An Introduction to the Theory of Non-Unique Factorization in Integral Domains and Monoids

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$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$

represents a nonunique factorization into products of irreducible elements.
Let \( D = \mathbb{Z} [\sqrt{-5}] \). In \( D \)

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\]

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Let’s consider this situation more closely.
Ideal Theory Behind the Example

In \( \mathbb{Z}[\sqrt{-5}] \) we have the following ideal decompositions:
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\[
\langle 2 \rangle = \langle 2, 1 + \sqrt{-5} \rangle^2 \\
\langle 3 \rangle = \langle 3, 1 - \sqrt{-5} \rangle \langle 3, 1 + \sqrt{-5} \rangle \\
\langle 1 + \sqrt{-5} \rangle = \langle 2, 1 + \sqrt{-5} \rangle \langle 3, 1 + \sqrt{-5} \rangle \\
\langle 1 - \sqrt{-5} \rangle = \langle 2, 1 + \sqrt{-5} \rangle \langle 3, 1 - \sqrt{-5} \rangle
\]
Ideal Theory Behind the Example

And so

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What is Happening in General?

Let $K = \mathbb{Q}(\alpha)$ be an algebraic extension of $\mathbb{Q}$ and $\mathcal{O}_K$ its ring of integers.
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Let $\text{Cl}(\mathfrak{O}_K)$ represent the ideal class group of $\mathfrak{O}_K$.

Hence, the class number of $\mathbb{Q}(\alpha)$ is $|\text{Cl}(\mathfrak{O}_K)|$. 
Proposition

$\mathcal{O}_K$ has unique factorization of elements into products of irreducibles (i.e., $\mathcal{O}_K$ is a UFD) if and only if $|\text{Cl}(\mathcal{O}_K)| \leq 1$. 
Famous Result of Carlitz (PAMS 1960)

Do higher order class numbers have an arithmetic interpretation?
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**Theorem**

The algebraic number field \( K = \mathbb{Q}(\alpha) \) has class number \( \leq 2 \) if and only if for every nonzero integer \( x \in \mathcal{O}_K \) the number of primes \( \pi_j \) in every factorization

\[
x = \pi_1 \pi_2 \cdots \pi_k
\]

only depends on \( x \).
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**Theorem**

The algebraic number field $K = \mathbb{Q}(\alpha)$ has class number $\leq 2$ if and only if for every nonzero integer $x \in \mathcal{O}_K$ the number of primes $\pi_j$ in every factorization

$$x = \pi_1\pi_2 \cdots \pi_k$$

only depends on $x$.

In general, an integral domain with this property (i.e., every irreducible factorization of a given element has the same length) is known as a half-factorial domain.
Some Notation

Let

\[ M = \text{commutative cancellative monoid} \]

written multiplicatively with identity element 1 and associated group of units \( M^\times \). Set \( M^* = M \setminus M^\times \).
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We use the usual conventions involving divisibility:

\[ x \mid y \text{ in } M \iff xz = y \text{ for some } z \in M. \]

If \( x \mid y \) and \( y \mid x \) in \( M \), then \( x \) and \( y \) are associates.
Some Notation

Call $x \in M^*$

1. *prime* if whenever $x \mid yz$ for $x$, $y$, and $z$ in $M$, then either $x \mid y$ or $x \mid z$.

2. *irreducible* (or an *atom*) if whenever $x = yz$ for $x$, $y$, and $z$ in $M$, then either $y \in M^\times$ or $z \in M^\times$. 

As usual, $x$ prime in $M$ implies $x$ irreducible in $M$ but not conversely.
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Some Notation

Set

\[ \mathcal{A}(M) = \text{the set of irreducibles of } M \]

and

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If \( M^* = \langle \mathcal{A}(M) \rangle \), then \( M \) is called \textit{atomic}.

If \( D \) is an integral domain, then we can apply these conventions to \( D \) by setting \( M = D^\bullet \).
Sets of Lengths

For $x \in M^*$, the set

$$\mathcal{L}(x) = \{ n \mid x = x_1, \ldots, x_n \text{ with each } x_i \in A(M) \}$$

is called \textit{the set of lengths of factorizations of} $x$ and

$$\mathcal{L}(M) = \{ \mathcal{L}(x) \mid x \in M^* \}$$

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Theorem (Geroldinger 1988)

*If $R$ is a ring of integers in a finite extension $K$ of $\mathbb{Q}$ and $x \in R^*$, then $\mathcal{L}(x)$ is an almost arithmetic multipropgression.*
Sets of Lengths

What this essentially means is that for $x \in R^\bullet$ there exists a set of finite arithmetic sequences $A_1, \ldots, A_t$ and positive integers $x_1 < x_2 < \ldots < x_s$ and $z_1 < z_2 < \ldots < z_r$ such that

$$L(x) = \{x_1, \ldots, x_s\} \cup \bigcup_{i=1}^{t} A_t \cup \{z_1, \ldots, z_r\}.$$ 

with $x_s < \min \bigcup_{i=1}^{t} A_t$ and $\max \bigcup_{i=1}^{t} A_t < z_1$. 
Let $M$ be an atomic monoid. Define for $x \in M^*$

$$L(x) = \sup \mathcal{L}(x) \quad \text{and} \quad l(x) = \inf \mathcal{L}(x),$$

and

$$\rho(x) = \frac{L(x)}{l(x)}$$

to be their quotient. $\rho(x)$ is called the *elasticity* of $x$. 

On the Elasticity

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We also define

$$\rho(M) = \sup \{ \rho(x) \mid x \in M^* \}.$$ 

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Variations on the Elasticity

If

$$\{ \rho(x) \mid x \in M^* \} = [1, \rho(M)] \cap \mathbb{Q},$$

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then $M$ is called *fully elastic*.

If there exists an $x \in M^*$ such that

$$\rho(M) = \rho(x)$$

then the elasticity of $M$ is said to be *accepted*. 
Elasticity and Rings of Integers

**Theorem (Valenza, *J. Number Theory* 1990 and others)**

If $R$ is the ring of integers in a finite extension of $\mathbb{Q}$, then

$$\rho(R) = \frac{D(C1(R))}{2}$$

and this elasticity is accepted. Moreover, $R$ is fully elastic.
Theorem (Valenza, *J. Number Theory* 1990 and others)

If $R$ is the ring of integers in a finite extension of $\mathbb{Q}$, then

$$\rho(R) = \frac{D(\text{Cl}(R))}{2}$$

and this elasticity is accepted. Moreover, $R$ is fully elastic.

If $G$ is a finite abelian group, then $D(G)$ represents the *Davenport Constant* of $G$. 
Theorem (Anderson-Anderson-Chapman-Smith *PAMS* 1993)

Let $M$ be a finitely generated commutative cancellative atomic monoid. Then $\rho(M)$ is both rational and accepted.
A Note on Accepted Elasticity

All monoids in the remainder of this talk will have accepted elasticity. Here is an example of a monoid that does not.
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Let $M = \{4, 10, 16, 22, 28, \ldots\} \cup \{1\} = 4 + 6\mathbb{N}_0$. 

Facts:
(proofs are homework for your algebra/number theory classes)

$\rho(M) = 2$.

For each $x > 1$ in $M$, $\rho(x) < 2$.

Hint: The atoms of $M$ come in two forms, those exactly divisible by 2 or those exactly divisible by 4.
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**Hint:** The atoms of \( M \) come in two forms, those exactly divisible by 2 or those exactly divisible by 4.
Let $M$ be an atomic monoid and $x \in M^*$. Suppose

$$\mathcal{L}(x) = \{x_1, x_2, \ldots, x_t\}$$

with $x_1 < x_2 < \ldots < x_t$. Set

$$\Delta(x) = \{x_i - x_{i-1} \mid 2 \leq i \leq t\}.$$ 

We call $\Delta(x)$ the difference set of $x$. 
The Difference Set

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Set

$$\Delta(M) = \bigcup_{x \in M^*} \Delta(x).$$

We call $\Delta(M)$ the *difference set of* $M$. 
Basic Difference Set Result

Theorem (Geroldinger 1991)

If $M$ is a reduced atomic monoid, then

$$\min \Delta(M) = \gcd \Delta(M).$$
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Hence, if $d = \gcd \Delta(M)$, then

$$\{d\} \subseteq \Delta(M) \subseteq \{d, 2d, \ldots, td\}$$

for some nonnegative integer $t$. 
The Delta Set of a Ring of Integers

Theorem (Geroldinger ≈ 1992)

Let \( R \) be the ring of integers in a finite extension of \( \mathbb{Q} \). Then

\[
\Delta(R) \subseteq \{1, 2, \ldots, D(\text{Cl}(R)) - 2\}.
\]

If the class number of \( R \) is a prime \( p \), then

\[
\Delta(R) = \{1, 2, \ldots, p - 2\}.
\]
Definition of a Block Monoid

Let $G$ be an abelian group and

$$\mathcal{F}(G) = \left\{ \prod_{g_i \in G} g_i^{n_i} \mid \text{only finitely many } n_i \neq 0 \right\}$$

be the free abelian monoid on $G$. 
Definition of a Block Monoid

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be the free abelian monoid on $G$.

Let

$$\mathcal{B}(G) = \left\{ \prod_{g_i \in G} g_i^{n_i} \mid \sum_{g_i \in G} n_i g_i = 0 \right\}$$

be the submonoid of $\mathcal{F}(G)$ consisting of zero-sum sequences. $\mathcal{B}(G)$ is known as the block monoid on $G$. 
An Example

Let $\mathbb{Z}_3 = \{0, 1, 2\}$. Then

$$\mathcal{B}(\mathbb{Z}_3) = \{0^{n_0}1^{n_1}2^{n_2} \mid n_1 + 2n_2 \equiv 0 \pmod{3}\}.$$
An Example

Let $\mathbb{Z}_3 = \{\overline{0}, \overline{1}, \overline{2}\}$. Then

$$B(\mathbb{Z}_3) = \left\{ \overline{0}^{n_0} \overline{1}^{n_1} \overline{2}^{n_2} \mid n_1 + 2n_2 \equiv 0 \pmod{3} \right\}.$$

Notice that the irreducible elements of $B(\mathbb{Z}_3)$ are

- $\overline{0}$,
- $\overline{1}^3$,
- $\overline{2}^3$, and
- $\overline{12}$. 
Let $R$ be a ring of integers in a finite extension of $\mathbb{Q}$ and $\alpha \in R^\times$. Suppose

$$\langle \alpha \rangle = P_1 \cdots P_k$$

for not necessarily distinct prime ideals of $R$. Let $[P_i]$ represent the image of the ideal $P_i$ in $\text{Cl}(R)$. 
The Connection with Rings of Integers

Let $R$ be a ring of integers in a finite extension of $\mathbb{Q}$ and $\alpha \in R^\bullet$. Suppose

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for not necessarily distinct prime ideals of $R$. Let $[P_i]$ represent the image of the ideal $P_i$ in $\text{Cl}(R)$.

Define

$$\varphi : R^\bullet \rightarrow B(\text{Cl}(R))$$

by

$$\varphi(\alpha) = [P_1] \cdots [P_k].$$
The Connection with Rings of Integers

Facts:

- $\varphi$ is a monoid homomorphism from $R^\bullet$ to $B(\Cl(R))$.
- $\mathcal{L}(\alpha) = \mathcal{L}([P_1] \cdots [P_k])$.
- Moreover, $\{\mathcal{L}(\alpha) \mid \alpha \in R^\bullet\} = \{\mathcal{L}(B) \mid B \in B(\Cl(R))\}$. 
The Connection with Rings of Integers

Facts:
- \( \varphi \) is a monoid homomorphism from \( R^\bullet \) to \( B(\text{Cl}(R)) \).
- \( L(\alpha) = L([P_1] \cdots [P_k]) \).
- Moreover, \( \{L(\alpha) \mid \alpha \in R^\bullet \} = \{L(B) \mid B \in B(\text{Cl}(R))\} \).

**Moral:** All the results so far for rings of integers are specializations of more general results on Block Monoids.
Question: If $G_1$ and $G_2$ are finite abelian groups and

$$\mathcal{L}(\mathcal{B}(G_1)) = \mathcal{L}(\mathcal{B}(G_2))$$

then must $G_1 \cong G_2$. 

Answer: NO, but very few exceptional cases are known (e.g., $\mathcal{L}(\mathbb{Z}_3) = \mathcal{L}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ is one).
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**Answer:** NO, but very few exceptional cases are known ($\mathcal{L}(\mathbb{Z}_3) = \mathcal{L}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ is one).
Types of Monoids Examined in the Literature

- Block Monoids (*Israel Math. J.* 1990)
- Semigroup Rings (*Archiv Math.* 1993)
- Congruence Monoids (*Colloq. Math.* 2007)
- The Ring of Integer-Valued Polynomials (*J. Algebra* 2005)
- Numerical Monoids (rest of talk)
Definition of a Numerical Monoid

If $n_1 < n_2 < \cdots < n_t$ are natural numbers, then set

$$\langle n_1, n_2, \ldots, n_t \rangle = \{x_1 n_1 + x_2 n_2 + \cdots + x_t n_t \mid \text{each } x_i \in \mathbb{N}_0\}.$$ 

$\langle n_1, n_2, \ldots, n_t \rangle$ is an additive submonoid of $\mathbb{N}_0$. 
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\( \langle n_1, n_2, \ldots, n_t \rangle \) is an additive submonoid of \( \mathbb{N}_0 \).

Such a submonoid is known as a *numerical monoid*. The integers \( n_1, \ldots, n_t \) are called the *generators* of \( \langle n_1, n_2, \ldots, n_t \rangle \) and if they are relatively prime, then \( \langle n_1, n_2, \ldots, n_t \rangle \) is called a *primitive numerical monoid*. 
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If \( n_1 < n_2 < \cdots < n_t \) are natural numbers, then set

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Simple Fact: Every numerical monoid is isomorphic to a primitive numerical monoid.
Let $S = \langle n_1, \ldots, n_t \rangle$ be a primitive numerical monoid. The largest $n \in \mathbb{N}$ such that $n \notin S$ is called the *Frobenius Number of S* and denoted $F(S)$. 

\begin{align*}
\text{Theorem (Sylvester 1884)} \\
F(S) &= (n_1 - 1)(n_2 - 1) - 1 = n_1 n_2 - n_1 - n_2.
\end{align*}
Let $S = \langle n_1, \ldots, n_t \rangle$ be a primitive numerical monoid. The largest $n \in \mathbb{N}$ such that $n \not\in S$ is called the *Frobenius Number* of $S$ and denoted $\mathcal{F}(S)$.

**Theorem (Sylvester 1884)**

Let $S = \langle n_1, n_2 \rangle$ be a primitive numerical monoid. Then

$$\mathcal{F}(S) = (n_1 - 1)(n_2 - 1) - 1 = n_1 n_2 - n_1 - n_2.$$

Let $S = \langle n_1, n_2, \ldots, n_t \rangle$ be a primitive numerical monoid whose given generating set is minimal with $t \geq 2$.

- $\rho(S) = \frac{n_t}{n_1}$.
- There exists a rational number $q$ with $1 < q < \frac{n_t}{n_1}$ such that $\rho(s) \neq q$ for all $s \in S$ (i.e., $S$ is not fully elastic).

- If $S = \langle n_1, \ldots, n_k \rangle$ then
  $\min \Delta(S) = \gcd\{n_i - n_{i-1} \mid 2 \leq i \leq k\}$.
- If $S = \langle n, n + k, n + 2k, \ldots, n + tk \rangle$, then $\Delta(S) = \{k\}$.
- For any positive integers $k$ and $t$, there exists a three generated numerical monoid $S$ such that
  $\Delta(S) = \{k, 2k, \ldots, tk\}$.
- For $n \geq 3$,
  $\Delta(\langle n, n + 1, n^2 - n - 1 \rangle) = \{1, 2, \ldots, n - 2\} \cup \{2n - 5\}$. 
Theorem (Trinity REU 2006 - *Integers* 2007)

Let $S$ and $S'$ be numerical monoids requiring at least two generators. Then $\mathcal{L}(S) = \mathcal{L}(S')$ does not imply that $S = S'$. In fact, set

$$S = \langle a, a + k, \ldots, a + wk \rangle,$$

with $0 \leq w < a$ and $\gcd(a, k) = 1$, and also

$$S' = \langle c, c + t, \ldots, c + vt \rangle,$$

where $v < c$, $\gcd(c, t) = 1$. If $S \neq S'$, then the following statements are equivalent:

- $\mathcal{L}(S) = \mathcal{L}(S')$
- $k = t$, $\frac{c}{a} = \frac{v}{w}$, $\gcd(a, w) \geq 2$, and $\gcd(c, v) \geq 2$. 
Theorem (Trinity REU 2007 - to appear in *Aequationes Math*)

Given a primitive numerical monoid $M = \langle n_1, \ldots, n_k \rangle$, we have for all $x \geq 2kn_2n_k^2$ that $\Delta(x) = \Delta(x + n_1n_k)$. 
Theorem (Trinity REU 2007 - to appear in Aequationes Math)

Given a primitive numerical monoid \( M = \langle n_1, \ldots, n_k \rangle \), we have for all \( x \geq 2kn_2n_k^2 \) that \( \Delta(x) = \Delta(x + n_1n_k) \).

Corollary

Let \( M = \langle n_1, \ldots, n_k \rangle \) be a primitive numerical monoid, then if we set \( N = 2kn_2n_k^2 + n_1n_k \), we have:

\[
\Delta(M) = \bigcup_{x \in M, x < N} \Delta(x)
\]