Excellence in Algebra

Before we can be successful in science and calculus, we need some comfort with algebra.
Here are two major algebra computations we do throughout this class (and you will do throughout your career!)

- exponential notation, and
- polynomial arithmetic

Here we review exponential notation.

Exponential Notation

About three centuries ago, scientists developed abbreviations for multiplication of a variable, replacing

- $x \cdot x$ by $x^2$,
- $x \cdot x \cdot x$ by $x^3$ and
- $x \cdot x \cdot x \cdot x \cdot x$ by $x^5$, etc.

This is just an abbreviation! The exponent merely counts the number of times the base appears in the product.

This leads to some basic “rules” consistent with the abbreviation.

- For example, $x^3 \cdot x^2 = (x \cdot x \cdot x) \cdot (x \cdot x) = x^5$
- So if we multiply objects with the same base ($x$) we should add the exponents:

$$\frac{x^3}{x^2} = \frac{x \cdot x \cdot x}{x \cdot x} = \frac{x}{1} = x$$

When we divide objects with the same base ($x$) we subtract the exponents.

- $(x^3)^2 = (x \cdot x \cdot x)(x \cdot x \cdot x) = x \cdot x \cdot x \cdot x \cdot x \cdot x = x^6$.

Repeated exponentiation leads to multiplying exponents.

More Exponential Notation

Our symbolic abbreviation involving exponents leads naturally to some basic observations, sometimes called algebra “rules”.

Multiplying by 1 leaves a number unchanged.
Since $x^n \cdot x^0 = x^{n+0} = x^n$ then multiplying by $x^0$ leaves a number unchanged.
So

$$x^0 = 1 \quad (1)$$

We can also extend our exponent notation to rational exponents.
Since $(x^{\frac{1}{2}})^2 = x^{\frac{1}{2} \cdot 2} = x^1 = x$ then

$$x^{\frac{1}{2}} = \sqrt{x}.$$  

More generally, denominators in exponents represent roots:

$$x^{\frac{1}{3}} = \sqrt[3]{x}.$$  

(2)
Practicing exponential notation

Let’s practice this with two exercises

**Exercise.** Simplify \(8^{\frac{2}{3}}\).

**Solution.**

\[
8^{\frac{2}{3}} = (8^{1/3})^2 = (\sqrt[3]{8})^2 = 2^2 = 4.
\]

**Exercise.** Simplify \(4^{\frac{3}{2}}\).

**Solution.**

\[
4^{\frac{3}{2}} = (4^{1/2})^3 = (\sqrt{4})^3 = 2^3 = 8.
\]

Kilobytes and powers of ten

Here is a problem common in computer science applications. We note that \(2^{10} = 1024\) while \(1000 = 10^3\). So \(2^{10} \approx 10^3\)

The electronics (on/off) of a computer means that a computer scientist works in base two.

Computer storage and computer memory is measured in powers of two. But the language of computer science uses traditional powers of ten:

- **kilo-** represent a thousand,
- **mega-** represents a million and
- **giga-** a billion (etc.)

To a computer scientist, kilo- represents \(2^{10}\), not \(10^3\).

*Computer scientists approximate powers of 2 as powers of 10!*

**Example.** Approximate \(2^{30}\) as a power of ten: Since

\[
2^{10} = (2^{10})^3 \text{ and } 2^{10} \approx 10^3
\]

then \(2^{30} = (2^{10})^3 \approx (10^3)^3 = 10^9\).

More bytes and exponents

**Exercise.** How many digits are there in \(2^{300}\)?

**Solution.** We write \(2^{300} = (2^{10})^{30} \approx (10^3)^{30} = 10^{90}\).

Now \(10^{90} = 1 \times 10^{90} = 1\) followed by 90 zeroes so it has 91 digits. Therefore \(2^{300}\) should have 91 digits.

WolframAlpha gives

\[
2^{300} = 2037035976334486086268445688409378161051468393665936250636
\]

\[
4493543812997633367061833397376
\]

You can check that this has 91 digits!

More Exponential Notation

To succeed in calculus, we need comfort with these algebra techniques.

Practice these techniques throughout the semester, so that you can indeed be comfortable with your algebra!

In the next slide presentation we practice problems involving exponents and polynomials.

(End)
Practicing with exponents

In this short lecture we practice some problems involving the algebra of exponents and the algebra of polynomials.

Practice. (Some sample problems from an old quiz.)

- Simplify the expression \( \frac{\sqrt[3]{x^6}}{\sqrt{x^2}} \).

**Solution.** Rewrite \( \sqrt[3]{x^6} \) as \( x^2 \) since (this is the meaning of cube root!)
\((x^2)^3 = x^6\).
Rewrite \( \sqrt{x^2} \) as \( x \). (This is merely the meaning of square root.)
So
\[
\frac{\sqrt[3]{x^6}}{\sqrt{x^2}} = \frac{x^2}{x} = x.
\]

More exponent practice

- Simplify \( \frac{(x^6)^{\frac{1}{3}} \cdot x^{-2}}{x^4} \).

**Solution.** Simplify the numerator,
\((x^6)^{\frac{1}{3}} = x^2\) and \( x^2 \cdot x^{-2} = \frac{x^2}{x^2} = 1 \).
So
\[
\frac{(x^6)^{\frac{1}{3}} \cdot x^{-2}}{x^4} = \frac{x^2}{x^4} = \frac{1}{x^2} \text{ or } x^{-4}
\]

More exponent practice

- Simplify the expression \( \frac{2x^3 + x^2}{x^4} \).

**Solution.** Factor the numerator: \( 2x^3 + x^2 = (2x)x^2 + (1)x^2 = x^2(2x + 1) \).
Also simplify \( \frac{x^2}{x^4} = \frac{1}{x^2} \). So
\[
\frac{2x^3 + x^2}{x^4} = \frac{x^2(2x + 1)}{x^4} = \frac{2x + 1}{x^2}.
\]
Review of polynomial arithmetic

Once we learn the algebra abbreviation called “exponentiation”, we can often use polynomial arithmetic (factoring polynomials) to simplify an expression.

Let’s simplify the complex fraction

\[
\frac{2x^3 + x^2}{x^4} \div \frac{1}{x^2}
\]

Dividing by \( \frac{1}{x^2} \) is the same as multiplying by \( x^2 \) so

\[
\frac{2x^3 + x^2}{x^4} \cdot x^2 = \frac{2x^3 + x^2}{x^2}
\]

We factor the numerator and simplify.

\[
x^2(2x + 1) - 2x + 1
\]

More Exponential Notation

Here are some other simplification problems involving factoring.

Recall that \((A - B)(A + B) = A^2 - B^2\) and so we know how to factor a difference of squares.

- Simplify \(\frac{x^2 - 9}{x + 3}\).

**Solution.** We recognize \(x^2 - 9 = x^2 - 3^2\) as a difference of squares

\[
\frac{x^2 - 9}{x + 3} = \frac{(x - 3)(x + 3)}{x + 3} = x - 3.
\]

Difference of squares

- Simplify \(\frac{x^4 - 9}{x^2 + 3}\).

**Solution.** We use the same pattern as before, recognizing that the difference of squares \(A^2 - B^2 = (A + B)(A - B)\).

\[
x^4 - 9 = \frac{(x^2 - 3)(x^2 + 3)}{x^2 + 3} = x^2 - 3.
\]

More algebra practice

One more....

- Simplify \(\frac{x - 9}{\sqrt{x} + 3}\).

**Two solutions.** (1) We could do this as a difference of squares! (Think of \(x\) as the square of \(\sqrt{x}\).)

\[
\frac{x - 9}{\sqrt{x} + 3} = \frac{(\sqrt{x} - 3)(\sqrt{x} + 3)}{\sqrt{x} + 3} = \sqrt{x} - 3.
\]

(2) Or maybe you were taught to multiply by the "conjugate" here:

\[
\frac{x - 9}{\sqrt{x} + 3} = \frac{(x - 9)(\sqrt{x} - 3)}{(x - 9)} = \frac{\sqrt{x} - 3}{x - 9}.
\]

Simplify the denominator and then cancel:

\[
\frac{x - 9}{\sqrt{x} + 3} \cdot \frac{\sqrt{x} - 3}{\sqrt{x} - 3} = \frac{(x - 9)(\sqrt{x} - 3)}{(x - 9)} = \sqrt{x} - 3.
\]
To succeed in calculus, we need *comfort* with these algebra techniques.

*Practice* these techniques throughout the semester, so that you can indeed be *comfortable* with your algebra!

(END)