

Solving Cubic Polynomials

1.1 The general solution to the quadratic equation

There are four steps to finding the zeroes of a quadratic polynomial.

1. First divide by the leading term, making the polynomial **monic**.
2. Then, given $x^2 + a_1x + a_0$, substitute $x = y - \frac{a_1}{2}$ to obtain an equation without the linear term. (This is the “depressed” equation.)
3. Solve then for y as a square root. (Remember to use both signs of the square root.)
4. Once this is done, recover x using the fact that $x = y - \frac{a_1}{2}$.

For example, let’s solve

$$2x^2 + 7x - 15 = 0.$$

First, we divide both sides by 2 to create an equation with leading term equal to one:

$$x^2 + \frac{7}{2}x - \frac{15}{2} = 0.$$

Then replace x by $x = y - \frac{a_1}{2} = y - \frac{7}{4}$ to obtain:

$$y^2 = \frac{169}{16}$$

Solve for y :

$$y = \frac{13}{4} \text{ or } -\frac{13}{4}$$

Then, solving back for x , we have

$$x = \frac{3}{2} \text{ or } -5.$$

This method is equivalent to “completing the square” and is the steps taken in developing the much-memorized quadratic formula. For example, if the original equation is our “high school quadratic”

$$ax^2 + bx + c = 0$$

then the first step creates the equation

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

We then write $x = y - \frac{b}{2a}$ and obtain, after simplifying,

$$y^2 - \frac{b^2 - 4ac}{4a^2} = 0$$

so that

$$y = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

and so

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

The solutions to this quadratic depend heavily on the value of $b^2 - 4ac$. We give this a name (the **discriminant**) and a symbol (Δ) and so discuss the discriminant

$$\Delta = b^2 - 4ac.$$

If the coefficients of the polynomial are integers and Δ is a perfect square integer, we have rational roots. If the discriminant is positive, we have real roots. If the discriminant is zero, we have a single root. If the discriminant is negative, we have imaginary roots.

We note for later that if the discriminant $\Delta = b^2 - 4ac$ is equal to zero then we have a single root and so our polynomial is a perfect square.

1.2 The general solution to the cubic equation

Every polynomial equation involves two steps to turn the polynomial into a slightly simpler polynomial.

1. First divide by the leading term, creating a **monic** polynomial (in which the highest power of x has coefficient one.) This does not change the roots.
2. Then, given $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_1x + a_0$, substitute $x = y - \frac{a_{n-1}}{n}$ to obtain an equation without the term of degree $n - 1$. (This is the **depressed** polynomial.) Since this step is reversible, solutions to the “depressed equation” give us solutions to the original equation.

Here are the first steps in Cardano’s method of solving the cubic. First, we do the two automatic steps:

1. Divide by the leading term, creating a cubic polynomial $x^3 + a_2x^2 + a_1x + a_0$ with leading coefficient one.
2. Then substitute $x = y - \frac{a_2}{3}$ to obtain an equation without the term of degree two. This creates an equation of the form $x^3 + Px - Q = 0$.

Cardano would rewrite this equation in the form $x^3 + Px = Q$. He then noticed (!) the following algebra identity:

$$(a - b)^3 + 3ab(a - b) = a^3 - b^3. \tag{1}$$

Given $x^3 + Px = Q$, set

$$ab = \frac{P}{3}$$

and

$$a^3 - b^3 = Q.$$

Solve this system by substitution and then assign $x := a - b$.

For example, we might replace a by $\frac{P}{3b}$ (using the first equation) and then substitute into the second equation to obtain

$$\left(\frac{P}{3b}\right)^3 - b^3 = Q.$$

Multiplying by b^3 , we have

$$\left(\frac{P}{3}\right)^3 - (b^3)^2 = Qb^3$$

which is a quadratic equation in b^3 . Rewrite this equation as

$$(b^3)^2 + Qb^3 - \left(\frac{P}{3}\right)^3 = 0$$

so that

$$b^3 = -\frac{Q}{2} \pm \sqrt{\Delta} \quad (2)$$

where the discriminant Δ is

$$\Delta := \left(\frac{P}{3}\right)^3 + \left(\frac{Q}{2}\right)^2.$$

Now there are three possible solutions to equation (2).

This is where Cardano struggled. The next steps sometimes involved “imaginary” numbers and he wasn’t sure what to do with them. We will press on and finish Cardano’s work by jumping ahead to Euler’s day and using his understanding of complex numbers.

Using Euler’s formula

$$e^{i\theta} = \cos \theta + i \sin \theta \quad (3)$$

we will create the element

$$\omega := e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

Notice that $\omega^3 = 1$ and that $(\omega^2)^3 = 1$ so that $1, \omega$ and ω^2 are *all* cube roots of 1!

Using ω , we can see that *any* real number has three cube roots. Given one cube root, β , the others will be $\omega\beta$ and $\omega^2\beta$.

We use this to finish equation 2 by picking on cube root of $-Q/2 \pm \sqrt{\Delta}$. We call that one particular solution β . The other solutions to the equation (2) are $b = \omega\beta$ and $b = \omega^2\beta$.

Each solution to (2) gives a value for a so that $ab = P/3$. Let $\alpha := \frac{P}{3\beta}$.

Then

$$\alpha - \beta, \quad \alpha\omega^2 - \beta\omega, \quad \text{and} \quad \alpha\omega - \beta\omega^2$$

are all solutions to the cubic equation.

Euler’s example

We work through an example due to Euler: We find all solutions to

$$x^3 - 6x = 4. \quad (4)$$

Here $P = -6$ and $Q = 4$ and so the discriminant is

$$\Delta = -8 + 4 = -4 \text{ so } \sqrt{\Delta} = 2i.$$

Therefore $b^3 = -2 \pm 2i$.

We choose a sign and solve the cubic equation $b^3 = -2 + 2i$. Write $-2 + 2i$ in polar form as $\sqrt{8} (e^{\frac{3\pi}{4}i})$. Then $b^3 = \sqrt{8} e^{\frac{\pi}{4}i}$ has a particular solution

$$\beta = \sqrt{2} e^{\frac{\pi}{4}i} = 1 + i.$$

Set

$$\alpha = \frac{-2}{\beta} = \frac{-2}{\sqrt{2}} e^{-\frac{\pi}{4}i} = -\sqrt{2} e^{-\frac{\pi}{4}i} = \sqrt{2} e^{\frac{3\pi}{4}i}.$$

Or, in cartesian form, $\alpha = -1 + i$.

Then

$$x = \alpha - \beta = (-1 + i) - (1 + i) = -2$$

is one solution to the cubic equation.

But there are others!

If $x_1 = \alpha - \beta$ is a solution then so are

$$x_2 = \alpha\omega^2 - \beta\omega = \sqrt{2}(e^{\frac{3\pi}{4}i}e^{\frac{4\pi}{3}i} - e^{\frac{\pi}{4}i}e^{\frac{2\pi}{3}i}) = \sqrt{2}(e^{\frac{25\pi}{12}i} - e^{\frac{11\pi}{12}i}) = \sqrt{2}(e^{\frac{\pi}{12}i} + e^{\frac{7\pi}{12}i})$$

and

$$x_3 = \alpha\omega - \beta\omega^2 = \sqrt{2}(e^{\frac{3\pi}{4}i}e^{\frac{2\pi}{3}i} - e^{\frac{\pi}{4}i}e^{\frac{4\pi}{3}i}) = \sqrt{2}(e^{\frac{17\pi}{12}i} - e^{\frac{19\pi}{12}i}) = \sqrt{2}(e^{-\frac{7\pi}{12}i} + e^{\frac{7\pi}{12}i})$$

By Euler's formula, $e^{\theta i} + e^{-\theta i} = 2 \cos \theta$. So our answers are equivalent to

$$x_2 = 2\sqrt{2} \cos\left(\frac{\pi}{12}\right) \text{ and } x_3 = 2\sqrt{2} \cos\left(\frac{7\pi}{12}\right).$$

The value of $\cos\left(\frac{\pi}{12}\right)$ is not immediate, but we can find it from a trig identity.

Since

$$\cos(2\theta) = 2 \cos^2 \theta - 1$$

then

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}.$$

Thus

$$\cos \frac{\pi}{12} = \sqrt{\frac{1}{2} + \frac{\sqrt{3}}{4}}$$

and

$$x_2 = \sqrt{8} \sqrt{\frac{1}{2} + \frac{\sqrt{3}}{4}} = \sqrt{4 + 2\sqrt{3}}.$$

In a similar manner we can find x_3 . Our final answers are

$$x_2 = \sqrt{4 + 2\sqrt{3}} \text{ and } x_3 = -\sqrt{4 - 2\sqrt{3}}.$$

Another example from Euler

We solve Euler's cubic:

$$x^3 - 6x = 9. \tag{5}$$

Since $(a - b)^3 + 3ab(a - b) = a^3 - b^3$, we set

$$3ab = -6 \text{ and } a^3 - b^3 = 9.$$

(Let $P := -6; Q := 9$.)

Then

$$a = -2/b; (-2/b)^3 - b^3 = 9$$

so

$$-8 - b^6 = 9b^3 \text{ or } b^6 + 9b^3 + 8 = 0.$$

View this as a quadratic in b^3 so that

$$(b^3)^2 + 9b^3 + 8 = 0 \implies (b^3 + 8)(b^3 + 1) = 0. \tag{6}$$

Therefore either $b^3 = -8$ or $b^3 = -1$.

Suppose $b^3 = -8$ and presumably $b = -2$.

Then $a = 1$ and $x = a - b = 3$.

If $b = -1$ then $a = -2$ and $x = a - b = 3$, so choosing the other factor does not give new information. We have found *one* solution, $x = 3$.

But what about other solutions?

Set $\beta = -2$ as a solution to the cubic equation (6), so that $\beta^3 = -8$ and let $\alpha = 1$ be the corresponding choice for a .

Then $b = \beta\omega = -2\omega$ is also a solution to that cubic and in this case $a = \alpha\omega^2 = \omega^2$ is the corresponding choice for a .

Then a second solution is

$$x = a - b = \omega^2 + 2\omega = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) + (-1 + \sqrt{3}i) = \frac{1}{2}(-3 + \sqrt{3}i).$$

If instead we have $a = \alpha\omega$ and $b = \beta\omega^2$ then the final solution is

$$x = a - b = \omega + 2\omega^2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + (-1 - \sqrt{3}i) = \frac{1}{2}(-3 - \sqrt{3}i).$$

So we have found all three solutions:

$$3, \frac{1}{2}(-3 + \sqrt{3}i), \text{ and } \frac{1}{2}(-3 - \sqrt{3}i).$$

1.3 Rational Solutions

We can avoid the lengthy computations, above, if we are lucky enough to find a rational solution to our polynomial. For example, in the last problem, if we had merely stumbled on the root $x = 3$, we could have divided the cubic polynomial $x^3 - 6x - 9$ by $x - 3$ and rewritten it as

$$x^3 - 6x - 9 = (x - 3)(x^2 + 3x + 3).$$

The quadratic equation, applied to $x^2 + 3x + 3$ would have given us the final two solutions without the extra work.

So how do we find these rational solutions when they occur?

If $x = \frac{p}{q}$ is a rational solution to the polynomial equation $f(x) = 0$ then $qx - p$ is a factor of the polynomial $f(x)$ and so we can use long division to write $f(x) = (qx - p)g(x)$ where $g(x)$ is a polynomial of smaller degree.

We teach a version of this method in high school when students learn to solve quadratic equations by factoring. For example, one might solve the equation $3x^2 - 2x - 8 = 0$ by factoring the left-hand side into $(3x + 4)(x - 2)$, obtain solutions $x = -\frac{4}{3}$ and $x = 2$ and so avoid the quadratic formula.

We can try this method on polynomials of higher degree with integer coefficients. Descartes would point out, in the 1600s, that if $f(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ has a root $x = p/q$ then q must divide a_n and p must divide a_0 (Notice how this works on the quadratic $3x^2 - 2x - 8$.) With then a short list of possible rational solutions, Descartes was willing to try these solutions and exhaust the possibilities for a rational root.

The modern version of this is to pull out a graphing calculator, graph the polynomial equation $y = f(x)$ and hope that the calculator identifies a nice rational (or even integer!) root.

For example, with Euler's cubic $x^3 - 6x - 9$, we discover that $x = 3$ is a root. When then divide the polynomial by $x - 3$ to obtain a quadratic polynomial and now we can go ahead and use the quadratic formula.

This method is much faster than the general method, but it requires that we be "lucky" and stumble upon a root.

1.4 Quartic Polynomials

After del Ferro and Tartaglia solved the general cubic equation (and the result was released to the public by Cardano), mathematicians then concentrated on the quartic equation. Cardano's student, Ferrari, around 1540, suggested a general method for solving the depressed quartic.

Given the depressed quartic

$$x^4 + px^2 + qx + r = 0$$

rewrite the problem so that it appears to involve a square:

$$(x^2 + p)^2 = px^2 - qx + (p^2 - r).$$

Now insert a new variable y and expand $(x^2 + p + y)^2$ with the hopes that the right side, $(p + 2y)x^2 - qx + (p^2 - r + 2py + y^2)$ is a perfect square.

By the quadratic formula, the right-hand side will be a perfect square when the discriminant $4(p + 2y)(p^2 - r + 2py + y^2) - q^2$ is zero.

So we first need to solve

$$4(p + 2y)(p^2 - r + 2py + y^2) - q^2 = 0 \tag{7}$$

for y .

Since equation (7) is a cubic equation, we know how to do that!

In other words, given a quartic, we make it monic, and then turn it into a depressed equation, insert y , and solve the associated cubic for y . Once this is done, we solve the quadratic equation $(x^2 + p + y)^2 = (p + 2y)x^2 - qx + (p^2 - r + 2py + y^2)$ for x .

This is lengthy and tedious, but it works in general! (Well, it works if one is willing to believe in complex numbers.)

By 1545, after Cardano published Ferrari's work (with his permission, apparently!) mathematicians began to search for solutions to the general fifth-degree or quintic equation. It would take the work of Abel, Galois and others to show that in fact a general solution to the quintic is impossible.

We will save that story for another time.

Exercises on cubic equations

- For each of the equations, below, do the appropriate substitution to turn the polynomial on the left-hand side into a “depressed” polynomial.
 - $2x^2 - 5x + 3 = 0$
 - $x^3 - 6x^2 + 2x + 3 = 0$
 - $x^4 + 8x^3 + 3x - 12 = 0$
- Complete your work in equation part (a), above, finding the solutions to the equation by completing the square.
- For parts (b) and (c) in problem 1, guess a rational solution and then use this to find all solutions to the polynomial.
- Find a rational solution for the following polynomials. Then use this solution to find all the roots.
 - Viète and Cardano-Tartaglia examined this polynomial: $x^3 + 63x - 316$.
 - Viète and Cardano-Tartaglia also did this one: $x^3 - 63x - 162$.
 - Euler’s cubic: $x^3 - 6x - 4$.

Miscellanea

1. Varadarajan, p. 47 and following.

2. $8x^3 - 6x - 1$

Solution. (one solution is the cosine of 20 degrees!) $P = -\frac{3}{4}, Q = \frac{1}{8}$

3. Cardano and $13x^2 = x^4 + 2x^3 + 2x + 1$. (Eves 293) This is “magic”: Cardano added $3x^2$ to both sides to get a perfect square.

4. Ferrari on page 293. $a + b + c = 10; ac = b^2, ab = 6$ leads to the quartic $b^4 + 6b^2 + 36 = 60b$. This can be turned into a cubic by Ferrari and Viete’s methods.

5. The quartic equation

(Ferrari on page 273 of Eves.)

6. Descarte’s solution: Use it on $x^4 - 2x^2 + 8x - 3 = 0$.

Or translate this by $x = y + 1$. (Eves p. 371)