Delta Sets of Numerical Monoids: A Progress Report

Scott Chapman

Sam Houston State University

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Papers This Talk is Based on


Delta Sets of Numerical Monoids

Introduction

Background

Initial Results

Delta Sets and Arithmetic Sequences of Integers

A Theorem on Singleton Delta Sets

Baginski’s Conjecture

Kaplan’s Conjecture

Questions

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$$\mathcal{L}(m) = \{ t \in \mathbb{N} \mid \exists x_1, \ldots, x_t \in \mathcal{I}(M) \text{ with } m = x_1 \cdots x_t \}.$$
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Some Known Results

If \( M \) is a Krull monoid with finite divisor class group \( \text{Cl}(M) = G \) where \( |G| \geq 3 \), then

\[ \Delta(M) \subseteq \{1, \ldots, D(G) - 2\} \]

where \( D(G) \) represents Davenport’s constant of \( G \).
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On the other hand, the exact delta set of a Krull monoid is known in very few other instances, regardless of the knowledge of the distribution of the prime divisors in $\text{Cl}(M)$.
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**Proposition**

Let $M$ be a commutative cancellative reduced atomic monoid (i.e., $M^\times = \{0\}$). Then

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$$\min \Delta(M) = \gcd \Delta(M).$$

Hence, if $d = \gcd \Delta(M)$ and $|\Delta(M)| < \infty$, then

$$\Delta(M) \subseteq \{d, 2d, \ldots, kd\}$$

for some $k \in \mathbb{N}$. 
Goal

Given a numerical monoid $S = \langle n_1, n_2, \ldots, n_k \rangle$ describe as best we can the set $\Delta(S)$. 
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$\Delta(\langle 7, 10, 12 \rangle) = \{1, 2\}$. 
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Consider the monoid

$$\sim s := \{(x_1, \ldots, x_t, y_1, \ldots, y_t) \in \mathbb{N}^t_0 \mid \sum_{j=1}^{t} x_j s_j = \sum_{j=1}^{t} y_j s_j \}. $$
Since $S$ can be viewed as a submonoid of $\mathbb{N}^k_0$ for some $k \in \mathbb{N}$, the monoid $\sim_S$ is itself atomic and finitely generated.
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Assume that $\mathcal{I}(\sim_S) = \{v_1, \cdots, v_r\}$. 

$$\delta(\tau) = x_1 + \cdots + x_t - y_1 - \cdots - y_t$$

$\delta$ is a monoid homomorphism from $(\sim_S, +)$ to $(\mathbb{Z}, +)$. 
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Let $\tau = (x_1, \ldots, x_t, y_1, \ldots, y_t)$ be any element of $\sim_S$ and define $\delta : \sim_S \to \mathbb{Z}$ by

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More Tools

Set

$$P(\sim S) = \{\delta(v_i) | v_i \in I(\sim S)\} \cap \mathbb{N}$$
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For each \( 1 \leq i \leq r \) set \( v_i = (v_{i1}, \ldots, v_{it}, w_{i1}, \ldots, w_{it}) \) and define
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For each \( 1 \leq i \leq r \) set \( v_i = (v_{i1}, \ldots, v_{it}, w_{i1}, \ldots, w_{it}) \) and define

\[ \alpha(v_i) = \sum_{j=1}^{t} v_{ij}s_j. \]
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Set

\[ H(S) = \bigcup_{i=1}^{r} \Delta(\alpha(v_i)). \]
First Result

Proposition

Let $S$ be an affine monoid.

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2. $\min \Delta(S) \geq \gcd P(\sim S)$
3. $\{\gcd P(\sim S)\} \cup H(S) \subseteq \Delta(S) \subseteq [\gcd P(\sim S), \max H(S)]$. 
Proposition

Let $S = \langle n_1, n_2, \ldots, n_t \rangle$ where $\{n_1, n_2, \ldots, n_t\}$ is the minimal set of generators. Then

$$\min \Delta(S) = \gcd \{n_i - n_{i-1} \mid i \in \{2, 3, \ldots, t\}\}.$$ 

Hence, if $d = \gcd \{n_i - n_{i-1} \mid i \in \{2, 3, \ldots, t\}\}$, then $\Delta(S) \subseteq \{d, 2d, \cdots, qd\}$ for some $q \in \mathbb{N}$. 
Two Interesting Examples

Proposition

Let $d \geq 1$ and set $D_t = \{d, 2d, \ldots, td\}$ for some $t \geq 1$. There exists a three generated numerical monoid $S_t$ so that $\Delta(S_t) = D_t$.
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2. If $S = \langle n, n + 1, n^2 - n - 1 \rangle$ with $n \geq 3$, then

$$\Delta(S) = [1, n - 2] \cup \{2n - 5\}.$$
Let $S = \langle n, n + k, \ldots, n + tk, n + (t + 1)k \rangle$ where $n, t, k \geq 1$. Then $\Delta(S) = \{k\}$. 
Alternate to Original Argument

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**Proposition**

Let $S = \langle a, a + k, \ldots, a + wk \rangle$, with $0 \leq w < a$ and $\gcd(a, k) = 1$,
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1. If $n \in S$, then $n = c_1a + c_2k$ with $c_1, c_2 \in \mathbb{N}$ and $0 \leq c_2 < a$.

2. Suppose $n = c_1a + c_2k \in S$ with $0 \leq c_2 < a$. Then

$$L(n) = \left\{ c_1 + kd \mid \left\lfloor \frac{c_2 - c_1w}{a + wk} \right\rfloor \leq d \leq 0 \right\}.$$
Singleton Delta Sets

Proposition

Let $S = \langle n_1, n_2, n_3 \rangle$ be a numerical monoid with $n_1, n_2, n_3$ a minimal set of generators. If $d = \gcd(n_2 - n_1, n_3 - n_2)$, then the following conditions are equivalent:

1. $\Delta(S)$ is a singleton.
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1. \( \Delta(S) \) is a singleton.
2. \( n_1 \in \left\langle \frac{n_2 - n_1}{d}, \frac{n_3 - n_1}{d} \right\rangle \) and \( n_3 \in \left\langle \frac{n_3 - n_2}{d}, \frac{n_3 - n_1}{d} \right\rangle \).
Theorem

If $M = \langle n_1, \ldots, n_k \rangle$ is a primitive numerical monoid, with $n_1 < n_2 < \cdots < n_k$, then for all $x \geq 2kn_2n_k^2$ we have $\Delta(x) = \Delta(x + n_1n_k)$. Hence, Delta sets of numerical monoids are eventually periodic.
Baginski’s Conjecture

**Theorem**

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**Corollary**

Let $M = \langle n_1, \ldots, n_k \rangle$ be a primitive numerical monoid, then if we set $N = 2kn_2n_k^2 + n_1n_k$, we have:

$$\Delta(M) = \bigcup_{x \in M, x < N} \Delta(x)$$
By earlier results, if \( S = \langle s, s + 1, 2s - 1 \rangle \) for \( s \geq 3 \) then \( \Delta(S) = \{1, 2, \ldots, \left\lfloor \frac{s}{3} \right\rfloor \} \).
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Now, fix the successive differences between the generators and set $M_n = \langle n, n + 1, n + (s - 1) \rangle$. Computer observations indicate that increasing $n$ will cause the size of the delta set to diminish.
An Observation

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For instance, if $s = 21$ we obtain the following.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$M_n$</th>
<th>$\Delta(M_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td>$\langle 21, 22, 41 \rangle$</td>
<td>${1, 2, 3, 4, 5, 6, 7}$</td>
</tr>
<tr>
<td>22</td>
<td>$\langle 22, 23, 42 \rangle$</td>
<td>${1, 2, 3, 4, 5}$</td>
</tr>
<tr>
<td>53</td>
<td>$\langle 53, 54, 73 \rangle$</td>
<td>${1, 2, 3}$</td>
</tr>
<tr>
<td>321</td>
<td>$\langle 321, 322, 341 \rangle$</td>
<td>${1, 2}$</td>
</tr>
<tr>
<td>$n \geq 322$</td>
<td>$\langle n, n + 1, n + 20 \rangle$</td>
<td>${1}$</td>
</tr>
</tbody>
</table>
Kaplan’s Conjecture

Theorem

Let $M_n = \langle n, n + r_1, \ldots, n + r_t \rangle$, where $\gcd(r_1, \ldots, r_t) = z$. Then there exists $N \in \mathbb{N}$ such that for all $n > N$,

$\Delta(M_n) = \left\{ \frac{z}{\gcd(n, z)} \right\}$. Specifically, the statement is true for

$N = r_t(r_t - 1)(t - 1) - 1$. 

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\]
Specifically, the statement is true for \( N = r_t(r_t - 1)(t - 1) - 1 \).

Proposition

Suppose \( r, s \in \mathbb{N} \), \( \gcd(r, s) = 1 \), and \( 0 < r < s \). Set \( M_n = \langle n, n + r, n + s \rangle \) and
\[
N = \max\{rs - r - s, s^2 - rs + r - 3s\}.
\]
Then \( \Delta(M_n) = \{1\} \) for \( n > N \) but \( \Delta(M_N) \neq \{1\} \).
Questions

1. How do the results change when an arithmetic sequence of integers is replaced by a generalized arithmetic sequence of integers (i.e., $S = \langle a, ha + d, ha + 2d, \ldots, ha + xd \rangle$ with $\gcd(a, d) = 1$).
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3. Does Kaplan’s Conjecture work if the *entire* numerical monoid $S$ (and not just the generators) is shifted to the right $n$ units?
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2. Are the catenary degree and tame degree functions also eventually periodic on a numerical monoid?

3. Does Kaplan’s Conjecture work if the entire numerical monoid \( S \) (and not just the generators) is shifted to the right \( n \) units?

4. What are necessary and sufficient conditions on \( n_1, n_2 \) and \( n_3 \) so that

\[
\Delta(S) = [d, \max \Delta(S)] \cap d\mathbb{N}.
\]