2.3 Zeroes of polynomials and long division

2.3.1 Zeroes of polynomial functions

The Fundamental Theorem of Algebra tells us that every polynomial of degree \(n\) has at most \(n\) zeroes. Indeed, if we are willing to count multiplicity of zeroes and also count complex numbers (more on that later) then a polynomial of degree \(n\) has exactly \(n\) zeroes!

A major goal to understanding a polynomial is to write it in terms of its zeroes. Each zero \(c\) corresponds to a factor \(x - c\) so understanding the zeroes of a polynomial is equivalent to completely factoring the polynomial.

Consider the polynomial graphed below.

![Figure 13. A certain polynomial](image)

From the graph, can we see the number of turning points, the degree of the polynomial and the zeroes of the polynomial? Furthermore, from this graph can we in fact write out the polynomial exactly?

**Solution.**

The graph apparently has two turning points and so probably has degree three. It has zeroes \(x = -1, x = 1\) and \(x = 2\) which agrees with our guess that the degree is 3.

Since the graph has zeroes at \(-1, 1\) and \(2\) and presumably has degree 3, then it should have form

\[
f(x) = a(x + 1)(x - 1)(x - 2)
\]

for some unknown \(a\). (The unknown \(a\) is the leading term of this polynomial.)

Can we guess the leading coefficient \(a\) from the graph? Since the graph goes through the point \((0, 2)\) then \(f(0) = 2\). We see by direct computation from the formula above that \(f(0) = 2a\) so \(a = 1\). Therefore the polynomial graphed in figure 13 above must be

\[
f(x) = (x + 1)(x - 1)(x - 2)\]

**Another Example.**

Find a polynomial \(f(x)\) of degree 3 with zeroes \(x = -1, x = 1\) and \(x = 2\) where the graph of \(y = f(x)\) goes through the point \((3, 16)\).

**Solution.** Because the zeroes are \(-1, 1, 2\) then factors of the polynomial should be \(x + 1, x - 1\) and \(x - 2\). If \(f(x) = a(x + 1)(x - 1)(x - 2)\) then \(16 = f(3) = a(3 + 1)(3 - 1)(3 - 2) = 8a\) so \(a = 2\). So the
answer is

\[ f(x) = 2(x + 1)(x - 1)(x - 2). \]

These examples are intended to demonstrate that our understanding of a polynomial is very closely related to our knowledge of its zeroes.

In the next section we concentrate on dividing polynomials by smaller ones, with an eye to eventually factoring the polynomial and finding all its zeroes.

### 2.3.2 The Division Algorithm

The Division Algorithm for polynomials promises that if we divide a polynomial by another polynomial, then we can do this in such a way that the remainder is a polynomial with degree smaller than that of the divisor.

We will make that precise in a moment, but let us first review the Division Algorithm for integers, and, as we do this, review “long division.”

Suppose that we wish to divide 23 by 5. We notice that 5 goes into 23 at most 4 times and that

\[ 20 = 5 \cdot 4. \]

So we may take 20 away from 23, leaving a remainder of 3. We write that (in the United States) as a long division problem in the following form:

\[
\begin{array}{c}
4 \\
5 \overline{23} \\
20 \\
3
\end{array}
\]

We say that dividing 5 into 23 leaves a **quotient** of 4 and a **remainder** of 3.

There are equivalent ways to write this. We can write

\[
\frac{23}{5} = 4 + \frac{3}{5}
\]

or

\[ 23 = (4)(5) + 3. \]

Let us do a more complicated example. Suppose we divide 231 by 5. The easiest way to do this is to take advantage of our decimal notation and first divide 23 by 5 (as before) and note that 5 goes into 23 4 times. If 5 goes into 23 4 times then 5 goes into 230 at least 40 times.

Indeed, 5 goes into 231 at least 40 times. If we use 40 as our (temporary) quotient, we have a remainder of 31.

However, this remainder 31 is at least as big as the divisor 5 so we can divide 5 into 31 a few more times (6) and get a remainder of 1. We write this long division as

\[
\begin{array}{c}
46 \\
5 \overline{231} \\
200 \\
31 \\
30 \\
1
\end{array}
\]

and say that 231 divided by 5 leaves a quotient of 46 and a remainder of 1. Thus

\[
\frac{231}{5} = 46 + \frac{1}{5}
\]

or

\[ 231 = (46)(5) + 1. \]
We can do the same computations (the “Division Algorithm”) with polynomials.

Let us divide the polynomial $2x^4 - 3x^3 + 5x - 36$ by $x^2 + x + 2$. Let’s write this as a long division problem:

\[
x^2 + x + 2 \overline{) 2x^4 - 3x^3 + 5x - 36}
\]

We keep things simple (as we did when dividing 231 by 5) by focusing on part of the problem. If we just try to divide $2x^4$ by $x^2$ we would get $2x^2$. Let’s use $2x^2$ as the first guess at our quotient.

\[
x^2 + x + 2 \overline{) 2x^4 - 3x^3 + 5x - 36}
\]

We multiply the divisor $x^2 + x + 2$ by the quotient $2x^2$ to obtain $2x^4 + 2x^3 + 4x^2$ and subtract this from the original polynomial. This leaves a remainder of $-5x^3 + 4x^2 + 5x - 36$.

\[
x^2 + x + 2 \overline{) 2x^4 - 3x^3 + 5x - 36}
\]

\[
-2x^4 - 2x^3 - 4x^2
\]

\[
-5x^3 - 4x^2 + 5x
\]

Are we done?

No. Recall that in our long division of 231 by 5, we got a temporary remainder of 31 which was larger than the divisor and so we divided again, dividing 5 into the new remainder. In a similar way, here our remainder is also larger than the divisor – the remainder has larger degree than the divisor – and so we can divide into it again.

Since $-5x^3$ divided by $x^2$ is $-5x$, let us guess that $x^2 + x + 2$ goes into the temporary remainder $-5x^3 + 4x^2 + 5x - 36$ about $-5x$ times. This gives another layer of our long division.

\[
x^2 + x + 2 \overline{) 2x^4 - 3x^3 + 5x - 36}
\]

\[
-2x^4 - 2x^3 - 4x^2
\]

\[
-5x^3 - 4x^2 + 5x
\]

We multiply the divisor $x^2 + x + 2$ by $-5x$ and subtract....

\[
x^2 + x + 2 \overline{) 2x^4 - 3x^3 + 5x - 36}
\]

\[
-2x^4 - 2x^3 - 4x^2
\]

\[
-5x^3 - 4x^2 + 5x
\]

\[
5x^3 + 5x^2 + 10x
\]

\[
x^2 + 15x - 36
\]

Are we done here?

No. Now the degree of the remainder is the same as the degree of the divisor, which means we can go one more step.

\[
x^2 + x + 2 \overline{) 2x^4 - 3x^3 + 5x - 36}
\]

\[
-2x^4 - 2x^3 - 4x^2
\]

\[
-5x^3 - 4x^2 + 5x
\]

\[
5x^3 + 5x^2 + 10x
\]

\[
x^2 + 15x - 36
\]

\[
x^2 - x - 2
\]

\[
14x - 38
\]
At this point we have a remainder now of degree smaller than the degree of the divisor and so we are forced to stop.

Our quotient is \(2x^2 - 5x + 1\) and our remainder is \(4x - 38\).

We may write this out as either
\[
\frac{2x^4 - 3x^3 + 5x - 36}{x^2 + x + 2} = 2x^2 - 5x + 1 + \frac{4x - 38}{x^2 + x + 2}
\]
or
\[
2x^4 - 3x^3 + 5x - 36 = (2x^2 - 5x + 1)(x^2 + x + 2) + 4x - 38.
\]

Let us do another example, this time one which is a little bit simpler. Let us divide \(2x^4 - 3x^3 + 5x - 36\) by just \(x - 2\). Here is the long division.

\[
\begin{array}{r|llll}
& 2x^3 + x^2 + 2x + 9 \\
\hline
x - 2) & 2x^4 - 3x^3 + 5x - 36 \quad -2x^4 + 4x^3 \\
\hline
& -x^3 + 2x^2 \\
& -x^3 + 2x^2 \\
\hline
& 2x^2 + 5x \\
& -2x^2 + 4x \\
\hline
& 9x - 36 \\
& 9x - 36 \\
\hline
& -18 \\
& -18 \\
\end{array}
\]

In this case
\[
\frac{2x^4 - 3x^3 + 5x - 36}{x - 2} = 2x^3 + x^2 + 2x + 9 - \frac{18}{x - 2}
\]
or
\[
2x^4 - 3x^3 + 5x - 36 = (2x^3 + x^2 + 2x + 9)(x - 2) - 18.
\]

We summarize our work in this section by explicitly stating the Division Algorithm as a theorem.

**Theorem.** (The Division Algorithm)
Suppose \(f(x)\) and \(d(x)\) are polynomials with real coefficients. We may divide \(f(x)\) by \(d(x)\) and obtain a quotient \(q(x)\) and a remainder \(r(x)\), so that
\[
f(x) = q(x)d(x) + r(x) \tag{9}
\]
where the degree of \(r(x)\) is strictly less than the degree of the divisor \(d(x)\).

**2.3.3 The Remainder Theorem**

We digress for a moment to discuss the value of the remainder in a long division problem.

Here is an example. Observe that if
\[
f(x) = 2x^4 - 3x^3 + 5x - 36 = (2x^3 + x^2 + 2x + 9)(x - 2) - 18
\]
then
\[
f(2) = (2 \cdot 2^3 + 2^2 + 2 \cdot 2 + 9)(2 - 2) - 18.
\]
Ignore the first expression on the right side involving sums of powers of 2. The critical concept here is that we have $2 - 2 = 0$ in the expression for $f(2)$ and any numbers times zero is zero. Thus this expression simplifies to

$$f(2) = q(2) \cdot (0) - 18 = -18.$$ 

More generally: if, by the division algorithm we divide a polynomial $f(x)$ by $d(x)$ and obtain a quotient $q(x)$ and a remainder $r(x)$, so that

$$f(x) = q(x)d(x) + r(x)$$

then if $c$ is a zero of $d(x)$ then $f(c) = q(c) \cdot 0 + r(c) = r(c)$.

This is especially important if the divisor polynomial is nice and linear. Suppose we divide a polynomial $f(x)$ by $x - c$ and obtain a quotient $q(x)$ and a remainder $r$. Then

$$f(x) = q(x)(x - c) = r$$

and so

$$f(c) = q(c)(c - c) + r = q(c) \cdot 0 + r = r.$$

**The Remainder Theorem.**

If $f(x)$ is a polynomial then $f(c)$ is the remainder obtained by dividing $f(x)$ by $x - c$.

**A Worked Example.**

The polynomial

$$f(x) = x^{100} - 3x^{98} - 2x^{97} + 5x^4 - 7x^2 + 3,$$

when divided by $x - 2$ gives a quotient $q(x)$ and a remainder $r(x) = 55$. (Just take my word for it – I did a computation on a computer algebra system at [WolframAlpha](https://www.wolframalpha.com).)

Given this information, what is $f(2)$? (Why?)

**Solution.** We can write $f(x) = (x - 2)q(x) + 55$. So if we evaluate this expression at $x = 2$ we have $f(2) = 0 \cdot q(2) + 55 = 55$. So $f(2) = 55$.

**Another Example.**

A certain polynomial $f(x)$ of degree 999, when divided by $x^2 - 9$ gives a quotient $q(x)$ and a remainder $r(x) = 4x + 11$. What is $f(3)$? (Why?)

**Solution.**

We are given that $f(x) = (x^2 - 9)q(x) + 4x + 11$. Then $f(3) = (3^2 - 9)q(3) + 4(3) + 11 = 0 \cdot q(3) + 12 + 11 = 23$. So $f(3) = 23$.

The Remainder Theorem helps motivate a shorthand notation for dividing a polynomial by a nice linear factor of the form $x - c$. This shorthand notation is called synthetic division.

**2.3.4 Synthetic division**

Let’s review our long division when we divided $2x^4 - 3x^3 + 5x - 36$ by $x - 2$. 

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If we really don’t want to write all these $x$’s down, what really matters is the coefficients appear in each row. If we take into account that we are subtracting, we might organize these coefficients as

$$
\begin{array}{cc}
-2 & 4 \\
-1 & 2 \\
-2 & 4 \\
-9 & 18 \\
\end{array}
$$

and, just for simplicity, not write down the powers of $x$. We can then condense all this work into a short array, something like this:

$$
\begin{array}{cccc}
2 & -3 & 0 & 5 & -36 \\
2 & 4 & 2 & 4 & 18 \\
\end{array}
$$

This short array (a couple of lines) captures all the work we did in our long division problem. This line of steps in long division is sometimes called “synthetic division”. Here is how this works.

If we want to divide $2x^4 - 3x^3 + 5x - 36$ by $x - 2$, we write down the coefficients of the larger polynomial across the first line. Then, since we are dividing by $x - 2$ (which has a zero at $x = 2$) we write 2 on the far left. (Note that we write 2, not $-2$! This 2 is a potential zero of the polynomial; it will be our $c$ in the computation $f(c)$.)

$$
\begin{array}{cccc}
2 & -3 & 0 & 5 & -36 \\
2 & 4 & 2 & 4 & 18 \\
\end{array}
$$

Then we pull down the first coefficient.

$$
\begin{array}{cccc}
2 & -3 & 0 & 5 & -36 \\
2 & 4 & 2 & 4 & 18 \\
\end{array}
$$

We multiply it by the original $c = 2$ to obtain 4 and place that below the next coefficient.
We add the coefficients. In this case $-3 + 4 = 1$.

$$
\begin{array}{cccccc}
2 & -3 & 0 & 5 & -36 \\
2 & 4 \\
\hline
2 & 1 & 2
\end{array}
$$

We again multiply 1 by $c = 2$ and place that below the next coefficient and add.

$$
\begin{array}{cccccc}
2 & -3 & 0 & 5 & -36 \\
2 & 4 & 2 \\
\hline
2 & 1 & 2 & 9
\end{array}
$$

We continue this process one coefficient at a time to get

$$
\begin{array}{cccccc}
2 & -3 & 0 & 5 & -36 \\
2 & 4 & 2 & 4 \\
\hline
2 & 1 & 2 & 9 & 18
\end{array}
$$

and finally

$$
\begin{array}{cccccc}
2 & -3 & 0 & 5 & -36 \\
2 & 4 & 2 & 4 & 18 \\
\hline
2 & 1 & 2 & 9 & -18
\end{array}
$$

Now we read off the meaning of the bottom row. At the far right is the remainder $r = -18$. The rest of the bottom row gives the quotient $2x^3 + x^2 + 2x + 9$.

Let’s work a few more examples.

**Worked Examples.**

1. Divide $3x^5 - 8x^3 - 2x + 10$ by $x - 2$.

   **Solution.**

   Here we use $c = 2$ in our problem. Don’t forget to write all the coefficients of $3x^5 - 8x^3 - 2x + 10$; they are 3, 0, -7, 0, -2, 10.

   $$
   \begin{array}{cccccc}
   3 & 0 & -8 & 0 & -2 & 10 \\
   2 & 6 & 12 & 8 & 16 & 28 \\
   \hline
   3 & 6 & 4 & 8 & 14 & 38
   \end{array}
   $$

   Answer: $3x^5 - 8x^3 - 2x + 10 = (3x^4 + 6x^3 + 4x^2 + 8x + 14)(x - 2) + 38$

2. Divide $3x^5 - 8x^3 - 2x + 10$ by $x + 2$.

   **Solution.**

   Here we use $c = -2$ in our problem.

   $$
   \begin{array}{cccccc}
   3 & 0 & -8 & 0 & -2 & 10 \\
   -2 & -6 & 12 & -8 & 16 & -28 \\
   \hline
   3 & -6 & 4 & -8 & 14 & -18
   \end{array}
   $$
Answer: \( 3x^5 - 8x^3 - 2x + 10 = (3x^4 - 6x^3 + 4x^2 - 8x + 14)(x - 2) - 18 \)

3. Let \( f(x) = 3x^5 - 8x^3 - 2x + 10 \). Compute

(a) \( f(2) \)
(b) \( f(-2) \)

**Solution.**

(a) From our work in problem 1, and our understanding of the Remainder Theorem, we see that \( f(2) = 38 \), the remainder when \( f(x) \) is divided by \( x - 2 \).

(b) From our work in problem 2, and our understanding of the Remainder Theorem, we see that \( f(-2) = -18 \).

### 2.3.5 Other resources on long division

In the free textbook, *Precalculus*, by Stitz and Zeager (version 3, July 2011, available at [stitz-zeager.com](http://stitz-zeager.com)) this material is covered in section 3.2.


There are lots of online resources for studying polynomial division and results about the zeroes of polynomials. Here are some I recommend.

1. [Wikipedia](https://en.wikipedia.org) article on long division of polynomials.
2. [Wikipedia](https://en.wikipedia.org) article on synthetic division.
3. [Khan Academy videos](https://www.khanacademy.org) on synthetic division.
4. [Paul Dawkin's online notes](http://tutorial.math.lamar.edu) on dividing polynomials.

**Worksheet to go with these notes.**

As class homework, please complete *Worksheet 2.3, Zeroes of polynomials*, available through the class webpage.