5.7 Euler’s formula and the right way to use complex numbers

5.7.1 Euler’s Equation

The value of complex numbers was recognized but poorly understood during the late Renaissance period. The number system was explicitly studied in the late 18th century. Euler used \( i \) for the square root of \(-1\) in 1779. Gauss used the term “complex” in the early 1800’s. The “Argand diagram” or “Gauss plane”, was introduced in a memoir by Argand in Paris in 1806, although it was implicit in the doctoral dissertation of Gauss in 1799 and in work of Caspar Wessel around the same time.

Notice the following remarkable fact that if
\[
z = \sqrt{\frac{3}{2}} + \frac{1}{2} i = \cos \frac{\pi}{6} + i \sin \frac{\pi}{6}
\]
then \( z^3 = i \). (Multiply it out and see!) Thus \( z^{12} = 1 \) and so \( z \) is a twelfth root of 1.

Now the polar coordinate form for \( z \) is \( r = 1, \theta = \frac{\pi}{6} \), that is, \( z \) is exactly one-twelfth of the way around the unit circle. \( z \) is a twelfth root of 1 and it is one-twelfth of the way around the unit circle. This is not a coincidence!

DeMoivre apparently noticed this and proved (by induction) that if \( n \) is an integer then
\[
(cos \theta + i sin \theta)^n = \cos n\theta + i \sin n\theta. \tag{42}
\]

Thus exponentiation, that is raising a complex number to some power, is equivalent to multiplication of the arguments. Somehow the angles in the complex number act like exponents.

Euler would explain why that was true. Using the derivative and infinite series, he would show that
\[
e^{i\theta} = \cos \theta + i \sin \theta \tag{43}
\]

By simple laws of exponents, \((e^{iz})^n = e^{inz}\) and so Euler’s equation \(e^{iz} = \cos(z) + i \sin(z)\) explains DeMoivre formula \(42\).

This explains the “coincidence” we noticed with the complex number \( z = \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \) which is one-twelfth of the way around the unit circle; raising \( z \) to the twelfth power will simply multiply the angle \( \theta \) by twelve and move the point to the point with angle zero: \( (1, 0) = 1 + 0i \).

5.7.2 Trig functions in terms of the exponential function

Euler’s formula \(e^{iz} = \cos(z) + i \sin(z)\) allows us to write the exponential function in terms of the two basic trig functions, sine and cosine. We may then use Euler’s formula to find a formula for \( \cos(z) \) and \( \sin(z) \) as a sum of exponential functions. Since \( e^{iz} = \cos(z) + i \sin(z) \) then
\[
e^{-iz} = \cos(-z) + i \sin(-z) = \cos(z) - i \sin(z). \tag{44}
\]

Add the expressions for \( e^{iz} \) and \( e^{-iz} \) to get
\[
e^{iz} + e^{-iz} = 2 \cos(z) \tag{45}
\]

and so
\[
\cos(z) = \frac{e^{iz} + e^{-iz}}{2}. \tag{46}
\]

If we subtract the equation \( e^{-iz} = \cos(z) - i \sin(z) \) from Euler’s equation \(43\) and then divide by \( 2i \), we have a formula for sine:
\[
\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}. \tag{47}
\]
Euler’s formula writes the exponential function in terms of trig functions; now we have written a trig function in terms of exponential functions. The exponential and trig functions are very closely related. Trig functions are, in some sense, really exponential functions in disguise!

Some worked examples.
Let’s try out some applications of Euler’s formula. Here are some worked problems.

1. Put the complex number \( z = e^{\pi i} \) in the “Cartesian” form \( z = a + bi \).

**Solution.** \( z = e^{\pi i} = 1(\cos(\pi) + i \sin(\pi)) = 1(-1 + 0i) = -1 \)

It seems remarkable that if we combine the three *strangest* math constants, \( e, i \) and \( \pi \) we get

\[ e^{\pi i} = -1. \]

Some rewrite this in the form 

\[ e^{\pi i} + 1 = 0 \]

(often seen on t-shirts for engineering clubs or math clubs.)

2. Put the complex number \( z = 2e^{13\pi i/6} \) in the “Cartesian” form \( z = a + bi \) where \( a, b \in \mathbb{R} \).

**Solution.** 

\[ z = 2e^{13\pi i/6} = 2 \cos(13\pi/6) + 2i \sin(13\pi/6) = 2 \cos(\pi/6) + 2i \sin(\pi/6) = \sqrt{3} + i. \]

3. Put the complex number \( z = 18 + 26i \) in the “polar” form \( z = re^{i\theta} \) where \( r, \theta \in \mathbb{R} \) and both \( r \) and \( \theta \) are positive.

**Solution.** The modulus of \( z = 18 + 26i \) is \( \sqrt{18^2 + 26^2} = 1000 \). So the polar coordinate form of 

\[ z = 18 + 26i = \sqrt{10^4} e^{i\theta} \text{ where } \theta = \tan^{-1}(\frac{13}{8}). \]

(The angle \( \theta \) is about 0.96525166319.)

4. Find a cube root of the number \( z = 18 + 26i \) and put this cube root in the “Cartesian” form \( z = a + bi \) where \( a, b \in \mathbb{R} \). (Use a calculator and get an exact value.)

**Solution.** Using the previous problem, we write \( z = 18 + 26i = \sqrt{10^4} e^{i\theta} \) where \( \theta = \tan^{-1}(\frac{13}{8}) \).

The cube root of \( \sqrt{10^4} e^{i\theta} \) is 

\[ \sqrt[3]{10^4} e^{i\theta/3} \]

(The angle \( \theta/3 \) is about 0.3217505544.) Using a calculator, we can see that this comes out to approximately

\[ \sqrt[3]{10} \cos(\theta/3) + i\sqrt[3]{10} \sin(\theta/3) = 3 + i. \]

One could check by computing \((3 + i)^3\) and see that we get 18 + 26i.

5. Find a complex number \( z \) such that \( \ln(-1) = z \).

**Solution.** Since \(-1\) in polar coordinate form is \(-1 = e^{i\pi}\) then \( z = \pi i \) is a solution to \( \ln(-1) \).

6. (A question found on the internet:) What is \( i^i \)?

**Solution.** We can find one answer if we write the base \( i \) in polar form \( i = e^{\pi i/2} \).

Then \( i^i = (e^{\pi i/2})^i = e^{\pi i^2/2} = e^{-\pi/2} \approx 0.207879576350761908546955619834978770033877841631769608075135... \)

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\(^3\)More carefully, we might note that \( i = e^{\pi i/2 + 2\pi ki} \), for any integer \( k \). This will give us an infinite number of possibilities for the expression \( i^i \).
5.7.3 Complex numbers v. Real numbers

Here are some things one can do with the real numbers:

1. Show that \( f(x) = \sin x \) is periodic with period \( 2\pi \), that is, \( f(x + 2\pi) = f(x) \).
2. Find an infinite set of numbers, \( x \), such that \( \sin(x) = 1/2 \).
3. Find a number \( x \) such that \( e^x = 200 \).
4. Compute \( \ln(2) \).

Here are some things that require complex numbers:

1. Show that \( f(x) = e^x \) is periodic with period \( 2\pi i \), that is, \( f(x + 2\pi i) = f(x) \).
2. Find an infinite set of numbers, \( x \), such that \( e^x = 1/2 \).
3. Find a number \( x \) such that \( \sin(x) = 200 \).
4. Compute \( \ln(-2) \).

These are all topics for further exploration in a course in complex variables.

5.7.4 Other resources on Euler’s famous equation

In the free textbook, *Precalculus*, by Stitz and Zeager (version 3, July 2011, available at [stitz-zeager.com](http://stitz-zeager.com)) DeMoivre’s Theorem is covered in section 11.7 but Euler’s formula is not covered.

In the free textbook, *Precalculus, An Investigation of Functions*, by Lippman and Rassmussen (edition 1.3, available at [www.opentextbookstore.com](http://www.opentextbookstore.com)) DeMoivre’s Theorem is covered in section 8.3 but Euler’s formula is not covered.


Here are some online resources:

1. [Wikipedia article](https://en.wikipedia.org/wiki/Euler%27s_identity) on Euler’s formula.
2. Khan Academy videos on Euler’s formula via Taylor series.

**Worksheet to go with these notes.**

Please work through Worksheet 5.7 on Euler’s equation and complex numbers, available on the class webpage and Blackboard.