4.7 Solving Problems with Inverse Trig Functions

4.7.1 Inverse trig functions create right triangles

An inverse trig function has an angle (\( y \) or \( \theta \)) as its output. That angle satisfies a certain trig expression and so we can draw a right triangle that represents that expression.

One can always draw a right triangle with an inverse trig function and think of the output as a certain angle in that triangle. For example, the equation \( \arcsin(z) = \theta \) implies that \( \sin \theta = z \) and so it can be viewed as corresponding to a right triangle with hypotenuse 1, with \( \theta \) one of the acute angles and \( z \) the length of the side opposite \( \theta \). (See Figure 34, below.)

![Figure 34. The triangle that appears with the equation arcsin(z) = \theta.](image)

We will practice this idea with some worked problems....

Some Worked Problems

1. Draw a right triangle with the appropriate lengths and use that triangle to find the sine of the angle \( \theta \) if
   
   (a) \( \cos(\theta) = \frac{2}{3} \)
   (b) \( \cos(\theta) = \frac{2}{5} \)
   (c) \( \cos(\theta) = 0.8 \)
   (d) \( \cos(\theta) = 0.6 \)

   Partial solutions.
   
   (a) If \( \cos(\theta) = \frac{2}{3} \) then draw a triangle with legs of length 2, \( \sqrt{5} \) and hypotenuse of length 3. If the cosine of \( \theta \) is \( \frac{2}{3} \) then the sine of \( \theta \) is then \( \frac{\sqrt{5}}{3} \).
   
   (b) If \( \cos(\theta) = \frac{2}{5} \) then draw a triangle with legs of length 2, \( \sqrt{11} \) and hypotenuse of length 5. The sine of \( \theta \) is then \( \frac{\sqrt{11}}{5} \).
   
   (c) If \( \cos(\theta) = 0.8 \) then draw a triangle with legs of length 3, 4 and hypotenuse of length 5. The sine of \( \theta \) is then \( \frac{3}{5} \).
   
   (d) If \( \cos(\theta) = 0.6 \) then draw a triangle with legs of length 3, 4 and hypotenuse of length 5. The sine of \( \theta \) is then \( \frac{4}{5} \).

When we work with inverse trig functions it is especially important to draw a triangle since the output of the inverse trig function is an angle of a right triangle. Indeed, one could think of inverse trig functions as “creating” right triangles.
The angle \( \theta \) in the drawing in Figure 34 is \( \arcsin(z) \). Notice that the Pythagorean theorem then gives us the third side of the triangle (written in blue); its length is \( \sqrt{1-z^2} \). This allows us to simplify expressions like \( \cos(\arcsin z) \), recognizing that
\[
\cos(\arcsin z) = \cos(\theta) = \sqrt{1-z^2}.
\]
In a similar manner, we can simplify \( \tan(\arcsin z) \) to
\[
\tan(\arcsin(z)) = \frac{z}{\sqrt{1-z^2}}.
\]

**Some worked problems.**

1. Simplify (without use of a calculator) the following expressions
   (a) \( \arcsin[\sin(\frac{\pi}{8})] \).
   (b) \( \arccos[\sin(\frac{\pi}{8})] \).
   (c) \( \cos[\arcsin(\frac{1}{3})] \).

**Solutions.**

(a) Since \( \arcsin \) is the inverse function of sine then \( \arcsin[\sin(\frac{\pi}{8})] = \frac{\pi}{8} \).

(b) If \( \theta \) is the angle \( \frac{\pi}{8} \) then the sine of \( \theta \) is the cosine of the complementary angle \( \frac{\pi}{2} - \frac{\pi}{8} \), which, after getting a common denominator, simplifies to \( \frac{3\pi}{8} \). In other words, the sine of \( \frac{\pi}{8} \) is the cosine of \( \frac{3\pi}{8} \) so \( \arccos[\sin(\frac{\pi}{8})] = \frac{3\pi}{8} \). (Notice that I’ve solved this problem this without ever having to figure out the value of \( \sin(\frac{\pi}{8}) \).

(c) To simplify \( \cos[\arcsin(\frac{1}{3})] \) we draw a triangle with hypotenuse of length 3 and one side of length 1, placing the angle \( \theta \) so that \( \sin(\theta) = \frac{1}{3} \). The other short side of the triangle must have length \( \sqrt{8} = 2\sqrt{2} \) by the Pythagorean theorem so the cosine of \( \theta \) is \( \frac{2\sqrt{2}}{3} \). So \( \cos[\arcsin(\frac{1}{3})] = \frac{2\sqrt{2}}{3} \).

2. Simplify (without the use of a calculator) the following expressions:
   (a) \( \arccos(\sin(\theta)) \), assuming that \( \theta \) is in the interval \([0, \frac{\pi}{2}] \).
   (b) \( \arccos(y) + \arcsin(y) \).

**Solutions.**

(a) To simplify \( \arccos(\sin(\theta)) \), we draw a triangle (on the unit circle, say) with an acute angle \( \theta \) and short sides of lengths \( x, y \) and hypotenuse 1. (See the figure below.)

![Figure 35](image_url)

**Figure 35.** A right triangle with \( \theta \) and its complementary angle \( \frac{\pi}{2} - \theta \).
The sine of $\theta$ is then $y$ and the arccosine of $y$ must be the complementary angle $\frac{\pi}{2} - \theta$. So $\arccos(\sin(\theta)) = \frac{\pi}{2} - \theta$.

(b) Notice in the triangle in Figure 35 that the sine of $\theta$ is $y$ and the cosine of $\frac{\pi}{2} - \theta$ is $y$. So $\arcsin(y) = \theta$ and $\arccos(y) = \frac{\pi}{2} - \theta$. Therefore

$$\arccos(y) + \arcsin(y) = \theta + \left(\frac{\pi}{2} - \theta\right) = \frac{\pi}{2}.$$  

Indeed, the expression $\arccos(y) + \arcsin(y)$ merely asks for the sum of two complementary angles! By definition, the sum of two complementary angles is $\frac{\pi}{2}$!

4.7.2 Drawing triangles to solve composite trig expressions

Some problems involving inverse trig functions include the composition of the inverse trig function with a trig function. If the inverse trig function occurs first in the composition, we can simplify the expression by drawing a triangle.

Here are some worked problems.

Worked problems.

1. Do the following problems without a calculator.

   Find the exact value of

   (a) $\sin(\arccos(-\frac{3}{4}))$
   (b) $\tan(\arcsin(-\frac{1}{2}))$
   (c) $\sin(2 \arctan(-\frac{1}{3}))$ (Use the trig identity $\sin 2\theta = 2 \sin \theta \cos \theta$.)

Solutions.

(a) To compute $\sin(\cos^{-1}(-\frac{3}{4}))$ draw a triangle with legs $3, \sqrt{7}$ and hypotenuse 4. The angle $\theta$ needs to be in the second quadrant so the sine will be positive. So the sine of the angle $\theta$ should be $\frac{\sqrt{7}}{4}$.

(b) To compute $\tan(\sin^{-1}(-\frac{3}{4}))$ draw a triangle with legs $3, \sqrt{7}$ and hypotenuse 4. The tangent of the angle $\theta$ should be $-\frac{3}{\sqrt{7}}$. But the angle $\theta$ is in the fourth quadrant so the final answer is $-\frac{3}{\sqrt{7}}$.

(c) To compute $\sin(2 \arctan(-\frac{1}{3})) = 2 \sin \theta \cos \theta$ where $\tan(\theta) = -\frac{4}{3}$. draw a triangle with legs $3, 4$ and hypotenuse 5. The cosine of the angle $\theta$ is $\frac{4}{5}$ and the sine of the angle $\theta$ is $\frac{3}{5}$. Since the original problem has a negative sine in it, we must be working with an angle in the fourth quadrant, so the sine is really $-\frac{4}{5}$. Now we just plug these values into the “magical” identity given us:

$$\sin(2\theta) = 2 \sin \theta \cos \theta = 2 \left(-\frac{4}{5}\right) \left(\frac{3}{5}\right) = -\frac{24}{25}.$$  

6See section 5.3 of these notes for an explanation of the “double angle” identity for sine.
2. Draw a right triangle with short legs of length 1 and \( x \) and then compute

(a) \( \sin(\arctan(x)) \)
(b) \( \tan(\arctan(x)) \)
(c) \( \cot(\arctan(x)) \)
(d) \( \sec(\arctan(x)) \)

**Solutions.**

(a) To compute \( \sin(\arctan(x)) \) draw a right triangle with sides 1, \( x \) and hypotenuse \( \sqrt{1 + x^2} \). The sine of the angle \( \theta \) is \( \frac{x}{\sqrt{1 + x^2}} \).

(b) To compute \( \tan(\arctan(x)) \) just recognize that \( \tan x \) and \( \arctan x \) are inverse functions and so \( \tan(\arctan(x)) = x \).

(c) To compute \( \cot(\arctan(x)) \) draw a right triangle with sides 1, \( x \) and hypotenuse \( \sqrt{1 + x^2} \). The cotangent of the angle \( \theta \) is \( \frac{1}{x} \).

(d) To compute \( \sec(\arctan(x)) \) draw a right triangle with sides 1, \( x \) and hypotenuse \( \sqrt{1 + x^2} \). The secant of the angle \( \theta \) should be \( \sqrt{1 + x^2} \).

### 4.7.3 More on inverting composite trig functions

Just like other functions, we can algebraically manipulate expressions to create an inverse function.

**Some worked problems.** Find the inverse function of the following functions.

1. \( y = \sin(\sqrt{x}) + 2 \)
2. \( y = \sin(\sqrt{x} + 2) \)
3. \( y = \sin(\sqrt{x} + 2) \)
4. \( y = e^{\sin(\sqrt{x} + 2)} \)
5. \( y = \sin(\arccos x) \)

**Solutions.**

1. To find the inverse function of \( y = \sin(\sqrt{x}) + 2 \), let’s exchange inputs and outputs:

\[ x = \sin(\sqrt{y}) + 2 \]

and then solve for \( y \) by subtracting 2 from both sides

\[ x - 2 = \sin(\sqrt{y}) \],

applying the \( \arcsin \) to both sides,

\[ \arcsin(x - 2) = \sqrt{y} \]

and then squaring both sides

\[ (\arcsin(x - 2))^2 = y \]

so that the answer is \( y = (\arcsin(x - 2))^2 \).
2. The inverse function of \( y = \sin(\sqrt{x} + 2) \) is \( y = (\arcsin(x) - 2)^2 \).

3. The inverse function of \( y = \sin(\sqrt{x} + 2) \) is \( y = (\arcsin x)^2 - 2 \).

4. The inverse function of \( y = e^{\sin(\sqrt{x}+2)} \) is \( y = (\arcsin(\ln x) - 2)^2 \).

5. The inverse function of \( y = \sin(\cos^{-1} x) \) is the inverse function of \( y = \sqrt{1-x^2} \). It happens that the inverse function of \( y = \sqrt{1-x^2} \) obeys the equation \( x = \sqrt{1-y^2} \) so \( x^2 = 1 - y^2 \) so \( y^2 = 1 - x^2 \) so \( y = \sqrt{1-x^2} \). (That is \( y = \sqrt{1-x^2} \) is its own inverse function!)

### 4.7.4 Other resources on sinusoidal functions

(Hard copy references need to be fixed.) In the free textbook, *Precalculus*, by Stitz and Zeager (version 3, July 2011, available at stitz-zeager.com) this material is covered in section 10.6.

In the free textbook, *Precalculus, An Investigation of Functions*, by Lippman and Rasmussen (Edition 1.3, available at www.opentextbookstore.com) this material is covered in section 6.4 and 6.5.

In the textbook by Ratti & McWaters, *Precalculus, A Unit Circle Approach*, 2nd ed., c. 2014 this material appears in section ??.

In the textbook by Stewart, *Precalculus, Mathematics for Calculus*, 6th ed., c. 2012 (here at Amazon.com) this material appears ??.

**Homework.**

As class homework, please complete **Worksheet 4.7, More Inverse Trig Functions** available through the class webpage.