The Fibonacci Numbers

The numbers are:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, …

Each Fibonacci number is the sum of the previous two Fibonacci numbers!

Let $n$ any positive integer. If $F_n$ is what we use to describe the $n^{th}$ Fibonacci number, then

$$F_n = F_{n-1} + F_{n-2}$$
We proved a theorem about the sum of the first few Fibonacci numbers:

**Theorem**

*For any positive integer \( n \), the Fibonacci numbers satisfy:*

\[
F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1
\]
What about the first few Fibonacci numbers with even index:

\[ F_2, F_4, F_6, \ldots, F_{2n}, \ldots \]

Let’s call them “even” Fibonaccis, since their index is even, although the numbers themselves aren’t always even!!
The “even” Fibonacci Numbers

Some notation: The first “even” Fibonacci number is $F_2 = 1$.

The second “even” Fibonacci number is $F_4 = 3$.

The third “even” Fibonacci number is $F_6 = 8$.

The tenth “even” Fibonacci number is $F_{20} = ??$.

The $n^{th}$ “even” Fibonacci number is $F_{2n}$. 
Tonight, try to come up with a formula for the sum of the first few “even” Fibonacci numbers.
Let's look at the first few:

\[ F_2 + F_4 = 1 + 3 = 4 \]

\[ F_2 + F_4 + F_6 = 1 + 3 + 8 = 12 \]

\[ F_2 + F_4 + F_6 + F_8 = 1 + 3 + 8 + 21 = 33 \]

\[ F_2 + F_4 + F_6 + F_8 + F_{10} = 1 + 3 + 8 + 21 + 55 = 88 \]

See a pattern?
It looks like the sum of the first few “even” Fibonacci numbers is one less than another Fibonacci number.

But which one?

\[ F_2 + F_4 = 1 + 3 = 4 = F_5 - 1 \]

\[ F_2 + F_4 + F_6 = 1 + 3 + 8 = 12 = F_7 - 1 \]

\[ F_2 + F_4 + F_6 + F_8 = 1 + 3 + 8 + 21 = 33 = F_9 - 1 \]

\[ F_2 + F_4 + F_6 + F_8 + F_{10} = 1 + 3 + 8 + 21 + 55 = 88 = F_{11} - 1 \]

Can we come up with a formula for the sum of the first few “even” Fibonacci numbers?
Let’s look at the last one we figured out:

\[ F_2 + F_4 + F_6 + F_8 + F_{10} = 1 + 3 + 8 + 21 + 55 = 88 = F_{11} - 1 \]

The sum of the first 5 even Fibonacci numbers (up to \( F_{10} \)) is the 11th Fibonacci number less one.

Maybe it’s true that the sum of the first \( n \) “even” Fibonacci’s is one less than the next Fibonacci number. That is,

**Conjecture**

*For any positive integer \( n \), the Fibonacci numbers satisfy:*

\[ F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1 \]
Trying to prove: \[ F_2 + F_4 + \cdots F_{2n} = F_{2n+1} - 1 \]

Let's try to prove this!!

We’re trying to find out what

\[ F_2 + F_4 + \cdots F_{2n} \]

is equal to.

Well, let’s proceed like the last theorem.
Trying to prove: \( F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1 \)

We know \( F_3 = F_2 + F_1 \).

So, rewriting this a little:

\[ F_2 = F_3 - F_1 \]

Also: we know \( F_5 = F_4 + F_3 \).

So:

\[ F_4 = F_5 - F_3 \]

In general:

\[ F_{\text{even}} = F_{\text{next odd}} - F_{\text{previous odd}} \]

Let's put these together:
Trying to prove: \( F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1 \)

\[
\begin{align*}
F_2 &= F_3 - F_1 \\
F_4 &= F_5 - F_3 \\
F_6 &= F_7 - F_5 \\
F_8 &= F_9 - F_7 \\
& \vdots \\
F_{2n-2} &= F_{2n-1} - F_{2n-3} \\
F_{2n} &= F_{2n+1} - F_{2n-1}
\end{align*}
\]

Adding up all the terms on the left sides will give us something equal to the sum of the terms on the right sides.
Trying to prove: \( F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1 \)

\[
F_2 = F_3 - F_1 \\
F_4 = F_5 - F_3 \\
F_6 = F_7 - F_5 \\
F_8 = F_9 - F_7 \\
\vdots \\
F_{2n-2} = F_{2n-1} - F_{2n-3} \\
+ F_{2n} = F_{2n+1} - F_{2n-1}
\]
Trying to prove: \( F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1 \)

\[
F_2 = F_3 - F_1 \\
F_4 = F_5 - F_3 \\
F_6 = F_7 - F_5 \\
F_8 = F_9 - F_7 \\
\vdots \quad \vdots \\
F_{2n-2} = F_{2n-1} - F_{2n-3} \\
F_{2n} = F_{2n+1} - F_{2n-1}
\]
Trying to prove: $F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1$

\[
\begin{align*}
F_2 &= F_3' - F_1 \\
F_4 &= F_3' - F_3 \\
F_6 &= F_4' - F_3 \\
F_8 &= F_5' - F_4 \\
& \quad \vdots \\
F_{2n-2} &= F_{2n-1} - F_{2n-3} \\
+ F_{2n} &= F_{2n+1} - F_{2n-1}
\end{align*}
\]
Trying to prove:  \( F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1 \)

\[
F_2 = F_3 - F_1 \\
F_4 = F_5 - F_3 \\
F_6 = F_7 - F_5 \\
F_8 = F_9 - F_7 \\
\vdots \\
F_{2n-2} = F_{2n-1} - F_{2n-3} \\
+ F_{2n} = F_{2n+1} - F_{2n-1}
\]
Trying to prove: \[ F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1 \]

\[
F_2 = F_3 - F_1 \\
F_4 = F_5 - F_3 \\
F_6 = F_7 - F_5 \\
F_8 = F_9 - F_7 \\
\vdots \quad \vdots \\
F_{2n-2} = F_{2n+1} - F_{2n+3} \\
+ \quad F_{2n} = F_{2n+1} - F_{2n+3}
\]
Trying to prove: \( F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1 \)

\[
F_2 = F_3 - F_1 \\
F_4 = F_5 - F_3 \\
F_6 = F_7 - F_5 \\
F_8 = F_9 - F_7 \\
\vdots \quad \vdots \\
F_{2n-2} = F_{2n-1} + F_{2n-3} - F_{2n-1} \\
\]

\[
F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - F_1
\]
Trying to prove:  \( F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1 \)

\[
F_2 = F_3 - F_1 \\
F_4 = F_5 - F_3 \\
F_6 = F_7 - F_5 \\
F_8 = F_9 - F_7 \\
\vdots \quad \vdots \quad \vdots \\
F_{2n-2} = F_{2n+1} - F_{2n+3} \\
+ \quad F_{2n} = F_{2n+1} - F_{2n+3}
\]

\[
F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1
\]
Trying to prove: \[ F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1 \]

This proves our second theorem!!

**Theorem**

*For any positive integer \( n \), the even Fibonacci numbers satisfy:*

\[ F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1 \]
The “odd” Fibonacci Numbers

What about the first few Fibonacci numbers with odd index:

\[ F_1, F_3, F_5, \ldots, F_{2n-1}, \ldots \]

Let’s call them “odd” Fibonaccis, since their index is odd, although the numbers themselves aren’t always odd!!
Some notation: The first “odd” Fibonacci number is $F_1 = 1$.

The second “odd” Fibonacci number is $F_3 = 2$.

The third “odd” Fibonacci number is $F_5 = 5$.

The tenth “odd” Fibonacci number is $F_{19} = ??$.

The $n^{th}$ “odd” Fibonacci number is $F_{2n-1}$. 
Let's look at the first few:

\[ F_1 + F_3 = 1 + 2 = 3 \]

\[ F_1 + F_3 + F_5 = 1 + 2 + 5 = 8 \]

\[ F_1 + F_3 + F_5 + F_7 = 1 + 2 + 5 + 13 = 21 \]

\[ F_1 + F_3 + F_5 + F_7 + F_9 = 1 + 2 + 5 + 13 + 34 = 55 \]

See a pattern?
It looks like the sum of the first few “odd” Fibonacci numbers is another Fibonacci number.

But which one?

\[ F_1 + F_3 = 1 + 2 = 3 = F_4 \]

\[ F_1 + F_3 + F_5 = 1 + 2 + 5 = 8 = F_6 \]

\[ F_1 + F_3 + F_5 + F_7 = 1 + 2 + 5 + 13 = 21 = F_8 \]

\[ F_1 + F_3 + F_5 + F_7 + F_9 = 1 + 2 + 5 + 13 + 34 = 55 = F_{10} \]

Can we come up with a formula for the sum of the first few “even” Fibonacci numbers?
Maybe it’s true that the sum of the first $n$ “odd” Fibonacci’s is the next Fibonacci number. That is,

**Conjecture**

*For any positive integer $n$, the Fibonacci numbers satisfy:*

$$F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$$
Let’s try to prove this!!

We know $F_2 = F_1$.

So, rewriting this a little:

$$F_1 = F_2$$

Also: we know $F_4 = F_3 + F_2$.

So:

$$F_3 = F_4 - F_2$$

In general:

$$F_{\text{odd}} = F_{\text{next even}} - F_{\text{previous even}}$$
Trying to prove: \( F_1 + F_3 + \cdots + F_{2n-1} = F_{2n} \)

\[
F_1 = F_2 \\
F_3 = F_4 - F_2 \\
F_5 = F_6 - F_4 \\
F_7 = F_8 - F_6 \\
\vdots \\
F_{2n-3} = F_{2n-2} - F_{2n-4} \\
+ F_{2n-1} = F_{2n} - F_{2n-2}
\]

Adding up all the terms on the left sides will give us something equal to the sum of the terms on the right sides.
Trying to prove: \( F_1 + F_3 + \cdots + F_{2n-1} = F_{2n} \)

\[
F_1 = F_{1/2}
\]

\[
F_3 = F_4 - F_{1/2}
\]

\[
F_5 = F_6 - F_4
\]

\[
F_7 = F_8 - F_6
\]

\[
\vdots \quad \vdots
\]

\[
F_{2n-3} = F_{2n-2} - F_{2n-4}
\]

\[
+ \quad F_{2n-1} = F_{2n} - F_{2n-2}
\]

\[
\text{-------------} \quad \text{-------------}
\]
Trying to prove: \( F_1 + F_3 + \cdots + F_{2n-1} = F_{2n} \)

\[
\begin{align*}
F_1 &= \mathbb{F}_2 \\
F_3 &= \mathbb{F}_4 - \mathbb{F}_2 \\
F_5 &= F_6 - \mathbb{F}_4 \\
F_7 &= F_8 - F_6 \\
&\quad\vdots \\
F_{2n-3} &= F_{2n-2} - F_{2n-4} \\
+\quad F_{2n-1} &= F_{2n} - F_{2n-2}
\end{align*}
\]
Trying to prove: \( F_1 + F_3 + \cdots + F_{2n-1} = F_{2n} \)

\[
F_1 = F_2 \\
F_3 = F_4 - F_2 \\
F_5 = F_6 - F_4 \\
F_7 = F_8 - F_6 \\
\vdots \quad \vdots \\
F_{2n-3} = F_{2n-2} - F_{2n-4} \\
+ \quad F_{2n-1} = F_{2n} - F_{2n-2}
\]
Trying to prove: \[ F_1 + F_3 + \cdots + F_{2n-1} = F_{2n} \]

\[
F_1 = \mathbb{F}_2
\]

\[
F_3 = \mathbb{F}_4 - \mathbb{F}_2
\]

\[
F_5 = \mathbb{F}_6 - \mathbb{F}_4
\]

\[
F_7 = \mathbb{F}_8 - \mathbb{F}_6
\]

\[
\vdots \quad \vdots
\]

\[
F_{2n-3} = F_{2n-2} - \mathbb{F}_{2n-4}
\]

\[
+ \quad F_{2n-1} = F_{2n} - F_{2n-2}
\]
Trying to prove: \[ F_1 + F_3 + \cdots + F_{2n-1} = F_{2n} \]

\[
F_1 = F_2
\]

\[
F_3 = F_4 - F_2
\]

\[
F_5 = F_6 - F_4
\]

\[
F_7 = F_8 - F_6
\]

\[ \vdots \quad \vdots \]

\[
F_{2n-3} = F_{2n-1} - F_{2n-3}
\]

\[ + F_{2n-1} = F_{2n} - F_{2n-1} \]

\[
F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}
\]
Trying to prove: \[ F_1 + F_3 + \cdots + F_{2n-1} = F_{2n} \]

This proves our third theorem!!

**Theorem**

*For any positive integer* \( n \), the Fibonacci numbers satisfy:

\[ F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n} \]
So far:

**Theorem (Sum of first few Fibonacci numbers.)**

For any positive integer \( n \), the Fibonacci numbers satisfy:

\[
F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1
\]

**Theorem (Sum of first few EVEN Fibonacci numbers.)**

For any positive integer \( n \), the Fibonacci numbers satisfy:

\[
F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1
\]

**Theorem (Sum of first few ODD Fibonacci numbers.)**

For any positive integer \( n \), the Fibonacci numbers satisfy:

\[
F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}
\]
One more theorem about sums:

**Theorem (Sum of first few SQUARES of Fibonacci numbers.)**

*For any positive integer $n$, the Fibonacci numbers satisfy:*

$$F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n \cdot F_{n+1}$$
Trying to prove: \[ F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n \cdot F_{n+1} \]

We’ll take advantage of the following fact:

For each Fibonacci number \( F_n \)

\[ F_{n+1} = F_n + F_{n-1} \]

\[ F_n = F_{n+1} - F_{n-1} \]

\[ F_n^2 = F_n \cdot F_n = F_n \cdot (F_{n+1} - F_{n-1}) \]

\[ F_n^2 = F_n \cdot F_{n+1} - F_n \cdot F_{n-1} \]
Trying to prove: \[ F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n \cdot F_{n+1} \]

Let’s check this formula for a few different values of \( n \):

\[ F_n^2 = F_n \cdot F_{n+1} - F_n \cdot F_{n-1} \]

\( n = 4 \):

\[ F_4^2 = F_4 \cdot F_5 - F_4 \cdot F_3 \]

\[ 3^2 = 3 \cdot 5 - 3 \cdot 2 \]

\[ 9 = 15 - 6 \]

It works for \( n = 4 \)!
Trying to prove: \( F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n \cdot F_{n+1} \)

Let’s check this formula for a few different values of \( n \):

\[
F_n^2 = F_n \cdot F_{n+1} - F_n \cdot F_{n-1}
\]

\( n = 8 \):

\[
F_8^2 = F_8 \cdot F_{9} - F_8 \cdot F_7
\]

\[
21^2 = 21 \cdot 34 - 21 \cdot 13
\]

\[
441 = 714 - 273 = 441
\]

It works for \( n = 8 \)!
Trying to prove: \[ F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n \cdot F_{n+1} \]

Let's use this formula:

\[ F_n^2 = F_n \cdot F_{n+1} - F_n \cdot F_{n-1} \]

To find out what the sum of the first few SQUARES of Fibonacci numbers is:

\[ F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = \text{???} \]
Trying to prove: \[ F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n \cdot F_{n+1} \]

We’re using:

\[ F_n^2 = F_n \cdot F_{n+1} - F_n \cdot F_{n-1} \]

\[ F_1^2 = F_1 \cdot F_2 \]
\[ F_2^2 = F_2 \cdot F_3 - F_2 \cdot F_1 \]
\[ F_3^2 = F_3 \cdot F_4 - F_3 \cdot F_2 \]
\[ F_4^2 = F_4 \cdot F_5 - F_4 \cdot F_3 \]
\[ \vdots \]
\[ F_{n-1}^2 = F_{n-1} \cdot F_n - F_{n-1} \cdot F_{n-2} \]
\[ + \quad F_n^2 = F_n \cdot F_{n+1} - F_n \cdot F_{n-1} \]
Trying to prove: \( F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n \cdot F_{n+1} \)

\[
\begin{align*}
F_1^2 &= F_1^2 = F_1 \\
F_2^2 &= F_2 \cdot F_3 - F_1^2 = F_2 \\
F_3^2 &= F_3 \cdot F_4 - F_2^2 = F_3 \\
F_4^2 &= F_4 \cdot F_5 - F_3^2 = F_4 \\
&\vdots \\
F_{n-1}^2 &= F_{n-1} \cdot F_n - F_{n-2}^2 \quad F_n^2 = F_n \cdot F_{n+1} - F_{n-1} \cdot F_{n-2} \\
\end{align*}
\]
Trying to prove: \[ F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n \cdot F_{n+1} \]
Trying to prove: \[ F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n \cdot F_{n+1} \]

\[
F_1^2 = F_1 \cdot F_2
\]

\[
F_2^2 = F_2 \cdot F_3 - F_1 \cdot F_4
\]

\[
F_3^2 = F_3 \cdot F_4 - F_2 \cdot F_5
\]

\[
F_4^2 = F_4 \cdot F_5 - F_3 \cdot F_6
\]

\[
\vdots
\]

\[
F_{n-1}^2 = F_{n-1} \cdot F_n - F_{n-2} \cdot F_n
\]

\[
+ \quad F_n^2 = F_n \cdot F_{n+1} - F_{n-1} \cdot F_{n-1}
\]
Trying to prove: \( F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n \cdot F_{n+1} \)

\[
F_1^2 = F_1 \cdot F_2 \\
F_2^2 = F_2 \cdot F_3 - F_1^2 \\
F_3^2 = F_3 \cdot F_4 - F_2^2 \\
F_4^2 = F_4 \cdot F_5 - F_3^2 \\
\vdots \quad \vdots \\
F_{n-1}^2 = F_{n-1} \cdot F_n - F_{n-2}^2 \\
+ \quad F_n^2 = F_n \cdot F_{n+1} - F_{n-1} \cdot F_n
\]
Trying to prove: \[ F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n \cdot F_{n+1} \]

\[
F_1^2 = F_1 \cdot F_2 \\
F_2^2 = F_2 \cdot F_3 - F_1^2 \\
F_3^2 = F_3 \cdot F_4 - F_2^2 \\
F_4^2 = F_4 \cdot F_5 - F_3^2 \\
\vdots \\
F_{n-1}^2 = F_{n-1} \cdot F_n - F_{n-2}^2 \\
+ F_n^2 = F_n \cdot F_{n+1} - F_{n-1} \cdot F_{n+1}
\]

\[
F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n \cdot F_{n+1}
\]
Our fourth theorem:

**Theorem (Sum of first few SQUARES of Fibonacci numbers.)**

*For any positive integer $n$, the Fibonacci numbers satisfy:*

$$F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n \cdot F_{n+1}$$
So far:

<table>
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<tr>
<th>Theorem (Sum of first few Fibonacci numbers.)</th>
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<td>[ F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n} ]</td>
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<td>[ F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n \cdot F_{n+1} ]</td>
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Examples

\[ 1 + 2 + 5 + 13 + 34 + 89 + 233 + 610 + 1597 + 4181 = 6765 \]

\[ 1 + 1 + 4 + 9 + 25 + 64 + 169 + 441 + 1156 + 3025 + 7921 = 89 \times 144 = 12816 \]

Now, complete this worksheet .....