Showing something is not an identity

To show that an equation is not an identity it suffices to find just one value of \( \theta \) for which the equation fails!

For example, is the equation

\[ \sin \theta = \cos \theta \]

an identity? No – try \( \theta = 0 \) and see that the left-hand side is not the same as the right-hand side.

\[ 0 \neq 1 \]

Is \( \cos \theta = \sqrt{1 - \sin^2 \theta} \) an identity?

Here we might want to try an angle like \( \theta = \pi \) to see that this equation can fail (since the left-hand side is negative while the right-hand side is positive.)

\[ -1 \neq 1 \]

Is \( 2 \cos x = \cos 2x \)?

Let’s try \( x = 0 \).

\[ 2 \cos 0 = 2(1) = 2 \]
\[ \cos(2 \cdot 0) = \cos 0 = 1 \]

\[ 2 \neq 1! \]

Is \( 2 \sin x = \sin 2x \)?

If I try \( x = 0 \), I get a true statement. But this just means I was unlucky.... Try \( x = \pi/2 \). (Always pick something fairly simple to compute.)

\[ 2 \neq 0 \]
In the next presentation, we will look in depth at the Pythagorean Identities.

(End)

The Pythagorean Identities

From the Pythagorean theorem we found the equation for the unit circle:

$$x^2 + y^2 = 1.$$ 

From that equation and from our definition of $\cos \theta$ as the $x$-value and $\sin \theta$ as the $y$-value of points on the circle, we discovered the identity

$$\cos^2 \theta + \sin^2 \theta = 1. \quad (1)$$

If we divide both sides of equation 1 above by $\cos^2 \theta$ we can rewrite that identity as

$$1 + \tan^2 \theta = \sec^2 \theta. \quad (2)$$

If we divide both sides of equation 1 above by $\sin^2 \theta$ we can rewrite that identity as

$$\cot^2 \theta + 1 = \csc^2 \theta. \quad (3)$$

This triplet of identities are called the Pythagorean identities. We use them to change even powers of one trig function into an expression involving even powers of another.

The Pythagorean Identities

Solve $\cos^2 x + 5 \sin x - 7 = 0$.

**Solution.** We replace $\cos^2 x$ by $1 - \sin^2 x$ and so consider the equation

$$1 - \sin^2 x + 5 \sin x - 7 = 0.$$ 

This simplifies to

$$-\sin^2 x + 5 \sin x - 6 = 0.$$ 

Multiply both sides by $-1$ to get the equation

$$\sin^2 x - 5 \sin x + 6 = 0.$$ 

Factor the lefthand side:

$$(\sin x - 2)(\sin x - 3) = 0.$$

and note that this requires $\sin x = 2$ or $\sin x = 3$, both of which are impossible. So there is **no solution** to this equation.
Solve the equation \( \sec^2 \theta + \tan \theta - 1 = 0 \)

**Solution.** Suppose \( \sec^2 \theta + \tan \theta - 1 = 0 \). Using a Pythagorean identity we replace \( \sec^2 \theta \) by \( \tan^2 \theta + 1 \) and write this as

\[
(tan^2 \theta + 1) + \tan \theta - 1 = 0.
\]

Simplify, so that

\[
tan^2 \theta + \tan \theta = 0.
\]

Factor out \( \tan \theta \) to get

\[
\tan \theta (\tan \theta + 1) = 0.
\]

So either

\[
\tan \theta = 0
\]

or

\[
\tan \theta = -1.
\]

In the first case, \( \theta = 0 + \pi k \) and in the second case, \( \theta = \frac{3\pi}{4} + \pi k \). So our answer is

\[
\{k \pi, \frac{3\pi}{4} + k \pi \}.
\]

The Pythagorean Identities

Solve \( \sqrt{3} \cos x = \sin x + 1 \).

**Solution.** Since cosine and sine are both in this equation, it would be nice if they were squared. We can use that idea by first squaring both sides:

\[
(\sqrt{3} \cos x)^2 = (\sin x + 1)^2
\]

and so

\[
3 \cos^2 x = \sin^2 x + 2 \sin x + 1.
\]

Replace \( \cos^2 x \) by \( 1 - \sin^2 x \) to get the equation

\[
3(1 - \sin^2 x) = \sin^2 x + 2 \sin x + 1.
\]

so (after moving everything to the righthand side) we have

\[
0 = 4 \sin^2 x + 2 \sin x - 2.
\]

Divide both sides by 2

\[
0 = 2 \sin^2 x + \sin x - 1
\]

and factor \( 2 \sin^2 x + \sin x - 1 = (2 \sin x - 1)(\sin x + 1) \). Thus we have

\[
\sin x = \frac{1}{2}
\]

or \( \sin x = -1 \).

In the first case

\[
x = \pi/6 + 2\pi k \text{ or } x = 5\pi/6 + 2\pi k
\]

and in the second case

\[
x = -\pi/2 + 2\pi k.
\]

The Pythagorean Identities

We attacked the equation \( \sqrt{3} \cos x = \sin x + 1 \) by squaring both sides and using a Pythagorean identity to change the equation into

\[
0 = (2 \sin x - 1)(\sin x + 1)
\]

and so either \( \sin x = \frac{1}{2} \) or \( \sin x = -1 \).

In the first case

\[
x = \pi/6 + 2\pi k \text{ or } x = 5\pi/6 + 2\pi k
\]

and in the second case

\[
x = -\pi/2 + 2\pi k.
\]

Now when we squared both sides we may have introduced additional elements into our solution set (for example, maybe we now have elements where \( \sin x = -\sqrt{3} \cos x \)) and so we should check our answers. (It is always good practice to check our answers!)

If \( x = \pi/6 \) then \( \sin x + 1 = \frac{3}{2} \) which is the same as \( \sqrt{3} \cos(\pi/6) \). But if \( x = 5\pi/6 \) then \( \sin x + 1 = \frac{3}{2} \) while \( \sqrt{3} \cos(5\pi/6) = -\frac{3}{2} \). So \( x = 5\pi/6 \) is not a solution to the original equation!

The other solutions check out so our final solution set is

\[
\{k \pi, \frac{3\pi}{4} + k \pi \}.
\]
In the next presentations we examine some more trig identities.

(End)