1.3 Transformations of functions

In this course we learn to identify a variety of functions: linear functions, quadratic and cubic functions, general polynomial and rational functions, exponential and logarithmic functions, trigonometric functions and inverse trig functions. Many of these functions can be identified by their “shape”, by general properties of their graph. Instead of trying to remember the shapes of millions of different functions, we will identify some basic functions and then recognize transformations of the functions that give (essentially) the same shape.

For example, the graphs of the functions \( f(x) = x^2 \) and \( f(x) = 3(x - 5)^2 + 7 \) are the same shape. Indeed, if one plots them on the \( x \)-interval \([-1000, 1000]\) one gets the following pictures (below, in figure 15.) The graph of \( y = x^2 \) is on the left; the graph of \( y = 3(x - 5)^2 + 7 \) is on the right. If one looks carefully, one can see that the labels on the \( y \)-axis have changed, otherwise the graphs are the same.

![Figure 13. Graphs of \( y = x^2 \) and \( y = 3(x - 5)^2 + 7 \) (Generated by the author using Sage.)](Generated by the author using Sage.)

There are four types of transformations we will study in this section. In the first two types, we simply shift the graph by a fixed amount, either vertically or horizontally. In the last two types of transformations, we expand/shrink the graph by a fixed ratio, either vertically or horizontally.

1.3.1 Vertical shifts

It is easy to shift the graph \( y = f(x) \) up by a fixed positive amount \( c \). Just add \( c \) to the \( y \)-value, that is, create the graph of \( y = f(x) + c \).

If we can shift up by a fixed amount then shifting down is also easy – just make \( c \) negative. If \( c \) is negative then the graph of \( y = f(x) + c \) shifts the graph down by \( |c| \). (For example, \( y = f(x) - 2 \) will shift the graph down by 2.)

Consider the graph of \( y = x^2 \). The graph of \( y = x^2 + 1 \) shifts the graph of \( y = x^2 \) up one unit. The graph of \( y = x^2 + 3 \) shifts the graph of \( y = x^2 \) up three units. The graph of \( y = x^2 - 2 \) shifts the graph down by two units. Let’s graph these all on one plane (see figure 14) to show the effect of the shifting.
1.3.2 Horizontal shifts

Horizontal shifts are very similar, but their is a subtlety here. Because the horizontal $x$-axis represents inputs to the function, if we want to shift the curve to the right (in the positive $x$-direction) by a positive amount $c$ then we need to “prepare” the input by subtracting the amount $c$ from $x$ before it is inserted into the function. This may be the opposite of what one expects, but by subtracting $c$ from $x$, we make an input $x - c$ on the left of $x$ act like the input $x$ and this shift, moving $x - c$ to $x$ is a shift to the right.

Below in figure 15, as an example, are the graphs of $y = x^2$, $y = (x - 1)^2$ and $y = (x - 3)^2$. Notice that by replacing $x$ by $x - 3$, we have shifted the graph of $y = x^2$ three to the right, in the positive $x$-direction.

If we want to shift the graph left by a positive amount $c$ then we add $c$ to $x$ before inserting it into the function. For example, the graph of $y = (x + 2)^2$ will shift the parabola of $y = x^2$ to the left by 2.

Figure 14. Graphs of $y = x^2$ (thick black curve), $y = x^2 + 1$ (green), $y = x^2 + 3$ (blue), $y = x^2 - 2$ (red), (Generated by the author using Sage.)

Figure 15. Graphs of $y = x^2$ (thick black curve), $y = (x - 1)^2$ (green), $y = (x - 3)^2$ (blue), $y = (x + 2)^2$ (red), (Generated by the author using Sage.)
1.3.3 Vertical expansions

What if we want to expand or shrink the image of our graph? We can do this in the vertical ($y$-direction) simply by multiplying our function by a constant. For example, if we have the graph $y = f(x)$ then the graph of $y = 3f(x)$ will stretch (expand) the graph by a factor of 3 in the $y$-direction. The graph of $y = \frac{1}{3}f(x)$ will contract (shrink) the graph by a factor of 3.

Multiplying $f(x)$ by $-1$ will flip the graph over, reflecting it across the $x$-axis, replacing positive $y$-values by negative ones and conversely, replacing negative $y$-values by positive ones. This is our first example of a reflection. The graph of $y = -f(x)$ is a reflection of $y = f(x)$ across the $x$-axis.

1.3.4 Horizontal expansions

We can also expand or contract a graph in the horizontal direction, along the $x$-axis. But, just like horizontal shifts, because the horizontal axis represents the input variable, the action may be the reverse of what one might expect. To expand the graph horizontally by a factor of 2, we must divide $x$ by 2 before inserting it into the function.

For example, here in thick black ink is the graph of $y = x^2$. In lighter blue ink is the graph of $y = (x^2)^2$. By dividing by two, we have stretched the graph in the horizontal direction by a factor of 2.

![Graphs of $y = x^2$ (thick black curve), $y = (\frac{x}{2})^2$ (thin blue),](image)

If instead we multiply the input variable $x$ by a constant, we will contract (shrink) the graph in the horizontal direction. In the picture below, the graph of $y = x^2$ is again a thick black curve; the graph of $y = (2x)^2$ is the thinner green curve and if we graph $y = (5x)^2$ we get the curve in red, shrunk even more in the horizontal direction.

If we replace $x$ by $-x$, we interchange the role of positive and negative $x$-values and so we reflect the graph across the $y$-axis. This is our second example of a reflection.
1.3.5 Combining these expansions

In summary,

1. To shift a function up by \( c \) units, replace \( y = f(x) \) by \( y = f(x) + c \).

2. To shift a function to the right by \( c \) units, replace \( y = f(x) \) by \( y = f(x - c) \).

3. To expand a function vertically by a factor of \( c \), replace \( y = f(x) \) by \( y = cf(x) \).

4. To expand a function horizontally by a factor of \( c \), replace \( y = f(x) \) by \( y = f\left(\frac{x}{c}\right) \).

We can combine these various transformations by creating a sequence of transformations. For example, we could translate a function in a diagonal direction, over to the right by 2 and then up by 2 by replacing \( f(x) \) by \( f(x - 2) + 2 \). Notice that replacing \( x \) by \( x - 2 \) moves the graph 2 units to the right; adding 2 to the entire function moves the graph up two units.

A sequence of transformations then combine into a single form, changing \( y = f(x) \) into the expression

\[
y = af\left(b(x - c)\right) + d
\]

where first subtracting \( c \) translates everything to the right by \( c \) units, then multiplying by \( b \) inside the function shrinks the graph in the horizontal direction about the point \((c, 0)\) by a factor of \( b \) while multiplying by \( a \) on the outside of the function expands the graph vertically by a factor of \( a \). Finally, adding \( d \) to the entire piece raises the graph \( d \) units up.

When in doubt about the type of transformation involved, it is always easy to pick several nice points in the new graph and ask where they came from in the old graph. For example, in the expression \( y = af\left(b(x - c)\right) + d \), the new \( x \) values \( x = c \) and \( x = c + 1 \) lead to the computation of \( f(0) \) and \( f(b) \) and so correspond to old points where \( x \) was zero and where \( x \) was equal to \( b \).
Some worked examples.

1. Consider the two graphs below. The first is the graph of $f(x) = |x|$. The second is a graph in which the original graph has been contracted horizontally by a factor of two and then shifted 2 units to the right and up 1 unit. What is the function graphed in the graph at the right?

![Figure 18. A transformation of the graph of the absolute value function.](image)

**Solutions.** We first contract the graph horizontally by a factor of 2, replacing $x$ by $2x$. Then we shift the graph to the right by 2, replacing $x$ by $x - 2$. Lastly we add 1 to the result. So the expression $|x|$ becomes $|2(x - 2)| + 1$. Therefore the graph in question is $f(x) = |2(x - 2)| + 1$.

2. What transformations (in order) must be done to the graph of $y = f(x)$ to create the graph of $y = 2f\left(\frac{x - 5}{3}\right) - 7$?

**Solutions.** Do the following steps, in this order:

(a) Shift right by 5,
(b) Expand horizontally by a factor of 3 about the point $(5, 0)$,
(c) Expand vertically by a factor of 2,
(d) Shift down 7.

3. The graph of $y = f(x)$ is drawn in red below.

![Figure 19. A particular graph, waiting to be transformed.](image)
For each of the graphs, below (drawn in blue) first describe the transformation that turns the above graph into the new graph and then express this transformation algebraically in terms of the original function $f(x)$. (For example, the answer to problem (a) is “The graph is shifted up 3 units” and “$y = f(x) + 3$”)

![Graphs](image)

**Solutions.**

(b) The graph is shifted right 2 units and up 4 units; $y = f(x - 2) + 4$
(c) The graph is shifted left 4 units and up 2 units; $y = f(x + 4) + 2$
(d) The graph is reflected across the $x$-axis; $y = -f(x)$.
(e) The graph is stretched horizontally by a factor of 2; $y = f\left(\frac{x}{2}\right)$.

**One more example.**

Here is an example that shows up in engineering applications. The “sinc” function is defined in terms of the sine function as

$$\text{sinc}(x) := \frac{\sin x}{x}.$$ 

But sometimes it is more natural to “normalize” it by replacing $x$ by $\pi x$ so that

$$\text{sinc}(x) := \frac{\sin \pi x}{\pi x}.$$ 

What is the effect of this “normalization”? 
Replacing $x$ by $\pi x$ creates a horizontal contraction by a factor of $\pi$. Here, below (from Wikipedia), are the two functions graphed together.
1.3.6 Resources for function transformations

In the free textbook, *Precalculus*, by Stitz and Zeager (version 3, July 2011, available at stitz-zeager.com) this material is covered in section 1.7.

In the free textbook, *Precalculus, An Investigation of Functions*, by Lippman and Rasmussen (Edition 1.3, available at www.opentextbookstore.com) this material is covered in section 1.5.

In the textbook by Ratti & McWaters, *Precalculus, A Unit Circle Approach*, 2nd ed., c. 2014, [here at Amazon.com](https://www.amazon.com) this material appears in section 1.5. In the textbook by Stewart, *Precalculus, Mathematics for Calculus*, 6th ed., c. 2012, [here at Amazon.com](https://www.amazon.com) this material appears in section 2.5. (In July 2013 the first textbook was $147 at Amazon.com and the second textbook was $136 at Amazon.com. They are even more expensive in campus bookstores.)

There are lots of online resources for studying the transformations of functions. Here are some I recommend.

1. This [Youtube video](https://www.youtube.com) describes all four types of transformations in a function in one basic formula \( (y = af(b(x - c)) + d) \) with nice animations showing the transformations.

2. This [applet at the Wolfram site](https://www.wolframalpha.com) allows one to experiment with changing the values of the \( af(b(x - c)) + d \) to see how they move the graph.

3. A webpage at [Purplemath forums](https://www.purplemath.com) has a tutorial and nice examples.

4. The webpage at [Mathisfun](https://www.mathisfun.com) is another webpage with a nice introduction to the topic of transformations.

Worksheet to go with these notes.

As class homework, please complete Worksheet 1.3, *Transformations of functions*, available through the class webpage.