1 Functions and Polynomials

1.0 Algebra excellence

Before we study the elementary functions of science and calculus, we need some comfort with algebra. In this brief lecture we review two important ideas: (1) computations using exponential notation and (2) operations of polynomial arithmetic. These are two major algebra computations we will do throughout this class (and scientists will use throughout their careers!)

This review is brief and is not intended to be comprehensive. Our precalculus class will assume that students are comfortable with most of the major concepts of elementary and intermediate algebra and we will not, in general, review those concepts in this class.

1.0.1 Exponential notation – merely an abbreviation!

We abbreviate \( x \cdot x \cdot x \) by \( x^3 \). This notation (merely an abbreviation!) quickly leads to some rules on how one should treat exponents. For example, since

\[
x^3 \cdot x^2 = (x \cdot x \cdot x) \cdot (x \cdot x) = x^5
\]

then when we multiply objects with the same base \( (x) \) we should add the exponents:

\[
x^m x^n = x^{m+n}.
\]

(1)

Similarly,

\[
\frac{x^3}{x^2} = \frac{x \cdot x \cdot x}{x \cdot x} = \frac{x x x}{x x} = x,
\]

so when we divide objects with the same base \( (x) \) we should subtract the exponents:

\[
\frac{x^m}{x^n} = x^{m-n}.
\]

(2)

What if we use exponents in sequence, that is, we raise \( x \) to a power and then raise that result to a second power? For example,

\[
(x^3)^2 = (x \cdot x \cdot x)^2 = (x \cdot x)(x \cdot x) = x \cdot x \cdot x \cdot x \cdot x \cdot x = x^6.
\]

Here our “abbreviation” leads us to multiplying exponents. We may generalize from this that

\[
(x^m)^n = x^{mn}.
\]

(3)

Repeated exponentiation leads to multiplying exponents.

Our understanding of the exponent “abbreviation” has quickly led us to three natural rules about manipulating exponents. These algebra “rules” are merely the effects of the algebraic symbolism.

There are other effects of our algebraic symbolism. Once we get used to the impact of this notation, we see that since multiplying by 1 leaves a number unchanged and since multiplication by \( x^0 \) also leaves a number unchanged \( (x^n x^0 = x^{n+0} = x^n) \) then 1 and \( x^0 \) must be the same:

\[
x^0 = 1.
\]

(4)
We can extend our exponent notation to rational exponents. Since
\[(x^{\frac{1}{2}})^2 = x^{\frac{1}{2} \cdot 2} = x^1 = x\]
and since
\[(\sqrt{x})^2 = x\]
then \(x^{\frac{1}{2}}\) must represent \(\sqrt{x}\). More generally, denominators in exponents represent roots:
\[x^{\frac{1}{q}} = \sqrt[q]{x}.\]

Some examples. First we practice our understanding of exponentiation:

1. Simplify \(8^{\frac{2}{3}}\).
   
   We solve this by recognizing that the fractional exponent \(\frac{2}{3}\) represents \((\frac{1}{3})(2)\), so we will take a cube root (that is the meaning of the exponent \(\frac{1}{3}\)) and then we will square the result.
   
   Solution.
   \[8^{\frac{2}{3}} = (8^{\frac{1}{3}})^2 = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4.\]

Here is another, similar example.

2. Simplify \(4^{\frac{3}{2}}\).
   
   Solution.
   \[4^{\frac{3}{2}} = (4^{\frac{1}{2}})^3 = \left(\sqrt{4}\right)^3 = 2^3 = 8.\]

1.0.2 Kilobytes and powers of ten (an application to computer science)

Here is an application appearing in a number of computer science computations. We note that \(2^{10} = 1024\) while \(1000 = 10^3\). The computer scientist works with computer registers which use bits (zeros and ones) and so storage and memory are measured in powers of two. (We say that computer science computations are done in base two.) Yet the language of computer science is often based on our traditional powers of ten, where Greek prefixes such as kilo- represent a thousand, mega- represents a million and giga- a billion.

However, to a computer scientist, the prefix kilo- really represents \(2^{10}\), not \(10^3\). A kilobyte is \(2^{10} = 1024\) bytes; a gigabyte is \(2^{30}\) bytes.

Let us approximate \(2^{30}\) as a power of ten: Since \(2^{30} = (2^{10})^3\) and since we approximate \(2^{10} = 10^3\) then
\[2^{30} = (2^{10})^3 \approx (10^3)^3 = 10^9.\]

Exercise. How many digits are there in \(2^{300}\)?

Solution. Write \(2^{300} = (2^{10})^{30} \approx (10^3)^{30} = 10^{90}\). Now \(10^{90} = 1 \times 10^{90}\) is 1 followed by 90 zeroes so \(10^{90}\) has 91 digits. Therefore \(2^{300}\) should have \(91\) digits.

(A detour to WolframAlpha and a quick computation indeed gives
\[2^{300} = 203703597633448608626844568840937816105146839366593625063614049354381299763336706183397376\]
You can check that this has 91 digits! But since \(2^{10} > 10^3\), it turns out that \(2^{300}\) is more closely approximated by \(2 \times 10^{90}\) than \(1 \times 10^{90}\).)
**Practice.** Here are some sample problems from an old precalculus quiz.

Simplify the following expressions:

1. \( \frac{\sqrt[3]{x^6}}{\sqrt{x^2}} \)
2. \( \frac{(x^6)^{\frac{1}{3}} x^{-2}}{x^4} \)

**Solutions.**

1. We follow the meaning of the exponent, rewriting expressions such as \( \sqrt[3]{x^6} \) as \( x^2 \) since \((x^2)^3 = x^6\).

   So
   \[ \frac{\sqrt[3]{x^6}}{\sqrt{x^2}} = \frac{x^2}{x} = x. \]

2. We first simplify the numerator, noting that \((x^6)^{\frac{1}{3}} = x^2\) and \(x^2 x^{-2} = \frac{x^2}{x^2} = 1\).

   So
   \[ \frac{(x^6)^{\frac{1}{3}} x^{-2}}{x^4} = \frac{x^2 x^{-2}}{x^4} = \frac{1}{x^4} \text{ or } x^{-4}. \]

1.0.3 **Polynomial arithmetic: a review of** \(A^2 - B^2\) **and other basic factoring**

There are some basic polynomial expansion concepts that will appear throughout precalculus, calculus, and computations in the sciences. For example, at one point we learned to use the distributive law to expand (“FOIL”) expressions like:

\[(x + 5)(x - 3) = x^2 - 3x + 5x - 15 = x^2 + 2x - 15\]

and then to factor expressions like \(x^2 + 2x - 15\) by reversing this process.

If we expand the expression \((A + B)^2 = (A + B)(A + B)\) we discover, in addition to the obvious squares \(A^2\) and \(B^2\) the “cross term” \(2AB\). However, if instead we expand \((A + B)(A - B)\) we obtain \(A^2 - B^2\); the cross term involved both \(AB\) and \(-AB\) and these cancelled out. We will use these basic patterns repeatedly in this course:

\[(A + B)^2 = A^2 + 2AB + B^2 \quad (6)\]

and

\[(A + B)(A - B) = A^2 - B^2 \quad (7)\]

In the first case (equation 6), notice the existence of a middle term caused by our polynomial expansion. Don’t make the “freshman” mistake of thinking that \((A + B)^2\) is just the sum of the squares of \(A\) and \(B\)!

In the second case (equation 7), we see that the difference of two squares nicely factors into the product of the sum and difference of the elements.

**Examples.** Here are some other simplification problems. Each involves factoring of some type. Notice that the expression we are factoring in problems 2 and 3 have the same pattern as problem 1; recognizing that pattern leads us to our solution.

Simplify:
1. \[ \frac{x^2 - 9}{x + 3} \]
2. \[ \frac{x^4 - 9}{x^2 + 3} \]
3. \[ \frac{x - 9}{\sqrt{x} + 3} \]

**Solutions.**
1. \[ \frac{x^2 - 9}{x + 3} = \frac{(x - 3)(x + 3)}{x + 3} = x - 3. \]
2. \[ \frac{x^4 - 9}{x^2 + 3} = \frac{(x^2 - 3)(x^2 + 3)}{x^2 + 3} = x^2 - 3. \]
3. \[ \frac{x - 9}{\sqrt{x} + 3} = \frac{(\sqrt{x} - 3)(\sqrt{x} + 3)}{\sqrt{x} + 3} = \sqrt{x} - 3. \]

**1.0.4 Simplifying complicated fractions**

Once we learn these abbreviations that we call “exponentiation”, we can often use polynomial arithmetic (factoring polynomials) to simplify an expression.

For example, we may need to simplify a complex fraction such as

\[
\frac{2x^3 + x^2}{x^4} \div \frac{1}{x^2} = \frac{2x^3 + x^2}{x^4} \div \frac{1}{x^2}.
\]

Recall that dividing by \( \frac{1}{x^2} \) is the same as multiplying by \( x^2 \) so

\[
\frac{2x^3 + x^2}{x^4} \div \frac{1}{x^2} = \left( \frac{2x^3 + x^2}{x^4} \right) (x^2) = \frac{2x^3 + x^2}{x^2}.
\]

We factor the numerator and simplify.

\[
\frac{x^2(2x + 1)}{x^2} = (\frac{x^2}{x^2})(2x + 1) = 1(2x + 1) = 2x + 1.
\]

**A Worked Problem.** Simplify \( \frac{2x^3 + x^2}{x^4} \)

**Solution.** Factor the numerator: \( 2x^3 + x^2 = x^2(2x + 1) \). Then simplify \( \frac{x^2}{x^4} = \frac{1}{x^2} \). So

\[
\frac{2x^3 + x^2}{x^4} = \frac{x^2(2x + 1)}{x^4} = \frac{2x + 1}{x^2}.
\]

To succeed in calculus, we need comfort with these algebra techniques. Practice these techniques throughout the semester, so that you can indeed be *comfortable* with your algebra!
1.0.5 Other resources for algebra review

The material in this section is review material in a precalculus class and is usually not explicitly covered in a textbook.

In the textbook by Ratti & McWaters, *Precalculus, A Unit Circle Approach*, 2nd ed., c. 2014, [here at Amazon.com](https://www.amazon.com) this material appears in appendices A.1, A.3, A.4. In the textbook by Stewart, *Precalculus, Mathematics for Calculus*, 6th ed., c. 2012, [here at Amazon.com](https://www.amazon.com) this material appears in sections 1.2, 1.3 and 1.4. (In July 2013 the first textbook was $147 at Amazon.com and the second textbook was $136 at Amazon.com They are even more expensive in campus bookstores.)

There are lots of online resources for reviewing algebra. Here are additional sets of resources, in addition to the class notes and class presentations.

1. [Dr. Paul’s Dawkins’ webpage of algebra notes](https://math.lamar.edu) (Dr. Dawkins is a professor at Lamar University.)
   There is good review material at
   (a) [integer exponents](https://math.lamar.edu) and
   (b) [rational exponents](https://math.lamar.edu) and
   (c) [radicals](https://math.lamar.edu)

2. There are some nice [algebra videos from Khan Academy](https://www.khanacademy.org)

Homework.

As class homework, please complete **Worksheet 1.0, Algebra Excellence**, available through the class webpage or Blackboard.