A Proof of the Two-path Conjecture

Herbert Fleischner
Institute of Discrete Mathematics
Austrian Academy of Sciences
Sonnenfelsgasse 19
A-1010 Vienna
Austria, EU
herbert.fleischner@oeaw.ac.at

Robert R. Molina
Department of Mathematics and Computer Science
Alma College
614 W. Superior St.
Alma MI, 48801
molina@alma.edu

Ken W. Smith
Department of Mathematics
Central Michigan University
Mt. Pleasant, MI 48859
ken.w.smith@cmich.edu

Douglas B. West
Department of Mathematics
University of Illinois
1409 W. Green St.
Urbana, IL 61801-2975
west@math.uiuc.edu

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Abstract

Let $G$ be a connected graph that is the edge-disjoint union of two paths of length $n$, where $n \geq 2$. Using a result of Thomason on decompositions of 4-regular graphs into pairs of Hamiltonian cycles, we prove that $G$ has a third path of length $n$.

The “two-path conjecture” states that if a graph $G$ is the edge-disjoint union of two paths of length $n$ with at least one common vertex, then the graph has a third subgraph that is also a path of length $n$. For example, the complete graph $K_4$ is an edge-disjoint union of two paths of length 3, each path meeting the other in four vertices. The cycle $C_6$ is the edge-disjoint union of two paths of length 3 with common endpoints. In the first case, the graph has twelve paths of length 3; in the second there are six such paths.

The two-path conjecture arose in a problem on randomly decomposable graphs. An $H$-decomposition of a graph $G$ is a family of edge disjoint $H$-subgraphs of $G$ whose union is $G$. An $H$-decomposable graph $G$ is randomly $H$-decomposable if any edge disjoint family of $H$-subgraphs of $G$ can be extended to an $H$-decomposition of $G$. (This concept was introduced by Ruiz in [7].)

Randomly $P_n$-decomposable graphs were studied in [1, 5, 6, 4]. In attempting to classify randomly $P_n$-decomposable graphs, in [5] and [6] it was necessary to know whether the edge-disjoint union of two copies of $P_n$ could have a unique $P_n$-decomposition. The two-path conjecture is stated as an unproved lemma in [3].

Our notation follows [2]. A path of length $n$ is a trail with distinct vertices $x_0, \ldots, x_n$ ([2], p. 5). We say that $G$ decomposes into subgraphs $X$ and $Y$ when $G$ is the edge-disjoint union of $X$ and $Y$.

**Theorem.** If $G$ decomposes into two paths $X$ and $Y$, each of length $n$ with $n \geq 2$, and $X$ and $Y$ have least one common vertex, then $G$ has a path of length $n$ distinct from $X$ and $Y$.

**Proof.** Label the vertices of $X$ as $x_0, x_1, \ldots, x_n$, with $x_{i-1}$ adjacent to $x_i$ for $1 \leq i \leq n$. Similarly, label the vertices of $Y$ as $y_0, y_1, \ldots, y_n$. Let $s$ be the number of common vertices; thus $G$ has $2n + 2 - s$ vertices.

If $s = 1$, then we may assume by symmetry that $x_i = y_j$ with $i \geq j$ and $i \geq 1$ and $j < n$. In this case, the vertices $x_0, \ldots, x_i, y_{j+1}, \ldots y_n$ form
a path of length at least $n$ having a subpath of length $n$ different from $X$ and $Y$.

Similarly, if $s = 2$, then we may let the common vertices be $x_{i_1}, x_{i_2}$ and $y_{i_1}, y_{i_2}$ with $x_{i_1} = y_{j_1}$ and $x_{i_2} = y_{j_2}$. Using symmetry again, we may assume that $i_1 < i_2, j_1 < j_2$, and $i_1 \geq j_1$. With this labeling, again the vertices $x_0, \ldots, x_{i_1}, y_{j_1+1}, \ldots y_n$ form a path with a subpath of length $n$ different from $X$ and $Y$.

Hence we may assume that $s \geq 3$. The approach above no longer works, since now the points of intersection need not occur in the same order on $X$ and $Y$. Suppose first that the intersection contains an endpoint of one of the paths. We may assume that $x_0 = y_k$ for some $k$ with $k < n$. Now we consider two cases. If $y_{k+1}$ is not a vertex of $X$, then we replace the edge $x_{n-1}x_n$ with the edge $y_{k+1}x_0$ to create a third path of length $n$. If $y_{k+1} = x_i$ for some $i$, then we replace the edge $x_ix_{i-1}$ with the edge $y_{k+1}x_0$ to create a new path of length $n$.

Therefore, we may assume that $s \geq 3$ and that none of $\{x_0, x_n, y_0, y_n\}$ is among the $s$ shared vertices. We apply a result of Thomason ([8], Theorem 2.1, pages 263-4): If $H$ is a regular multigraph of degree 4 with at least 3 vertices, then for any two edges $e$ and $f$ there are an even number of decompositions of $H$ into two Hamiltonian cycles $C_1$ and $C_2$ with $e$ in $C_1$ and $f$ in $C_2$.

From the given graph $G$, we construct a 4-regular multigraph $H$. We first add the edges $e_0 = x_0x_n$ and $f_0 = y_0y_n$. We then “smooth out” all vertices of degree 2; that is, we iteratively contract edges incident to vertices of degree 2 until no such vertices remain. Since every vertex of $G \cup \{e_0, f_0\}$ has degree 2 or degree 4, the resulting multigraph $H$ is regular of degree 4. Since $s \geq 3$, $H$ has at least three vertices.

In $H$, the edge $e_0$ is absorbed into an edge $e$, and $f_0$ is absorbed into an edge $f$. The cycles $X \cup \{e_0\}$ and $Y \cup \{f_0\}$ have been contracted to become Hamiltonian cycles in $H$. Together they decompose $H$. By the theorem of Thomason, there is another Hamiltonian decomposition $C_1, C_2$ of $H$ with $e$ in $C_1$ and $f$ in $C_2$.

Now we reverse our steps. Restore the vertices of degree 2 and remove the edges $e_0$ and $f_0$. The cycle $C_1$ becomes a path from $x_0$ to $x_n$, and $C_2$ becomes a path from $y_0$ to $y_n$. Neither of these paths is the original $X$.
or $Y$. Since $G$ has $2n$ edges and is the edge-disjoint union of these two paths, one of the paths has length at least $n$. It contains a new path of length $n$.

References


