Euclid’s book *The Elements* (in 300 BC!) introduces a “geometric progression” as a progression in which the ratio of any element to the previous element is a constant. (Geometric progressions are the main topic of Book VIII of *The Elements*.)

The Greeks, over two thousand years ago, considered sequences such as

\[
\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^k}, \ldots
\]

and their sums, such as

\[
\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + \frac{1}{2^k} + \ldots
\]

The sum, above, at any finite step, is always less than the number 1. The partial sums have the form

\[
\sum_{k=1}^{m} \frac{1}{2^k} = 1 - \frac{1}{2^m}.
\]

Since the sum is less than 1 at any finite step but eventually exceeds any number greater than 1, we can conclude that the series converges in the limit to 1.

Note that this series has the property that each term is exactly \(\frac{1}{2}\) of the previous term and so it is “geometric”. Geometric series abound in science and mathematics.

**Introduction**

A sequence \(a, ar, ar^2, \ldots, ar^k, \ldots\) in which each new term is exactly \(r\) times the previous term, is said to be a geometric sequence with *ratio* \(r\).

For example, the sequence \(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^k}, \ldots\) is a geometric sequence with ratio \(\frac{1}{2}\). We are interested in finding values for *sums* such as

\[
\sum_{n=1}^{10} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + \frac{1}{1024}.
\]

or even

\[
\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + \frac{1}{2^k} + \ldots
\]

If we are adding up the terms of the geometric sequence then we have a geometric sum or geometric series. (See [http://en.wikipedia.org/wiki/Geometric_series](http://en.wikipedia.org/wiki/Geometric_series) for a general discussion of these series, including modern applications.)

In general, a geometric series has form

\[
\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \ldots + ar^{k-1} + \ldots
\]

**Geometric series are nice!**

There is a nice way to work out a formula for the partial sum of a geometric series. Suppose \(a\) is the first term of a geometric sum and \(r\) is the common ratio between terms. Let us write out the sum of the first \(m\) terms:

\[
S := a + ar + ar^2 + ar^3 + \ldots + ar^{m-1} = \sum_{k=0}^{m-1} ar^k.
\]
Notice that
\[ rS = ar + ar^2 + ar^3 + \ldots + ar^{m-1} + ar^m = \sum_{k=1}^{m} ar^k \]
and so
\[ rS - S = ar^m - a. \]
(Notice how most terms cancel!)

If we solve for \( S \), we factor out \( r - 1 \) and then divide by it to get:
\[ S = \frac{a(r^m - 1)}{r - 1} = \frac{a(1 - r^m)}{1 - r}. \]

Therefore the sum of the first \( m \) terms of a geometric sum is
\[ S = \frac{a(1 - r^m)}{1 - r}. \]

What about the infinite series \( \sum_{k=1}^{\infty} ar^k \)?

The series \( \sum_{k=1}^{\infty} ar^k \) is the limit, as \( m \) goes to infinity, of the finite sum \( \frac{a(1 - r^m)}{1 - r} \). If \( |r| < 1 \) then the expression \( r^m \) converges to zero and so \( \frac{a(1 - r^m)}{1 - r} \) converges to \( \frac{a}{1 - r} \). Therefore the geometric series converges to \( \frac{a}{1 - r} \) and we get a legitimate value to the infinite sum. For example, one can check that the geometric series starting at 1 with ratio \( r = \frac{1}{2} \) converges, so
\[ \sum_{k=0}^{\infty} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + \frac{1}{2^k} + \ldots = \frac{1}{1 - \frac{1}{2}} = 2 \]

On the other hand, if \( |r| > 1 \) then the expression \( \frac{a(1 - r^m)}{1 - r} \) does not converge and so the geometric series does not converge. If \( r = 1 \) then the expression \( \frac{a(1 - r^m)}{1 - r} \) is undefined but it is easy to check that the partial sums are \( S_m = ma \) and so the series diverges to infinity. If \( r = -1 \) then the \( \frac{a(1 - r^m)}{1 - r} \) does not converge and so the geometric series does not converge. The partial sums alternate between \( a \) and 0.

This result is important enough to emphasize as a theorem.

**Theorem (Limit of a geometric series)**

If \( |r| < 1 \) then
\[ a + ar + ar^2 + ar^3 + \ldots + ar^{k-1} + \ldots = \sum_{k=0}^{\infty} ar^k = \frac{a}{1 - r}. \]

But if \( |r| \geq 1 \) then \( \sum_{k=0}^{\infty} ar^k \) diverges.

The geometric series are central to the study of infinite series. Indeed, even if a series is not geometric, we may attempt (in a certain way) to “pretend” it is geometric anyway! Sometimes this “pretense” gives us very useful information. (This motivates the “ratio” and “root” tests.)
Some applications of geometric series

Geometric series seem to underlie many applications of infinite series. Here are some of those applications.

First, we give a fairly elementary application. Suppose a number, written in decimal notation, repeats a block of digits. Then the geometric series viewpoint allows us to write the decimal as a fraction! For example, the number with repeating digits

0.23 23 23... = 0.\overline{23}

is a rational number. Can we prove this? Can we explicitly write out the rational number as a fraction of two integers?

The number 0.232323... = \frac{23}{100} + \frac{23}{100^2} + ... represents a geometric series with first term \(a = \frac{23}{100}\) and ratio \(r = \frac{1}{100}\). Therefore it converges to

\[
\frac{\frac{23}{100}}{1 - \frac{1}{100}} = \frac{23}{99}.
\]

(We multiplied both numerator and denominator in the left side of the equation by 100 to obtain the simpler expression on the right side.)

Another example, a harder one. Can we write the number 3.77142587 142587... as a fraction of integers?

3.77142587 142587... = 3.77142587 142587 142587... = 3.77 + (\frac{142587}{10^8} + \frac{142587}{10^{14}} + \frac{142587}{10^{20}} + ...)

After the initial integer 3.77 we have a geometric series with initial term \(a = \frac{142587}{10^8}\) and ratio \(r = 10^{-6}\). So our answer is

3.77 + \frac{142587}{1 - 10^{-6}} = 3 + \frac{77}{100} + \frac{142587}{10^8 - 10^2} = 3 + \frac{77}{100} + \frac{142587}{99999900}.

Using the common denominator 99999900, this can then be written as

\[
\frac{377142480}{99999900}.
\]

One more example, a little puzzle a teacher asked me in high school: what number does 0.9999... represent?

Writing the decimal 0.999... as a geometric series and using our sum we have

\[
0.999... = \frac{\frac{9}{10}}{1 - \frac{1}{10}} = \frac{9}{9} = 1.
\]

(There are a lot of webpages, math tutorials, and FAQs out there on the internet trying to explain this result to lots of disbelieving high school students! There is even a very entertaining webpage devoted to claiming mathematicians got it all wrong!! ☺️☺️)
Other series from geometric series

We saw, from the previous theorem about the limit of a geometric series, that if $|x| < 1$ then

$$1 + x + x^2 + x^3 + \ldots + x^{k-1} + \ldots = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}. \tag{4}$$

In other words, in the interval $(-1, 1)$, we can *equate* the function $f(x) = \frac{1}{1-x}$ with the sum $\sum_{k=0}^{\infty} x^k$.

European mathematicians such as Euler (c. 1730) would note that derivative of $\ln(1 - x)$ is $-\frac{1}{1-x}$ and so take the antiderivative of this series and write

$$x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \ldots + \frac{1}{k}x^k + \ldots = -\ln(1-x). \tag{5}$$

It turns out that this series converges if $x = -1$ and so the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + (-1)^{k-1} \frac{1}{k} + \ldots$$

converges to $\ln 2 \approx 0.69314 718055 99453 09417\ldots$

We might instead substitute $-x^2$ for $x$ in the original series in equation (4) and write (for $-1 \leq x \leq 1$)

$$1 - x^2 + x^4 - x^6 + \ldots + (-1)^k x^{2k} + \ldots = \sum_{k=0}^{\infty} (-1)^k x^{2k} = \frac{1}{1+x^2}. \tag{6}$$

Now we might take the antiderivative of this expression and recall that $\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$. Suddenly we have an infinite series representing the arctangent function!

$$x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots + (-1)^k \frac{1}{2k+1} x^{2k+1} + \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} x^{2k+1} = \tan^{-1}(x)! \tag{7}$$

We can evaluate this at $x = 1$ and get a (very slowly) converging series for $\pi/4$.

The series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots + (-1)^k \frac{1}{2k+1} + \ldots = \frac{\pi}{4}$$

was first discovered by James Gregory around 1670. Versions of it were eventually used to expand $\pi$ to seventy or one hundred decimal places. In 1873 a mathematician (Shanks) used a version of this formula to calculate $\pi$ to 527 decimal places. (He really calculated 707 decimal places but made a mistake at the 528th position and the rest of his work was incorrect!)