Elastic Properties of Some Semirings

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Happy Halloween

BOO!
Authors Involved

This talk is based on the work of one of the 2008 Trinity REU Algebra Groups.
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With the More than Valuable Input: Stephen McAdam (UT-Austin)
Motivation

Let

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and

\[ \mathbb{N}_0 = \text{the set of nonnegative integers}. \]
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At a recent Conference on the theory of non-unique factorization, Ulrich Krause posed several questions which were motivated by his familiarity with the theory of positive systems in control theory.

1. If

$$\mathbb{R}^+[X] = \{ f(X) \mid f(X) = \sum_{i=0}^{t} a_i x^i \in \mathbb{R}[X] \text{ with } a_i \geq 0 \text{ for every } i \}$$

then what are the relative factorization properties of this multiplicative monoid?
2. Let $d > 0$ be a squarefree integer. If

$$\mathbb{N}_0[\sqrt{d}] = \{a + b\sqrt{d} \mid a, b \in \mathbb{N}_0\}$$

then what factorization properties does this multiplicative monoid inherit from the regular ring of integers in $\mathbb{Q}(\sqrt{d})$?
So, let

\[ M = \text{commutative cancellative monoid} \]

written multiplicatively with identity element 1 and associated group of units \( M^\times \). Set \( M^* = M \setminus M^\times \).
Definitions

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We use the usual conventions involving divisibility:

\[ x \mid y \text{ in } M \iff xz = y \text{ for some } z \in M. \]
Basic Factorization Notation

Set

\[ \mathcal{A}(M) = \text{the set of irreducibles (or atoms) of } M \]
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For \( x \in M^* \), the set

\[ \mathcal{L}(x) = \{ n \mid x = x_1, \ldots, x_n \text{ with each } x_i \in \mathcal{A}(M) \} \]

is called the set of lengths of factorizations of \( x \).
Define for \( x \in M^* \)

\[
L(x) = \sup \mathcal{L}(x) \quad \text{and} \quad l(x) = \inf \mathcal{L}(x),
\]

and

\[
\rho(x) = \frac{L(x)}{l(x)}
\]

to be their quotient. \( \rho(x) \) is called the *elasticity* of \( x \).
On the Elasticity

Define for $x \in M^*$

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We also define

$$\rho(M) = \sup\{\rho(x) \mid x \in M^*\}.$$ 

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On the Delta Set

Given $x \in M \setminus M^\times$, write its length set in the form

$$\mathcal{L}(x) = \{n_1, n_2, \ldots, n_k\}$$

where $n_i < n_{i+1}$ for $1 \leq i \leq k - 1$. 

The $\Delta$-set of $x$ is defined by

$$\Delta(x) = \{n_i - n_{i-1} \mid 2 \leq i \leq k\}$$

and the delta set of $M$ by

$$\Delta(M) = \bigcup_{x \in M \setminus M^\times} \Delta(x).$$
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Examples


*If* $M$ *is a finitely generated atomic monoid then* $\rho(M)$ *is rational and* $\rho(M) = \rho(x)$ *for some* $x \in M^*$.
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Corollary

*If* $D$ *is a Krull domain with finite divisor class group, then* $\rho(D^\bullet)$ *is rational and* $\rho(D^\bullet) = \rho(x)$ *for some* $x \in D^\bullet$.  

Examples

Proposition (Geroldinger *Math. Zeit.* 1988)

*If D is a Krull domain with divisor class group \( \mathbb{Z}_n \), then*

\[
\Delta(D^\bullet) \subseteq \{1, 2, ..., n - 2\}.
\]

*If each divisor class of the class group contains a height-one prime ideal, then*

\[
\Delta(D^\bullet) = \{1, 2, ..., n - 2\}.
\]
Definition

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Set

$$A^+[X] = \{ f(x) \in A[x] \mid f(x) = \sum_{i=0}^{t} a_i x^i \text{ with } a_i \geq 0 \text{ for every } i \}.$$
Elastic Properties of Some Semirings

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Introduction

Background

The Language of Non-Unique Factorizations

What is Known?

Solution to Problem # 1

Definition

Clearly

\[ \mathbb{Z}^+[X] \subseteq A^+[X] \subseteq \mathbb{R}^+[X]. \]
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We note that if \( f(X) \) is a monic nonconstant polynomial in \( A^+[X] \) which is irreducible in \( \mathbb{R}^+[X] \), then \( f(X) \) is irreducible in \( A^+[X] \).
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The Language of Non-Unique Factorizations

What is Known?

Solution to Problem # 1

Theorem

*The monoid $A^+[x]^\bullet$ has infinite, full elasticity, and $\Delta(A^+[x]^\bullet) = \mathbb{N}$.***
Sketch of Proof for Problem #1

Lemma (McAdam’s Lemma)

Let \( n > 1 \) be an integer, and let \( c \geq n \). Then \((x + c)^n(x^2 - x + b) \in A^+[X] \) if and only if \( nb \geq c \).
Sketch of Proof for Problem #1

Lemma (McAdam’s Lemma)

Let \( n > 1 \) be an integer, and let \( c \geq n \). Then
\[
(x + c)^n(x^2 - x + b) \in A^+[X] \text{ if and only if } nb \geq c.
\]

Corollary

If \( b > 1/4 \) and \( nb \geq c \), but \( (n - 1)b < c \), then
\[
(x + c)^n(x^2 - x + b) \text{ is irreducible in } \mathbb{R}^+[x] \text{ and hence in } A^+[X].
\]
Problem #1 Answer

Consider

\[ f(x) = (x + n)^n(x^2 - x + 1)(x + 1)^a, \text{ where } n, a \geq 2. \]
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Thus in $A^+[X]$ we have,

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Thus in $A^+[X]$ we have,

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Thus in $A^+[X]$ we have,

- $\mathcal{L}(f) = \{1 + a, n + a\}$
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Thus in \( A^+[X] \) we have,

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For appropriate choices of \( n \) and \( a \), all rational elasticities are possible, and all possible \( \Delta \) set values are attained.
Let $\alpha$ be an algebraic integer of degree 2 in a real quadratic field. The set

$$\mathbb{N}_0[\alpha] = \{ \beta \in \mathbb{Z}[\alpha] \mid \beta = \sum_{i=0}^{1} u_i \alpha^i \}$$

with $u_i \in \mathbb{N}_0$ for every $i$}

forms a semiring under the usual operations if and only if $\alpha^2 \in \mathbb{N}_0[\alpha]$. One can reduce this to $\alpha = q + r \sqrt{d}$ where $q, r \geq 0$. 
We find a similar result to that of Problem # 1.

**Theorem**

*With notation as above, \( \mathbb{N}_0[\alpha]^\bullet \) has infinite elasticity, full elasticity, and \( \Delta(\mathbb{N}_0[\alpha])^\bullet = \mathbb{N} \).*
Argue that $\mathbb{N}_0[\alpha]^\bullet$ is atomic.
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- The *fundamental unit* $\eta$ of $\mathbb{Z}[\tau]$, is actually irreducible in $\mathbb{N}_0[\tau]$.
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- Argue that $\mathbb{N}_0[\alpha]^*$ is atomic.
- The fundamental unit $\eta$ of $\mathbb{Z}[\tau]$, is actually irreducible in $\mathbb{N}_0[\tau]$. 
- Using a theorem of Hans Rademacher, we show that for any positive integer $k$ there is a prime $\pi$ of the ring $\mathbb{Z}[\tau]$ such that $\pi \in \mathbb{N}_0[\tau]$ and $k$ is least with $\eta \mid \pi^k$ in $\mathbb{N}_0[\tau]$. 
**Sketch of Proof**

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- The *fundamental unit* $\eta$ of $\mathbb{Z}[\tau]$, is actually irreducible in $\mathbb{N}_0[\tau]$.
- Using a theorem of Hans Rademacher, we show that for any positive integer $k$ there is a prime $\pi$ of the ring $\mathbb{Z}[\tau]$ such that $\pi \in \mathbb{N}_0[\tau]$ and $k$ is least with $\eta \mid \pi^k$ in $\mathbb{N}_0[\tau]$.
- Exploiting the unique factorization of $\pi^k$ in $\mathbb{Z}[\tau]$, we will show that the length set of $\pi^k$ is either $\{2, k\}$ or $\{3, k\}$ (with the latter occurring when $\tau = \eta$). Along with the existence of a prime element of $\mathbb{N}_0[\tau]$ this is adequate to prove the theorem.