3 Exponential and logarithmic functions

3.1 Introduction to exponential functions

An exponential function is a function of the form \( f(x) = b^x \) where \( b \) is a fixed positive number. The constant \( b \) is called the base of the exponent.

For example, \( f(x) = 2^x \) is an exponential function with base 2.

Most applications of mathematics in the sciences and economics involve exponential functions. They are probably the most applicable class of functions we will study in this course.

3.1.1 Review of basic properties of exponents

Practice Problem. \( f(x) = 2^x \). Fill in the table for the values of \( x \) and \( f(x) \) and then graph \( y = f(x) \) using the points you have plotted.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( -2 )</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>( -2 )</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>( -6 )</td>
<td>( \frac{1}{64} )</td>
</tr>
<tr>
<td>5</td>
<td>32</td>
</tr>
<tr>
<td>( -5 )</td>
<td>( \frac{1}{32} )</td>
</tr>
<tr>
<td>( -10 )</td>
<td>( \frac{1}{1024} )</td>
</tr>
<tr>
<td>( DNE )</td>
<td>0</td>
</tr>
</tbody>
</table>

Let’s use this work to graph the function \( f(x) = 2^x \). We plot some of the points we found in the table and “connect the dots”, assuming that the exponential function is continuous. If we plot enough points, our graph should look something like this:

![Figure 1. The graph of \( y = 2^x \)](image-url)
3.1.2 Domain and range of exponential functions

With exponential functions such as $y = b^x$, will always assume that our base $b$ is positive. (This is necessary if we are to make sense of expressions like $b^{1/2}$. In general we will also assume that our base $b$ is greater than 1. (If $b$ is between 1 and 0, replace $b^x$ by $b^{-x} = \left(\frac{1}{b}\right)^x$ so that the base is greater than 1.)

The domain of $f(x) = b^x$ is then the entire real line, $(-\infty, \infty)$. Since $b$ is a positive number, $b^x$ will be positive if $x$ is positive. And if $x$ is negative then $b^x = \frac{1}{b^{-x}}$ which is the reciprocal of a positive number and so is still positive. Therefore $y = b^x$ is always positive. In addition, for any small positive number $y$, we can make $b^x$ smaller than $y$ just by making $x$ a negative number with large absolute value. So the range of $f(x) = b^x$ is the set of positive reals, $(0, \infty)$.

3.1.3 Shifting, stretching, translating exponential functions

Given the graph of $y = b^x$, the graph of $y = b^{x+1}$ is just a translation left by one unit. But by properties of exponents, $y = b^{x+1} = b^x b^1 = b(b^x)$ and so this translation left by one unit is the same as a vertical stretch by a factor of $b$. Horizontal translations of exponential functions can be reinterpreted as vertical stretches or vertical contractions.

Practice problem. Describe the transformation used to move the graph of $y = 2^x$ onto the graph of $y = 2^{x+2}$.

Solution. Shift the graph of $y = 2^x$ two units to the left. (Alternatively, since $2^{x+2} = (2^2)(2^x)$, one could stretch the graph by a factor of four in the y-direction.)

3.1.4 Focusing on properties of the exponent

Whenever we work with exponential functions, we will eventually work with the inverse function. In preparation for that, here is an exercise which focuses on properties of the exponent.

Suppose that $8^a = 3$ and $8^b = 5$. Find the exponent on 8 that gives

1. $2$
2. $15$
3. $25$
4. $10$

(Answers in (b), (c), and (d) will involve the unknowns $a$ and/or $b$.)

Solutions. Since $8^a = 3$ and $8^b = 5$ then

1. $2 = 8^{1/3}$. So our answer is $\frac{1}{3}$.
2. $15 = (3)(5) = (8^a)(8^b) = 8^{a+b}$. So our answer is $a + b$.
3. $25 = (5)^2 = (8^b)^2 = 8^{2b}$. So our answer is $2b$.
4. $10 = (2)(5) = (8^{1/3})(8^b) = 8^{1/3+b}$. So our answer is $\frac{1}{3} + b$.

Here are some sample problems from [Dr. Paul’s online math notes on logarithms](https://tutorial.math.lamar.edu/), at Lamar University.

Example 1. Solve the following exponential equations for $x$.

1. $5^{3x} = 5^{7x-2}$
2. $4^{t^2} = 4^{6-t}$
3. $3^z = 9^{z+5}$
4. $4^{5-9x} = \frac{1}{8^{x-2}}$

**Solutions.**

1. To solve $5^{3x} = 5^{7x-2}$, we note that the bases are the same and so (since $f(x) = 5^x$ is a one-to-one function) then we must have $3x = 7x - 2$. This is a simple linear equation in $x$ and a quick step or two leads to $4x = 2$ so $x = \frac{1}{2}$.

2. To solve $4^{t^2} = 4^{6-t}$, we again note that the bases are the same so $t^2 = 6 - t$. This is a quadratic equation in $t$. If we get zero on one side and write

\[ t^2 + t - 6 = 0 \]

we can factor this quadratic equation into

\[ (t + 3)(t - 2) = 0 \]

and so $t = -3$ or $t = 2$.

3. To solve $3^z = 9^{z+5}$ in the same manner as before, we need to get the bases to be equal. Let’s write $9 = 3^2$ and make this problem one involving only base 3. So

\[ 3^z = (3^2)^{z+5} \]

and by properties of exponents

\[ 3^z = (3^{2(z+5)}) \]

Therefore $z = 2(z + 5)$. Now we have a simple linear equation. As step or two yields $z = -10$.

4. To solve $4^{5-9x} = \frac{1}{8^{x-2}}$ we seek a common base. Let’s use base 2 and write $4 = 2^2$ and $8 = 2^3$ so that the equation becomes

\[ (2^2)^{5-9x} = \frac{1}{(2^3)^{x-2}}. \]

We then use properties of exponents to write

\[ 2^{2(5-9x)} = \frac{1}{2^{3(x-2)}}. \]

The expression on the righthand side (since $2^3(x-2)$ is in the denominator) can be rewritten as

\[ 2^{2(5-9x)} = 2^{-3(x-2)}. \]

Now that our bases are the same, we solve the (easy!) linear equation $2(5 - 9x) = -3(x - 2)$ to find $4 = 15x$ so $x = \frac{4}{15}$. 

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3.1.5 We can pick the base!

Suppose one person is working with an exponential function with base \( b_1 \) and another person is working with an exponential function with base \( b_2 \). Since the range of \( f(x) = b_1^x \) is the set of positive real numbers and so includes the real number \( b_2 \), then there is a positive real number \( c \) such that \( b_2 = b_1^c \).

The first person works with (and graphs) the function \( f(x) = b_1^x \). What if we want to work with base \( b_2 \)?

Changing the base is easy! If \( b_2 = b_1^c \) then \( b_2^x = (b_1^c)^x \). But by properties of exponents, \( (b_1^c)^x = b_1^{cx} \).

The graph of \( b_2^x \) is just a horizontal contraction of the graph of \( b_1^x \) by the factor \( c \).

We stress this:

\[
b_2^x = (b_1^c)^x = b_1^{cx}.
\] (1)

Changing the base merely expands or contracts the graph horizontally.

Two Examples.

1. Suppose we have the graph of \( f(x) = 4^x \). How does this graph compare with the graph of \( g(x) = 2^x \).

**Solution.**

Since \( 4 = 2^2 \), the graph of \( y = 4^x \) is the same as \( y = (2^2)^x = 2^{2x} \) and so the graph of \( y = 4^x \) is created by transforming the graph of \( y = 2^x \) with a horizontal contraction by a factor of 2.

2. Suppose we have a graph of \( y = 2^x \) and we wish to create a graph of \( y = 10^x \). What transformation changes the graph of \( y = 2^x \) into a graph of \( y = 10^x \)?

**Solution.** It turns out that 2 is approximately \( 10^{0.30103} \). (Check this out on a calculator or via WolframAlpha) So

\[
2^x = (10^{0.30103})^x = 10^{0.30103x}.
\]

So, we can change the graph of \( y = 10^x \) into the graph of \( y = 2^x \) by expanding by the ratio 0.30103.

Since 0.30103 is smaller than 1, this means the graph of \( y = 2^x \) is more stretched out than the graph of \( y = 10^x \).

3.1.6 We choose base \( e \)!

There is a number between 2 and 3 that turns out to be the perfect base for exponential functions. It naturally occurs in our universe; exponential functions with this base have the simplest properties.

This number is now denoted by the letter \( e \) and is called the “natural base.”

\[
e \approx 2.718281828459045235360...
\]

The number \( e \) has a role in mathematics similar to that of \( \pi \); we can work without it but working without using \( e \) is a lot harder and more complicated than computations that involve \( e \).

If possible, we will assume the base of most exponential functions is \( e \) and write \( f(x) = e^x \) as the “standard” (or “natural”) exponential function.

A digression.

A story is told to help people remember the first few digits of \( e \). A mathematician needs to remember just two things about President Andrew Jackson. First, he was the seventh president and second, he was elected to office in 1828.

Remember

2. (Two things to remember about Jackson)
7 (He was the seventh president)

1828 (He was elected to the presidency in 1828)

1828 (Which is one of two things – so let’s repeat that!)

45°-90°-45° (This has nothing to do with Jackson, but every mathematician should remember the 45°–90°–45° isosceles right triangle!)

2 (How many things should we remember about Jackson?!)

Put them all together and we have \( e \approx 2.7182818284590452 \)

Since these class notes originate from a particular university\(^1\) we might digress further to ask this Andrew Jackson question:

“Which of Andrew Jackson’s followers ran for governor of Tennessee just before Jackson was elected President?”

This young army lieutenant\(^2\) was elected governor of Tennessee but did not serve out his term and instead resigned the governorship in 1829. After leaving the governor’s office, he went to live in the Cherokee nation in Oklahoma. Later in life he would be elected governor of a different state, the state of Texas. An ardent supporter of Jackson all his life, he was the only person to be elected governor of two different U.S. states!

Just as he did not serve out his term as governor of Tennessee, he would not serve out his Texas governorship either; he was kicked out of the office in 1861\(^3\) and retired to Huntsville, Texas, where he died in 1863.

More about \( e \) and the exponential function \( f(x) = e^x \) can be found (of course) at Wikipedia.

### 3.1.7 Other resources for exponential functions

In the free textbook, *Precalculus*, by Stitz and Zeager (version 3, July 2011, available at stitz-zeager.com) this material is covered in section 6.1 (which also introduces logarithms.)


There are lots of online resources on exponential functions. Here are some I recommend.

1. Dr. Paul’s online math notes on exponential functions
2. Videos on exponential growth from Khan Academy.

**Homework.**

As class homework, please complete Worksheet 3.1, Exponential Functions available through the class webpage.

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\(^1\)Sam Houston State University

\(^2\)Sam Houston, of course!

\(^3\)for not supporting the Confederacy