

Vladislav Babenko*, Yuliya Babenko** and
Dmytro Skorokhodov*

* *Dnepropetrovsk National University*

** *Sam Houston State University*

On the linear approximation of
a quadratic function of two variables
in L_p -metric and applications

Sam Houston State University

Huntsville, Texas

October 26 – 27, 2007

Let $G \subset \mathbb{R}^2$ be a bounded connected domain which can be represented as a finite union of closed triangles.

For $1 \leq p \leq \infty$ and for a weight function $\Omega(x, y)$ set $L_{p, \Omega}(G)$, be the space of functions with the weighted norm:

$$\|f\|_{p, \Omega} = \begin{cases} \left(\int_G |f(x, y)|^p \Omega(x, y) dx dy \right)^{1/p} & \text{if } p < \infty, \\ \text{esssup} \{ |f(x, y)| \Omega(x, y) : (x, y) \in \mathcal{M} \} & \text{if } p = \infty. \end{cases}$$

Let $z = f(x, y) \in C^2(G)$. Denote the Hessian of $f(x, y)$ by

$$H(x, y) := (f_{xx}f_{yy} - f_{xy}^2)(x, y).$$

The Hessian $H(x, y)$ and the Gaussian curvature $K(x, y)$ of the surface $z = f(x, y)$ are connected by the formula:

$$K(x, y) = \frac{H(x, y)}{(1 + (f_x(x, y))^2 + (f_y(x, y))^2)^2}.$$

A collection $\Delta_N = \Delta_N(G) = \{T_i\}_{i=1}^N$ of N triangles in the plane is called a triangulation of a set G provided that:

(i) any pair of triangles from Δ_N intersect at most at a common vertex or along a common edge,

(ii) $G = \bigcup_{i=1}^N T_i.$

Let \mathcal{P}_1 be the set of polynomials

$$p(x, y) = ax + by + c, \quad a, b, c \in \mathbb{R}.$$

Set

$$\mathcal{S}_1^0(\Delta_N) := \{f \in C(G) : \forall i = 1, \dots, N \exists p \in \mathcal{P}_1 \text{ s.t. } f|_{T_i} = p|_{T_i}\}.$$

Main purpose

For each function f we need to find an "optimal" in some sense triangulation Δ_N^* of the set G which provides us with the lowest error of approximation of f by $\mathcal{S}_1^0(\Delta_N^*)$ among all other triangulations Δ_N .

L. Fejes Toth, 1972: he indicated that the distance from a body $C \in \mathbb{R}^3$ with boundary of differentiability class C^2 and positive Gaussian curvature to its best inscribed polytope with at most n vertices in the Hausdorff metric is

$$d_H(C) = \frac{1 + o(1)}{3\sqrt{3}} \left(\int_{\partial C} K(x, y)^{1/2} d\sigma(x, y) \right) \frac{1}{n}$$

and the volume of the difference between C and the polytope is

$$d_1(C) = \frac{1 + o(1)}{4\sqrt{3}} \left(\int_{\partial C} K(x, y)^{1/2} d\sigma(x, y) \right)^2 \frac{1}{n}$$

as $n \rightarrow \infty$, where σ is the surface measure in \mathbb{R}^3 .

Gruber, 1992 generalized Toth's result on the case of convex bodies in \mathbb{R}^d .

Nadler, 1986 solved the problem of asymptotically optimal choice of a sequence of triangulations for approximation of C^3 functions by piecewise linear splines (which are not globally continuous) of best L_2 -approximation.

D'Azevedo and Simpson, 1989 studied the question of triangulating a given set of vertices for interpolation of a convex quadratic surface by piecewise linear functions.

This problem was studied by a lot of other mathematicians such as Pottmann, Hamann, Huang, Sun, Chen, V. Babenko, Yu. Babenko, Ligun, Shumeiko, etc.

Let $D = [0, 1]^2$.

Theorem 1 (Yu. Babenko, 2006) *Let $f \in C^2(D)$ and $H(f; x, y) \geq C^+ > 0$ for all $(x, y) \in D$. Let also the positive continuous weight function $\Omega(x, y)$ be given. Then for all $1 \leq p \leq \infty$*

$$\limsup_{N \rightarrow \infty} N \|f - s(f, \Delta_N^*)\|_{p, \Omega}$$

$$\leq \frac{C_p^+}{2} \left(\int_D H(f; x, y)^{\frac{p}{2(p+1)}} \Omega(x, y)^{\frac{1}{p+1}} dx dy \right)^{\frac{p+1}{p}}$$

where

$$C_p^+ = \min_T \frac{L_p - \text{error of linear interpolation of } x^2 + y^2 \text{ on } T}{|T|^{1 + \frac{1}{p}}}.$$

Theorem 2 (Yu. Babenko, 2006) *Let $f \in C^2(D)$ and $H(f; x, y) \geq C^+ > 0$ for all $(x, y) \in D$. Let also the positive continuous weight function $\Omega(x, y)$ be given. Then for all $1 \leq p \leq \infty$ and for any sequence $\{\Delta_N\}_{N=1}^{\infty}$ of triangulations satisfying*

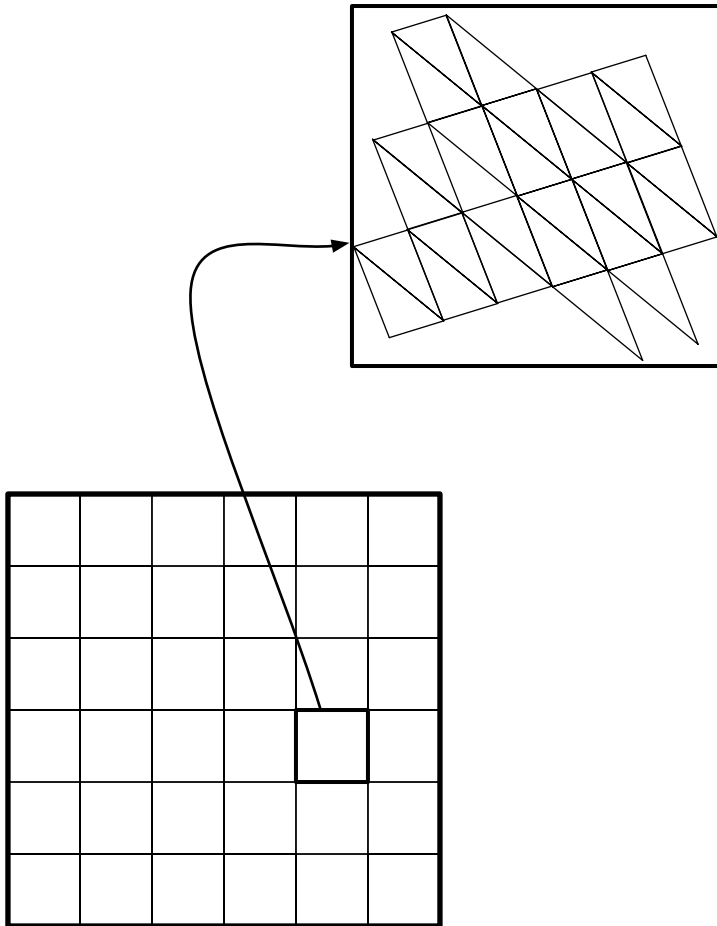
$$\sup_N \sqrt{N} \max_{T \in \Delta_N} \text{diam}(T) < \infty$$

we have

$$\begin{aligned} & \liminf_{N \rightarrow \infty} N \|f - s(f, \Delta_N)\|_{p, \Omega} \\ & \geq \frac{C_p^+}{2} \left(\int_D H(f; x, y)^{\frac{p}{2(p+1)}} \Omega(x, y)^{\frac{1}{p+1}} dx dy \right)^{\frac{p+1}{p}}. \end{aligned}$$

Construction of optimal sequence of linear splines

1. Divide D into a number m_N^2 of equal subregions D_i^N .
On each D_i^N instead of f , consider its Taylor polynomial $P_{N,i}$ of second degree taken at the center of D_i^N .

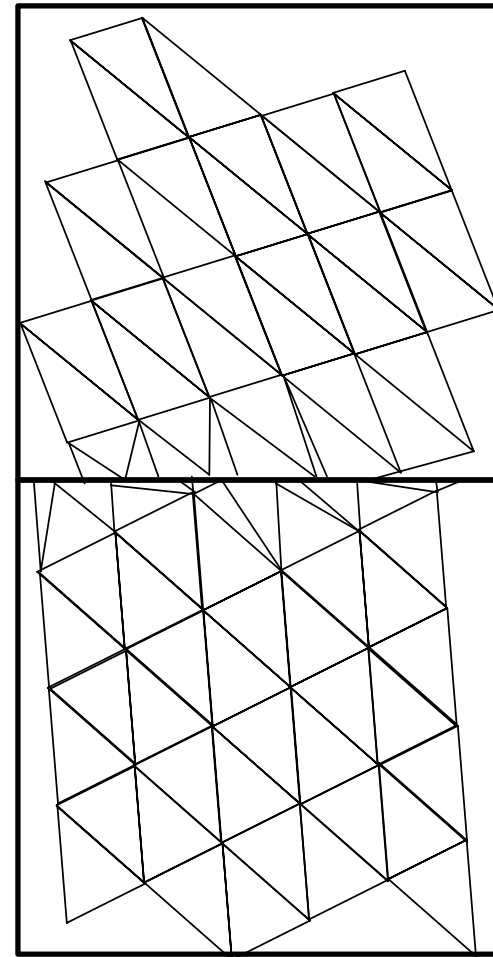
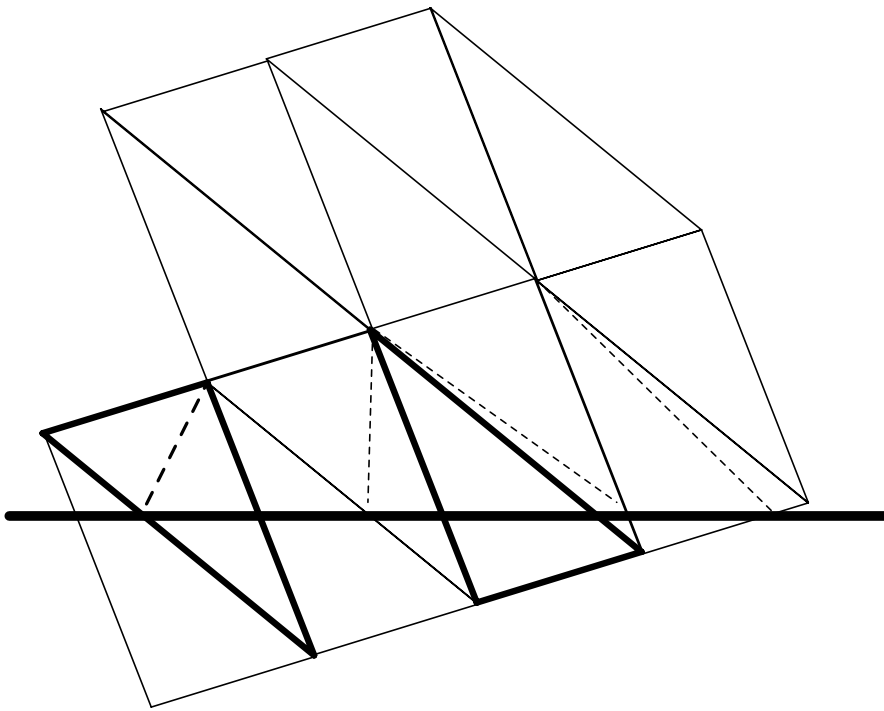


2. To find an appropriate triangulation of D_i^N , first take any triangle T , which provides minimum in the problem

$$\frac{L_p - \text{error of linear interpolation of } x^2 + y^2 \text{ on } T}{|T|^{1+\frac{1}{p}}} \rightarrow \min_T. \quad (1)$$

Then we reshape T and choose the size of T in such a way that the overall number n_i^N of triangles used for D_i^N is approximately N .

3. We obtain the final triangulation of D by "gluing" together triangulations of each region D_i^N possibly subdividing (without adding new vertices) triangles which have nonempty intersection with $\cup_i \partial D_i^N$, where ∂D_i^N denotes the boundary of D_i^N .



What is the constant C_p^+ ?

The constant C_p^+ was known in for the following cases:

$p = \infty$ V. Babenko, Yu. Babenko, *et al* (2006),

$p = 1, 2, 3$ Yu. Babenko (2007).

We have found the exact value of C_p^+ for $p > 0$:

$$C_p^+ = \left(\frac{4}{3\sqrt{3}} \right)^{p+1} \left[\frac{\pi}{p+1} - 6 \int_{1/2}^1 x(1-x^2)^p \arccos \frac{1}{2x} dx \right].$$

Moreover, we have established that the minimum in the right hand part of (1) is attained by equilateral triangles only.