This talk is based on a recent paper of the same title (currently under review) by Bill Smith and myself. This paper is based on a 1990 paper of Erdős and Zaks (Journal of Number Theory 36, 89–94).
Definitions of the Erdős-Zaks Paper

Let $a_1, \ldots, a_k, n_1, \ldots, n_k$ be positive integers and set $s = \sum_{i=1}^{k} \frac{a_i}{n_i}$.

Call $s'$ a subsum of $s$ if $s' = \sum_{i=1}^{k} \frac{a'_i}{n_i}$ where the $a_i$'s are integers such that $0 \leq a'_i \leq a_i$ for $i = 1, \ldots, k$.

A sum $s$ is admissible if for some positive integers $b_1, \ldots, b_k$ with $\frac{b_i}{n_i} < 1$ for each $i$, then the sum $\sum_{i=1}^{k} \frac{b_i}{n_i}$ is an integer.

The sum $s$ is reducible if a proper subsum $s'$ exists for which $s' = 1$. The sum $s$ is otherwise irreducible.

The set $\{n_1, \ldots, n_k\}$ is splittable if and only if whenever $s > 1$ and $s$ is an integer, then $s$ is reducible.
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Example

The set \( \{2, 3, 5, 30\} \) is not splittable. The sum

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s = \frac{1}{2} + \frac{2}{3} + \frac{4}{5} + \frac{1}{30} = 2,
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but no subsum of \( s \) equals 1.

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The set \( \{2, 5, 10\} \) is splittable. This is easily seen by considering the inequality

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\frac{x}{10} + \frac{y}{5} + \frac{z}{2} > 1.
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Hence \( x + 2y + 5z > 10 \) and we need only consider the finitely many cases where \( 0 \leq x \leq 9, \ 0 \leq y \leq 4 \) and \( 0 \leq z \leq 1 \).
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Theorem 1

For every integer $N$, $N > 1$, there exists a non-splittable set $\{n_1, \ldots, n_k\}$ such that $s = \sum_{i=1}^{k} \frac{a_i}{n_i}$ is an integer, $s \geq N$ and $s$ is irreducible.

Problem A

For a given $\alpha$ (not necessarily an integer) what is the smallest possible $k$ for which an irreducible sum $s$ exists whose length is $k$?

Problem B

For a given $k$, what is the smallest $\alpha$ that will imply that every sum of length $k$, for which $s \geq \alpha$ is reducible?
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An admissible sum $s$ of length $k$ is reducible whenever

$$s \geq (k - 1) + \frac{1}{p_{k-1}}$$

where $p_{k-1}$ is the $(k - 1)$th prime number.

Theorem 3

For every $i$, $2 \leq i \leq k - 2$, an admissible sum $s$ of length $k$ exists which is irreducible and for which $s = i$. 
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Motivation?

**Question**

What is the motivation for studying admissible sums and splittable sets?

**Answer**

The connection between these sums and the theory of non-unique factorizations in algebraic number rings and more general Dedekind domains.
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Let $D$ be a Krull (or Dedekind) domain with divisor class group $\text{Cl}(D) = G$.

Let $S$ be the set of divisor classes of $D$ which contain height-one prime ideals of $D$.

Set $\mathcal{F}(G)$ be the free abelian monoid on $G$ whose elements we represent as $\prod_{g \in G} g^{x_g}$.

Set $\mathcal{B}(G, S) = \{ \prod_{g \in G} g^{x_g} \mid \sum x_g g = 0 \text{ and } x_g = 0 \text{ if } g \notin S \}$.
Definitions and Notation

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Motivating Fact

Let $D^\bullet$ represent the multiplicative monoid of $D$.

In $D$ suppose $x \in D^\bullet$ is represented uniquely as $(x) = (P_1 \cdots P_k)_v$ where $P_1, \ldots, P_k$ are not necessarily distinct height-one prime ideals of $D$.

The map

$$\varphi : D^\bullet \to B(G, S)$$

defined by

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To study the factorization properties of $D$, we can consider the factorization properties of the Block Monoid $\mathcal{B}(G, S)$.

Example

Let $D$ be an algebraic ring of integers with class number 2. The map from the previous slide is of the form

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\varphi : D^* \rightarrow \mathcal{B}(\mathbb{Z}_2, \mathbb{Z}_2).
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If $B = 0^{x_0}1^{x_1}$ is in $\mathcal{B}(\mathbb{Z}_2, \mathbb{Z}_2)$ then its factorization into irreducible blocks is of the form $B = (0)^{x_0}(1^2)^{x_1/2}$. Hence, given $x \in D$, every irreducible factorization of $x$ contains the same number of irreducible factors. An integral domain with this property is called a half-factorial domain.
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What about the Irreducibles?

Set $\mathcal{B}(\mathbb{Z}_n, \mathbb{Z}_n) = \mathcal{B}(\mathbb{Z}_n)$.

If $n$ is very small, then the irreducibles (or atoms) of $\mathcal{B}(\mathbb{Z}_n)$ can be written out and closely studied.

Even if $n$ is relatively small, the atomic structure of $\mathcal{B}(\mathbb{Z}_n)$ can be chaotic.

Nick Baeth (Central Missouri State) and his group has shown that $\mathcal{B}(\mathbb{Z}_{99})$ has over 30,000,000,000 atoms!
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Can we find sets $S$ in $\mathbb{Z}_n$ for which the atomic structure of $B(\mathbb{Z}_n, S)$ is “nice”?

Answer

YES, using the proof of Theorem 1 in the Erdős-Zaks paper.
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Nice Atomic Structure

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The EZADS Method

We choose \( \{a_1, \ldots, a_k\} \), a set of pairwise relatively prime positive integers greater than or equal to 2 with \( a_i < a_{i+1} \) for \( 1 \leq i \leq k - 1 \). We shall refer to such a set as an EZADS input.

Set

\[
q = \prod_{1}^{k} a_i
\]

and for each \( 1 \leq i \leq k \) set

\[
q_i = \frac{q}{a_i}.
\]

Note by our construction that \( q_1 > q_2 > \cdots > q_k > 1 \) and for each \( i \) we have \( \gcd(q_i, a_i) = 1 \). We finally set

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S = \{q_1, \ldots, q_k, 1\} \subseteq \mathbb{Z}_q.
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What Do the Atoms Look Like?

We describe all the atoms of $B(\mathbb{Z}_q, S)$.

To begin, there are $k + 1$ primary irreducible blocks of the form $q_i^{a_i}$ and $1^q$.

To find the remaining irreducible Blocks, we note from basic Number Theory, for any integer $y$, that the equation

$$q_1 x_1 + \cdots + q_k x_k \equiv y \pmod{q} \quad (*)$$

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has a solution.
A solution to (*) of the form

\[ q_1 x_1 + \cdots + q_k x_k \equiv -d \pmod{q} \]

with all \( x_i \geq 0 \) corresponds to

\[ q_1^{x_1} \cdots q_k^{x_k} 1^d \]  

(\*)

being an element of \( B(\mathbb{Z}_q, S) \).

A necessary (but not sufficient) condition that (**) be irreducible in \( B(\mathbb{Z}_q, S) \) is that \( 0 \leq x_i < a_i \) (we later show that this condition is not sufficient).
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Thus, the non-primary irreducible blocks come from solving (*) for each $d$, $1 \leq d \leq q - 1$ under the constraint $0 \leq x_i < a_i$ and then determining which are reducible.

The solution method, using the Chinese Remainder Theorem, is well known. Moreover, under our constraints it produces a unique solution.
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The Definition

**Definition**

Let $x_1, \ldots, x_k$ be the unique solutions described above. For each $d$ with $1 \leq d \leq q - 1$, set

$$M_d = q_1^{x_1} \cdots q_k^{x_k} 1^d.$$  

Hence, $M_d$ is the unique block in $B(\mathbb{Z}_q, S)$ where the exponents $x_i$ satisfy the constraint $0 \leq x_i < a_i$ for each $1 \leq i \leq k$.

Thus, there are at most $q - 1$ non-primary irreducible blocks of $B(\mathbb{Z}_q, S)$. Hence, the atomic structure of $B(\mathbb{Z}_q, S)$ is simple when compared to that of $B(\mathbb{Z}_q)$.
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For the EZADS input \( \{2, 3, 5\} \) we obtain the following sequence of Blocks \( M_d \) in the Block Monoid \( B(\mathbb{Z}_{30}, \{15, 10, 6, 1\}) \).
<table>
<thead>
<tr>
<th>$M_1$</th>
<th>$M_2$</th>
<th>$M_3$</th>
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<td>$15^010^16^31^2$</td>
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</tbody>
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A Nice Theorem

Theorem

Let $B(\mathbb{Z}_q, S)$ be the Block Monoid obtained from an EZADS input $\{a_1, \ldots, a_k\}$. For each $1 \leq d \leq q-1$, $M_d$ is reducible if and only if $M_d = M_r M_s$ where $r + s = d$. Hence, $M_d$ is irreducible if and only if $M_{d'} \nmid M_d$ for each $1 \leq d' < d$. 
Using the Lemma, the following Blocks are irreducible: $M_1, M_2, M_3, M_4, M_5, M_6, M_8, M_9, M_{10}, M_{12}, M_{14}, M_{15}, M_{18}, M_{20}$ and $M_{24}$. The remaining Blocks in the table are reducible (for instance, $M_{11} = M_6 \cdot M_5$). Note that since $\kappa(M_1) = 2$, the Block Monoid $B(\mathbb{Z}_q, S)$ is not half-factorial. Moreover, $\kappa(M_d) = 1$ for all irreducibles $M_d$ with $d \neq 1$. 
Hence, the block $B = q_1^{y_1} \cdots q_k^{y_k} 1^z$ of $B(\mathbb{Z}_q, S)$ produced by the EZADS input $\{a_1, \ldots, a_k\}$ yields a sum of the form

$$s = \sum_{1}^{n} \frac{y_i}{a_i} + \frac{z}{q}$$

where $q = \prod_{1}^{n} a_i$ and $s = \kappa(B)$ is the cross number of $B$.

### Lemma

Let $I = \{a_1, \ldots, a_k\}$ be an EZADS input and $B = q_1^{y_1} \cdots q_k^{y_k} 1^z$ be as above.

1. The sum $s = \sum_{1}^{n} \frac{y_i}{a_i} + \frac{z}{q}$ associated to $B$ is admissible.
2. If the set $\{a_1, \ldots, a_k, q\}$ is splittable, then $B(\mathbb{Z}_q, S)$ is half-factorial.
Hence, the block $B = q_1^{y_1} \cdots q_k^{y_k} 1^z$ of $\mathcal{B}(\mathbb{Z}_q, S)$ produced by the EZADS input $\{a_1, \ldots, a_k\}$ yields a sum of the form

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**Lemma**

Let $I = \{a_1, \ldots, a_k\}$ be an EZADS input and $B = q_1^{y_1} \cdots q_k^{y_k} 1^z$ be as above.

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The Results

**Theorem**

If \( B = q_1^{y_1} \cdots q_k^{y_k} 1^z \) is an atom of \( B(\mathbb{Z}_q, S) \) then its admissible sum is irreducible in the Erdős-Zaks sense, but NOT conversely.

The smallest counterexample we could find lies in the EZADS input \{3, 5, 11, 13, 17, 19, 127\} where \( q = 87,990,045 \).

We have

\[
s = \frac{2}{3} + \frac{2}{5} + \frac{4}{11} + \frac{12}{13} + \frac{12}{17} + \frac{8}{19} + \frac{66}{127} + \frac{2}{q} = 4
\]

and \( s = t + t \) where

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t = \frac{1}{3} + \frac{1}{5} + \frac{2}{11} + \frac{6}{13} + \frac{6}{17} + \frac{4}{19} + \frac{33}{121} + \frac{1}{q} = 2
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but \( s \) is not divisible by a sum of value 1. Here we also have that \( M_1M_1 = M_2 \).
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Theorem

The EZADs input \( I = \{a_1, \ldots, a_n\} \) for \( n \geq 2 \) yields \( \mathcal{M}_1 = q_1^{1} \cdots q_n^{1} \) if and only if \( a_1, \ldots, a_n \) is the sequence \( a_1 = 2 \) and \( a_k = \prod_{i=1}^{k-1} a_i + 1 \) for \( k \geq 2 \).

Theorem

For every \( n \), there exists an EZADS input \( I = \{a_1, \ldots, a_n\} \) for which \( \mathcal{M}_1 = q_1^{a_1-1} q_2^{a_2-1} \cdots q_n^{a_n-1} \).

Note: The last Theorem leads to shorter proofs of Theorems 1 and 3 in the Erdős-Zaks paper.
More Results

**Theorem**

The EZADs input $I = \{a_1, \ldots, a_n\}$ for $n \geq 2$ yields $M_1 = q_1^{a_1} \cdots q_n^{a_n}1^1$ if and only if $a_1, \ldots, a_n$ is the sequence $a_1 = 2$ and $a_k = \prod_{i=1}^{k-1} a_i + 1$ for $k \geq 2$.

**Theorem**

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