unique.

If $x \in M^{\bullet}$, then *the set of lengths of x* is

$$\mathcal{L}(x) = \{k \in \mathbb{N} \mid x = a_1 a_2 \cdots a_k \text{ where } a_i \in \mathcal{A}(M(a,n))\}.$$

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The Notation Continues Coming

Set

$$\ell(x) = \min \mathcal{L}(x) \text{ and } L(x) = \max \mathcal{L}(x).$$

If $\mathcal{L}(x) = \{n_1, \ldots, n_t\}$ with the n_i's listed in increasing order, then set

$$\Delta(x) = \{n_i - n_{i-1} \mid 2 \leq i \leq t\}$$

and

$$\Delta(M) = \bigcup_{1 \neq x \in M^\bullet} \Delta(x).$$

If $\Delta(M) \neq \emptyset$, then,

$$\min \Delta(M) = \gcd \Delta(M).$$

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The Elasticity of Factorization

The elasticity of an element $x \in M^\bullet$, denoted $\rho(x)$, is given by

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The *elasticity of M* is then defined as

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Numerical Monoids

Let S be an additive submonoid of $\mathbb{N} \cup \{0\}$. S is called a *numerical monoid*.

If $\{n_1, \ldots, n_t\}$ is a set of elements of S such that every $x \in S$ can be written in the form

$$x = x_1 n_1 + \cdots x_t n_t$$

then $\{n_1, \ldots, n_t\}$ is called a *generating set of* S.

This is commonly denoted by

$$S = \langle n_1, \ldots, n_t \rangle.$$

It follows from Elementary Number Theory that every numerical monoid S possesses a unique minimal set of generators. If $\gcd\{ s \mid s \in S\} = 1$, then S is called *primitive*. It again follows easily from Number Theory that every numerical monoid S is isomorphic to a primitive numerical monoid.

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What is known?

Much is known about the factorization properties of numerical monoids.

- If $S = \langle n_1, \ldots, n_k \rangle$, then $\rho(S) = \frac{n_k}{n_1}$.
- Many results are known about $\Delta(S)$, one of the nicest being the following.

Theorem

If $S = \langle n_1, \ldots, n_k \rangle$ is a primitive numerical monoid, with $n_1 < n_2 < \cdots < n_k$, then for all $x \geq 2kn_2n_k^2$ we have $\Delta(x) = \Delta(x + n_1 n_k)$.

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Introduction

Let S be a numerical monoid minimally generated by $\{n_1, \ldots, n_p\}$. Consider the monoid homomorphism

$$\varphi : \mathbb{N}^p \to S,\ \varphi(a_1, \ldots, a_p) = a_1 n_1 + \cdots + a_p n_p,$$

known as the factorization morphism of S.

The *set of factorizations* of an element $n \in S$ is

$$\mathbb{F}(n) = \varphi^{-1}(n) = \{(a_1, \ldots, a_p) \in \mathbb{N}^p \mid a_1 n_1 + \cdots + a_p n_p = n\}.$$

Let $(a_1, \ldots, a_p) \in \mathbb{F}(n)$. The *length* of the factorization $a = (a_1, \ldots, a_p)$ is $|a| = a_1 + \cdots + a_p$.

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More Definitions

For $z = (z_1, \ldots, z_p), z' = (z'_1, \ldots, z'_p) \in \mathbb{N}^p$ write

$$\gcd(z, z') = (\min\{z_1, z'_1\}, \ldots, \min\{z_p, z'_p\}),$$

and

$$\frac{z}{z'} = z - z'.$$

Define

$$\mathrm{d}(z, z') = \max\left\{\left|\frac{z}{\gcd(z, z')}\right|, \left|\frac{z'}{\gcd(z, z')}\right|\right\},$$

to be the *distance* between z and z'. If $Z' \subseteq Z(s)$, then set

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The Distance is Amazing

The distance function acts as a metric. The following for a numerical monoid can easily be shown (and are in fact true in general).

Theorem

Let z_1, z_2 and z_3 be factorizations of x in a numerical monoid S.

1. $\mathrm{d}(z_1, z_2) = 0$ *if and only if* $z_1 = z_2$.
2. $\mathrm{d}(z_1, z_2) = \mathrm{d}(z_2, z_1)$.
3. $\mathrm{d}(z_1, z_2) \leq \mathrm{d}(z_1, z_3) + \mathrm{d}(z_3, z_2)$.
4. $\mathrm{d}(z_3 z_1, z_3 z_2) = \mathrm{d}(z_1, z_2)$.
5. $\mathrm{d}(z_1^k, z_2^k) = k\mathrm{d}(z_1, z_2)$.

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More Definitions

Given $n \in S$ and $z, z' \in \mathbb{F}(n)$, an *N-chain of factorizations* from z to z' is a sequence $z_0, \ldots, z_k \in \mathbb{F}(n)$ such that $z_0 = z$, $z_k = z'$ and $\mathrm{d}(z_i, z_{i+1}) \leq N$ for all i.

The *catenary degree* of n, $\mathrm{c}(n)$, is the minimal $N \in \mathbb{N} \cup \{\infty\}$ such that for any two factorizations $z, z' \in \mathbb{F}(n)$, there is an N-chain from z to z'.

The catenary degree of S, $\mathrm{c}(S)$, is defined by

$$\mathrm{c}(S) = \sup\{\mathrm{c}(n) \mid n \in S\}.$$

Note: If S does not have unique factorization, then $\mathrm{c}(S) \geq 2$ and if $\mathrm{c}(S) = 2$, then S is half-factorial.

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Comments

- There is a method to compute the catenary degree of a numerical monoid. In general, it is difficult to apply.
- The GAP Numerical Monoid Package can compute both the catenary degree of an element in a Numerical Monoid, and the catenary degree of the entire monoid.

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Variations on the Catenary Degree

1. The *monotone catenary degree* of an element $c_{\text{mon}}(s)$ is the minimal number such that for any $z, z' \in Z(s)$ with $|z| \leq |z'|$, there exists a $c_{\text{mon}}(s)$-chain $z = z_1, z_2 \dots z_k = z'$ with the added restriction that $|z_i| \leq |z_{i+1}|$.
2. The *equivalent catenary* degree $c_{\text{eq}}(s)$ of an element $s \in S$ is the minimal number such that given $z, z' \in Z(S)$ with $|z| = |z'|$, there exists a $c_{\text{eq}}(s)$-chain $z = z_1, \dots z_k = z'$ with the added restriction that $|z_i| = |z_{i+1}|$.
3. We say that $a, b \in \mathcal{L}(s)$ (with $a < b$) are *adjacent* if $[a, b] \cap \mathcal{L}(s) = \{a, b\}$. Let $Z_l(s) = \{z \in Z(s) || z| = l\}$. The *adjacent catenary degree* $c_{\text{adj}}(s)$ of an element $s \in S$ is the minimal number such that $d(Z_a(s), Z_b(s)) \leq c_{\text{adj}}(s)$ for all adjacent a, b.

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1. The *monotone catenary degree* of an element $c_{\mathrm{mon}}(s)$ is the minimal number such that for any $z, z' \in Z(s)$ with $|z| \leq |z'|$, there exists a $c_{\mathrm{mon}}(s)$-chain $z = z_1, z_2 \ldots z_k = z'$ with the added restriction that $|z_i| \leq |z_{i+1}|$.
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Main Theorem

Theorem

If $S = \langle n_1, \cdots, n_k \rangle$ is a numerical monoid, then the sequences

$$\{c(s)\}_{s \in S}, \{c_{mon}(s)\}_{s \in S}, \{c_{eq}(s)\}_{s \in S}, \text{ and } \{c_{adj}(s)\}_{s \in S}$$

are all eventually periodic with fundamental period a divisor of L.

Ideas behind the Proofs

Lemma

Let $S = \langle n_1, \cdots, n_k \rangle$ be a numerical monoid and $f : S \to \mathbb{N}_0$ a function. If there exists positive integers N and M such that $s \in S$ and $s > M$ implies that $f(s - N) \geq f(s)$, then $\{f(s)\}_{s \in S}$ is eventually periodic with fundamental period a divisor of N.

A Needed Definition

Definition

Let s be an element of a numerical monoid $S = \langle n_1, ...n_k \rangle$ such that $s - L \in S$. For each i, with $1 \le i \le k$, define a map

$$\varphi_i : Z(s - L) \to Z(s)$$

by

$$\varphi_i : z \to z + (0, ..., 0, \frac{L}{n_i}, 0, ...0).$$

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A Key Proposition

Proposition

If $S = \langle n_1, ...n_k \rangle$ *and* $s \in S$ *are as in Definition 5 with* $s \geq L(kn_k)$*, then*

$$Z(s) = \bigcup_{i \leq k} \varphi_i(Z(s-L)).$$

Individual Values

Theorem

Given $S = \langle a, a+d \ldots a+kd \rangle$, *where* $\gcd(a,d) = 1$, $1 < k < a$, *and* $s \in S$, *then*

$$c(s) = \begin{cases} 0 & \textit{if } |Z(s)| = 1, \\ 2 & \textit{if } |Z(s)| > 1 \textit{ and } |\mathcal{L}(s)| = 1, \\ \left\lceil \frac{a}{k} \right\rceil + d & \textit{if } |\mathcal{L}(s)| > 1. \end{cases}$$

The Dissonance

Theorem

If $s \in S$ with $s > a \cdot c(S) + \mathcal{F}(S)$, then $c(s) = c(S)$. Thus the sequence $\{c(s)\}_{s\in S}$ is eventually constant.

Definition

If $s \in S$ is the biggest element in S such that $c(s) \neq c(S)$, then we call s the **dissonance** of S and we denote it by $\mathrm{dis}(S) = s$.

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The Dissonance

Theorem

$$\mathrm{dis}(S) = \begin{cases} a \cdot c(S) + \mathcal{F}(S) & \textit{if } 1 \leq k < 2 + [a-1 \bmod k] + [a-2 \bmod k] \\ a \cdot c(S) + \mathcal{F}(S) - a & \textit{if } k \geq 2 + [a-1 \bmod k] + [a-2 \bmod k]. \end{cases}$$