Markov-type inequalities for homogeneous polynomials on nonsymmetric star-like domains.

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Abstract

Let H_n^d be the set of homogeneous polynomials of degree n and d variables. Consider a compact set $K \subset \mathbb{R}^d$. Denote by $\|\cdot\|_K$ the usual sup norm on K. It was proved by Harris [4] that if K is a $\mathbf{0}$-symmetric convex body then for every $h \in H_n^d$ with $\|h\|_K \leq 1$ we have $\|D_{\mathbf{u}}h\|_K \leq Cn \log n$ where $D_{\mathbf{u}}h$ is the derivative of h in the direction $\mathbf{u} \in S^{d-1}$. In this paper we extend Harris' result for nonsymmetric star-like domains.

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1 Introduction and New Results.

Let K be a compact set in $\mathbb{R}^d$, and F be a family of differentiable functions on K. As usual, $S^{d-1} = \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| = 1\}$ stands for the unit sphere in $\mathbb{R}^d$ in Euclidean norm. (Here and in what follows $|\mathbf{x}|$ denotes the Euclidean norm of $\mathbf{x}$ in $\mathbb{R}^d$.) Denote by $D_{\mathbf{u}}f$ the derivative of f in direction $\mathbf{u} \in S^{d-1}$. Let

$$\|f\|_K = \sup_{\mathbf{x}\in K} |f(\mathbf{x})|$$

be the usual supremum norm on K.

Then the Markov Factor of F on K is given by

$$M(F,K) := \sup\left\{\|D_{\mathbf{u}}f\|_K : f \in F, \|f\|_K \leq 1, \mathbf{u} \in S^{d-1}\right\}. \tag{1}$$

This quantity measures the size of the derivatives of functions in F compared to their sup norms on K. The problem of estimating $M(F,K)$ originates from the classical Markov inequality which gives

$$M(P_n^1, [a,b]) = \frac{2n^2}{b-a}, \tag{2}$$

where

$$P_n^d := \left\{\sum_{|\mathbf{k}|_1 \leq n} a_{\mathbf{k}}\mathbf{x}^{\mathbf{k}} : a_{\mathbf{k}} \in \mathbb{R}\right\}, \quad \mathbf{x} \in \mathbb{R}^d, \quad d \geq 1, \quad n \geq 1, \tag{3}$$

is the space of polynomials of total degree at most n in d variables. (Here $|\cdot|_1$ denotes the l_1-norm.)

Numerous extensions of Markov inequality for various families of univariate and multivariate polynomials are known. For an overview of univariate inequalities see [2] or [3]; a survey of multivariate Markov-type inequalities can be found in [6]. In particular, it is known that for convex bodies $K \subset \mathbb{R}^d$ we have

$$M(P_n^d, K) \asymp n^2, \tag{4}$$

while for cuspidal domains in $\mathbb{R}^d$ the Markov Factors of P_n^d are, in general, of higher order. The size of the Markov Factors is essentially different for the set of homogeneous polynomials of degree n defined by

$$H_n^d := \left\{\sum_{|\mathbf{k}|_1 = n} a_{\mathbf{k}}\mathbf{x}^{\mathbf{k}} : a_{\mathbf{k}} \in \mathbb{R}\right\}, \quad \mathbf{x} \in \mathbb{R}^d, \quad d \geq 2, \quad n \geq 1. \tag{5}$$

It was first shown by Harris [4] that for $\mathbf{0}$-symmetric convex bodies K in $\mathbb{R}^d$ we have

$$M(H_n^d, K) \leq Cn \log n. \tag{6}$$

For the sharpness of this upper bound see [5] and [7].

Thus, the rates of Markov Factors for homogeneous polynomials are substantially smaller than for ordinary polynomials. It is also shown in [5] that for smooth $\mathbf{0}$-symmetric convex bodies the $\log n$ in (6) can be dropped, i.e., $M(H_n^d, K) = O(n)$.

In all of the papers on homogeneous polynomials mentioned above the symmetry of the domain played an essential role. The goal of the present note consists of extending the results on Markov Factors for homogeneous polynomials to nonsymmetric domains K for which $\mathbf{0}$ is on the boundary ∂K of K, rather than inside K. The consideration of nonsymmetric domains K will require a more delicate study of the geometry of K around the origin. Also we shall relax the assumption of convexity of the domain and replace it by the more general star-like property. Since $|h(\mathbf{x})| = |h(-\mathbf{x})|$ for $h \in H_n^d$, it is natural to consider star-like domains contained in the half-space

$$\mathbb{R}_+^d := \left\{ \mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 \geq 0 \right\}.$$

Let

$$S_+^{d-1} := \left\{ \mathbf{x} = (x_1, \ldots, x_d) \in S^{d-1} : x_1 \geq 0 \right\}$$

be the upper halfsphere, $r : S_+^{d-1} \to \mathbb{R}_+$ be a continuous nonnegative mapping. Then a star-like domain in the halfspace $\mathbb{R}_+^d$ associated with r is given by

$$K_r := \left\{ t\mathbf{x} : \mathbf{x} \in S_+^{d-1},\ 0 \leq t \leq r(\mathbf{x}) \right\}. \tag{7}$$

We shall need to impose some mild smoothness conditions on ∂K_r at the origin.

Assume that there exist $C_1, C_2 > 0$, $\beta > 0$, and $0 < \epsilon < 1$ such that for $\mathbf{x} = (x_1, \ldots, x_d) \in S_+^{d-1}$

$$C_2 e^{-x_1^{\epsilon - 1}} \leq r(\mathbf{x}) \leq C_1 x_1^{\beta}. \tag{8}$$

The right inequality in (8) insures that $r(\mathbf{x})$ tends to 0 (as $\mathbf{x} \to \mathbf{0}$) with a polynomial rate. On the other hand the left inequality requires that $r(\mathbf{x})$ does not vanish at $\mathbf{0}$ exponentially. Thus if, for example,

$$C_2 x_1^{\alpha} \leq r(\mathbf{x}) \leq C_1 x_1^{\beta}$$

with some $C_1, C_2 > 0$, $0 < \beta < \alpha$ $(0 < x_1 \le 1)$ then (8) will hold. We shall also require that $r \in \mathrm{Lip}_M\alpha$ on S_+^{d-1}, for some $0 < \alpha \le 1$, $M > 0$, i.e. for any $\mathbf{x}_1, \mathbf{x}_2 \in S_+^{d-1}$

$$|r(\mathbf{x}_1) - r(\mathbf{x}_2)| \le M|\mathbf{x}_1 - \mathbf{x}_2|^\alpha.$$

Main Theorem. *Let K_r be a star-like domain (7) with $r \in Lip_M\alpha$, $0 < \alpha \le 1$, satisfying condition (8). Then for any $n \ge 2$*

$$M(H_n^d, K_r) \le C_0 \rho_\alpha(n), \tag{9}$$

where

$$\rho_\alpha(n) := \begin{cases} n^{1/\alpha}, & \alpha < 1; \\ n \log n, & \alpha = 1, \end{cases} \tag{10}$$

and constant $C_0 > 0$ depends only on K_r.

For $\alpha = 1$ the above theorem is a generalization of Harris' result to nonsymmetric star-like domains without cusps. In particular, it yields that $M(H_n^d, K_r) = O(n \log n)$ for a wide family of nonsymmetric convex bodies. The cuspidal case $(0 < \alpha < 1)$ is new.

While we can not prove the necessity of condition (8) it can be shown that some kind of smoothness at the origin is needed, in general. Indeed, let

$$h_n(x, y) := x^n \tilde{T}_n\left(\frac{y}{x}\right) \in H_n^2,$$

where $\tilde{T}_n(t) = \cos(n \arccos(2t - 1))$ is the Chebyshev polynomial on $[0, 1]$. Consider the triangle

$$\Delta := \{(x, y) \in \mathbb{R}^2 : 0 \le y \le x \le 1\}.$$

Clearly, $\|h_n\|_\Delta = 1$, and

$$\frac{\partial}{\partial y} h_n(1, 1) = \tilde{T}_n'(1) = 4n^2.$$

Thus, the $O(n \log n)$ bound does not hold for Δ. (Note that condition (8) fails for this triangle.)

Now let us address the question of sharpness of estimate (9). For $\alpha = 1$, i.e. the $n \log n$ bound, the estimate is known to be sharp even in **0**-symmetric case (see [5], [7]). Consider now the domain

$$D_\alpha := \{(x, y) \in \mathbb{R}^2 : x^2 \le y \le 1 - x^\alpha\}, \quad 0 < \alpha.$$

Clearly, this star-like set satisfies (8), and the $\mathrm{Lip}_M\alpha$ condition holds for the function r associated with this set. Consider now $g_n(x,y) := xy^n \in H^2_{n+1}$. An easy calculation yields that

$$\|g_n\|_{D_\alpha} \le \|(1-|x|^\alpha)^n x\|_{[-1,1]} = \left(\frac{\alpha n}{\alpha n+1}\right)^n \frac{1}{(1+\alpha n)^{1/\alpha}} \le Cn^{-1/\alpha}.$$

On the other hand $\|\frac{\partial g_n}{\partial x}\|_{D_\alpha} = 1$. Thus, $M(H^2_n, D_\alpha) \ge \frac{n^{1/\alpha}}{C}$ which is also the rate of the upper bound of (9) for $0 < \alpha < 1$. Hence, the estimate (9) is sharp, in general.

2 Proof of the Main Theorem in the case $d = 2$.

Let the star-like domain $K_r \subset \mathbb{R}^2, \mathbf{0} \in \partial K_r$, be parameterized by polar coordinates:

$$x = \tilde\rho\cos\phi, \quad y = \tilde\rho\sin\phi, \quad 0 \le \tilde\rho \le r(\cos\phi, \sin\phi), \quad |\phi| \le \frac{\pi}{2}. \tag{11}$$

Set

$$\tilde r(t) := r\left(\frac{1}{\sqrt{1+t^2}}, \frac{t}{\sqrt{1+t^2}}\right), \quad t = \tan\phi, \quad t \in \mathbb{R}.$$

Since $r \in \mathrm{Lip}_M\alpha$ on S^1, it immediately follows that $\tilde r \in \mathrm{Lip}_M\alpha$ on $\mathbb{R}$. Moreover, by (8) (with $x_1 = \cos\phi = 1/\sqrt{1+t^2}$)

$$e^{-C_3|t|^{1-\epsilon}} \le \tilde r(t) \le C_1|t|^{-\beta}, \quad t \in \mathbb{R}, \tag{12}$$

where $C_3 > 0$ depends only on K_r. Suppose we have a homogeneous polynomial $p \in H^2_n$

$$p(x,y) = \sum_{k=0}^n a_k x^{n-k} y^k = x^n \sum_{k=0}^n a_k \left(\frac{y}{x}\right)^k,$$

satisfying

$$\|p\|_{K_r} \le 1. \tag{13}$$

Setting $t = \tan\phi, |\phi| \le \pi/2$, we obtain a weighted univariate polynomial on $\mathbb{R}$

$$p(r(\cos\phi,\sin\phi)\cos\phi, r(\cos\phi,\sin\phi)\sin\phi) = \left(\frac{\tilde r(t)}{\sqrt{1+t^2}}\right)^n \sum_{k=0}^n a_k t^k. \tag{14}$$

Denote $q_n(t) := \sum_{k=0}^{n} a_k t^k$. Then by (13) and (14)

$$\left\| \left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}} \right)^n q_n(t) \right\|_{\mathbb{R}} = \|p\|_{\partial K_r} \leq 1. \tag{15}$$

Observe that for $(x, y) \in \partial K_r$

$$\frac{\partial p(x,y)}{\partial x} = n \sum_{k=0}^{n} a_k x^{n-k-1} y^k - \sum_{k=1}^{n} k a_k x^{n-k-1} y^k$$

$$= n \left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}} \right)^{n-1} q_n(t) - \left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}} \right)^{n-1} t q_n'(t). \tag{16}$$

and

$$\frac{\partial p(x,y)}{\partial y} = x^{n-1} \sum_{k=1}^{n} k a_k \left(\frac{y}{x} \right)^{k-1} = \left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}} \right)^{n-1} q_n'(t). \tag{17}$$

In order to estimate the weighted polynomials appearing in (16) and (17) we shall need the following two lemmas.

Lemma 1. *Let $\tilde{r} \in C(\mathbb{R})$, $\tilde{r} > 0$, satisfy the right inequality in (12) for some $\beta > 0$. Then there exists a $t^* > 0$ depending only on $\tilde{r}$ such that for any $n \geq 2 + 2/\beta$ and $q_n \in P_n^1$ we have*

$$\left\| \left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}} \right)^{n-1} q_n(t) \right\|_{\mathbb{R}} = \left\| \left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}} \right)^{n-1} q_n(t) \right\|_{[-t^*, t^*]}.$$

Proof. Note that $\tilde{r}(t) \leq C_1 |t|^{-\beta}$ and $n \geq 2 + 2/\beta$ ensure that

$$\left\| \left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}} \right)^{n-1} q_n(t) \right\|_{\mathbb{R}} < \infty.$$

Thus, without loss of generality we can assume

$$\left\| \left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}} \right)^{n-1} q_n(t) \right\|_{\mathbb{R}} = 1.$$

It is known (for example, see [2]) that for any $q_n(t) \in P_n^1$

$$|q_n(t)| \leq \|q_n\|_{[-1,1]}|T_n(t)|, \quad |t| > 1, \tag{18}$$

where

$$T_n(t) := \frac{1}{2}\left((t+\sqrt{t^2-1})^n + (t-\sqrt{t^2-1})^n\right)$$

is the Chebyshev polynomial. Evidently,

$$|T_n(t)| \leq 2^n|t|^n, \quad |t| > 1. \tag{19}$$

To estimate the norm $\|q_n\|_{[-1,1]}$, observe that

$$\inf_{|t|\leq 1} \tilde{r}(t) > 0$$

and, hence,

$$|q_n(t)| \leq \left(\frac{\sqrt{1+t^2}}{\tilde{r}(t)}\right)^{n-1} \leq C_4^{n-1}, \quad |t| \leq 1, \tag{20}$$

with some positive constant C_4 depending only on $\tilde{r}$. Therefore, combining estimates (18), (19) and (20), we obtain that for $|t| > 1$

$$|q_n(t)| \leq 2^n|t|^n C_4^{n-1} = C_5^n|t|^n.$$

Taking into consideration that

$$\tilde{r}(t) \leq C_1|t|^{-\beta},$$

we obtain for $|t| > 1$

$$\left|\left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}}\right)^{n-1} q_n(t)\right| \leq \frac{\tilde{r}^{n-1}(t)C_5^n|t|^n}{|t|^{n-1}} \leq \frac{C_6^n}{|t|^{\beta(n-1)-1}},$$

where C_6 depends only on $\tilde{r}$. Note that assumption $n \geq 2 + 2/\beta$ yields that $\beta(n-1) - 1 \geq \beta n/2$. Thus, setting $t^* := C_6^{2/\beta} + 1$ we obtain for $|t| > t^* > 1$

$$\left|\left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}}\right)^{n-1} q_n(t)\right| \leq \left(\frac{C_6}{|t|^{\beta/2}}\right)^n < 1.$$

Hence, the norm of the weighted polynomial is achieved on the finite interval $[-t^*, t^*]$. $\square$

In what follows $C_k,\ k \in \mathbb{N}$ will stand for constants depending only on $\tilde{r}$.

Lemma 2. *Let $\tilde{r} \in C(\mathbb{R})$ satisfy (12). Then for any $q_n \in P_n^1$ satisfying (15)*

$$\left\| \left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}} \right)^{n-1} q_n'(t)| \right\|_{\mathbb{R}} \leq C_7 \rho_\alpha(n), \tag{21}$$

where $\rho_\alpha(n)$ is defined in (10).

Proof.

First, observe that condition (15) can be rewritten in the following form:

$$|e^{-nQ(t)} q_n(t)| \leq 1 \tag{22}$$

where

$$Q(t) := \log \Omega(t) \text{ and } \Omega(t) := \frac{\sqrt{1+t^2}}{\tilde{r}(t)}.$$

By a well-known inequality (see, for example, [1], p. 92) for any $\xi \in [-t^*, t^*]$, where t^* is defined in Lemma 1, and any $z = u + iv \in \mathbb{C}$ such that $|z - \xi| \leq \rho$, $0 < \rho < 1/e$,

$$\begin{aligned} \log |q_n(z)| \leq \frac{|v|}{\pi} \int_{\mathbb{R}} \frac{\log |q_n(t)|}{(t-u)^2 + v^2} dt \leq \frac{n|v|}{\pi} \int_{\mathbb{R}} \frac{Q(t)}{(t-u)^2 + v^2} dt \\ = \frac{n|v|}{\pi} \int_{\mathbb{R}} \frac{Q(t_1 + \xi)}{(t_1 - u_1)^2 + v^2} dt_1, \end{aligned} \tag{23}$$

where $t_1 := t - \xi$, $u_1 := u - \xi$. Note that $|u_1| = |u - \xi| \leq \rho$, $|v| \leq \rho$.

Since

$$\int_{\mathbb{R}} \frac{1}{(t-u_1)^2 + v^2} dt = \frac{\pi}{|v|},$$

we have that the last expression in (23) is equal to

$$\frac{n|v|}{\pi} \int_{\mathbb{R}} \frac{Q(t+\xi) - Q(\xi)}{(t-u_1)^2 + v^2} dt + nQ(\xi).$$

Hence,

$$\log(e^{-nQ(\xi)} |q_n(z)|) \leq \frac{n|v|}{\pi} \int_{\mathbb{R}} \frac{\log \Omega(t+\xi) - \log \Omega(\xi)}{(t-u_1)^2 + v^2} dt. \tag{24}$$

The numerator in the integrand on the right-hand side can be rewritten as follows

$$\log \Omega(t+\xi) - \log \Omega(\xi) = \log \frac{\Omega(t+\xi)}{\Omega(\xi)} = \log(\frac{\Omega(t+\xi) - \Omega(\xi)}{\Omega(\xi)} + 1). \tag{25}$$

Let us introduce the following modulus of continuity of function Ω:

$$\tilde{\omega}(\Omega, t) := \max\{|\Omega(x_1) - \Omega(x_2)| : x_1 \in [-t^*, t^*], |x_1 - x_2| \leq t\}, \quad t > 0.$$

It has the following properties:

(i) $\tilde{\omega}(\Omega, t)$ is an increasing function of t.

(ii) For $|t| \geq 1$, by property (12), we have

$$\tilde{\omega}(\Omega, t) \leq 2 \max_{x \in [-t^*-t, t^*+t]} |\Omega(x)| \leq e^{C_8(t^*+t)^{1-\varepsilon}}. \tag{26}$$

This yields

$$\log \tilde{\omega}(\Omega, t) = O(t^{1-\varepsilon}), \quad \text{for} \quad |t| \geq 1. \tag{27}$$

(iii) $\tilde{\omega}(\Omega, t) \leq C_9 t^\alpha$ for $|t| \leq 1$, because $\Omega(t) := \frac{\sqrt{1+t^2}}{\tilde{r}(t)}$, where $\tilde{r}(t) \in \mathrm{Lip}_M \alpha$ on $\mathbb{R}$ and $\tilde{r}(t) \geq C_{10}$ for $|t| \leq 1 + t^*$ by property (12).

Thus, using (24) and substituting $t = u_1 + |v|y,\ t_2 = \rho(1+y)$, we have

$$\log(e^{-nQ(\xi)}|q_n(z)|) \leq \frac{n|v|}{\pi} \int_{\mathbb{R}} \frac{\log\left(\frac{\Omega(t+\xi)-\Omega(\xi)}{\Omega(\xi)} + 1\right)}{(t-u_1)^2 + v^2} dt \leq \frac{n|v|}{\pi} \int_{\mathbb{R}} \frac{\log\left(\frac{\tilde{\omega}(\Omega,|t|)}{C_{11}} + 1\right)}{(t-u_1)^2 + v^2} dt$$

$$= \frac{n}{\pi} \int_{\mathbb{R}} \frac{\log\left(\frac{\tilde{\omega}(\Omega,|u_1+|v|y|)}{C_{11}} + 1\right)}{y^2+1} dy \leq \frac{2n}{\pi} \int_0^\infty \frac{\log\left(\frac{\tilde{\omega}(\Omega,\rho(1+y))}{C_{11}} + 1\right)}{y^2+1} dy$$

$$\leq \frac{4n}{\pi} \int_0^\infty \frac{\log(\frac{\tilde{\omega}(\Omega,\rho(1+y))}{C_{11}} + 1)}{(y+1)^2} dy = \frac{4n\rho}{\pi} \int_\rho^\infty \frac{\log(\frac{\tilde{\omega}(\Omega,t_2)}{C_{11}} + 1)}{t_2^2} dt_2$$

$$= \frac{4n\rho}{\pi} \int_\rho^1 \frac{\log(\frac{\tilde{\omega}(\Omega,t_2)}{C_{11}} + 1)}{t_2^2} dt_2 + \frac{4n\rho}{\pi} \int_1^\infty \frac{\log(\frac{\tilde{\omega}(\Omega,t_2)}{C_{11}} + 1)}{t_2^2} dt_2. \tag{28}$$

We can estimate each integral as follows. Using property (iii) of $\tilde{\omega}(\Omega, t)$ we obtain for the first term in (28):

$$\frac{4n\rho}{\pi} \int_\rho^1 \frac{\log(\frac{\tilde{\omega}(\Omega,t)}{C_{11}} + 1)}{t^2} dt \leq C_{12} n\rho \int_\rho^1 t^{\alpha-2} dt \leq C_{13} n \gamma_\alpha(\rho), \tag{29}$$

where

$$\gamma_\alpha(\rho) := \begin{cases} \rho^\alpha, & \alpha < 1; \\ \rho \log \frac{1}{\rho}, & \alpha = 1. \end{cases} \tag{30}$$

To estimate the second term in (28) we use (ii) which yields

$$\frac{4n\rho}{\pi}\int_1^\infty \frac{\log(\frac{\tilde{\omega}(\Omega,t)}{C_{11}}+1)}{t^2}dt \leq C_{14}n\rho\int_1^\infty \frac{t^{1-\varepsilon}}{t^2}dt = \frac{C_{14}}{\varepsilon}n\rho. \tag{31}$$

Therefore, by (29) and (31) for the whole sum in (28) we obtain:

$$\log(e^{-nQ(\xi)}|q_n(z)|) \leq C_{15}n(\gamma_\alpha(\rho)+\rho) \leq 2C_{15}n\gamma_\alpha(\rho). \tag{32}$$

Setting in (32) $\rho := 1/\rho_\alpha(n)$ where $\rho_\alpha(n)$ is defined in (10), we obtain

$$\log(e^{-nQ(\xi)}|q_n(z)|) \leq C_{16}. \tag{33}$$

Thus, by (33) for all z such that $|z-\xi| \leq \rho = 1/\rho_\alpha(n)$

$$|q_n(z)| \leq C_{17}e^{nQ(\xi)}. \tag{34}$$

By the Cauchy Integral Formula

$$q_n'(\xi) = \frac{1}{2\pi i}\int_{|z-\xi|=\rho}\frac{q_n(z)}{(z-\xi)^2}dz.$$

Estimating this integral using (34), we have

$$|q_n'(\xi)| \leq \max_{z:|z-\xi|\leq\rho}|q_n(z)|\frac{1}{\rho} \leq C_{17}e^{nQ(\xi)}\frac{1}{\rho} = C_{17}e^{nQ(\xi)}\rho_\alpha(n).$$

Hence,

$$\left(\frac{\tilde{r}(\xi)}{\sqrt{1+\xi^2}}\right)^n |q_n'(\xi)| = e^{-nQ(\xi)}|q_n'(\xi)| \leq C_{17}\rho_\alpha(n). \tag{35}$$

Recall that this estimate holds for $\xi \in [-t^*, t^*]$, where t^* is defined in Lemma 1. Using Lemma 1

$$\left\|\left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}}\right)^{n-1} q_n'(t)\right\|_{\mathbb{R}} = \left\|\left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}}\right)^{n-1} q_n'(t)\right\|_{[-t^*,t^*]}.$$

Furthermore, by (12)

$$\left\|\frac{\sqrt{1+t^2}}{\tilde{r}(t)}\right\|_{[-t^*,t^*]} \leq C_{18}. \tag{36}$$

Finally, by last two relations and (35)

$$\left\| \left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}} \right)^{n-1} q_n'(t) \right\|_{\mathbb{R}} \le C_{17}\rho_\alpha(n) \left\| \frac{\sqrt{1+t^2}}{\tilde{r}(t)} \right\|_{[-t^*,t^*]} \le C_{19}\rho_\alpha(n).$$

This completes the proof of Lemma 2.

□

Now in order to prove the statement of the Theorem for $d = 2$ we need to estimate partial derivatives in (16) and (17) under the assupmtion (15). Note that for $n < 2 + 2/\beta$ the statement of the Theorem is obviously true by the equivalence of norms in finite-dimensional spaces, i.e., we may assume that $n \ge 2 + 2/\beta$ and, hence, Lemma 1 is applicable. Then using (16), (15) and (36) together with Lemmas 1 and 2, we obtain

$$\left\| \frac{\partial p}{\partial x} \right\|_{K_r} = \left\| \frac{\partial p}{\partial x} \right\|_{\partial K_r} \le n \left\| \left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}} \right)^{n-1} q_n(t) \right\|_{[-t^*,t^*]} + t^* \left\| \left(\frac{\tilde{r}(t)}{\sqrt{1+t^2}} \right)^{n-1} q_n'(t) \right\|_{[-t^*,t^*]}$$

$$\le n \left\| \frac{\sqrt{1+t^2}}{\tilde{r}(t)} \right\|_{[-t^*,t^*]} + t^* C_7 \rho_\alpha(n) \le C_{20}\rho_\alpha(n).$$

Similarly, by (17), (15) and Lemma 2

$$\left\| \frac{\partial p}{\partial y} \right\|_{K_r} = \left\| \frac{\partial p}{\partial y} \right\|_{\partial K_r} \le C_7 \rho_\alpha(n).$$

The last two estimates complete the proof of the Main Theorem for $d = 2$.

3 Proof of the Main Theorem for $d > 2$.

First let us observe that for any $h \in H_n^d$ we have $h(t\mathbf{x}) = t^n h(\mathbf{x})$, $t \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^d$, i.e. $D_{\mathbf{u}} h(\mathbf{0}) = 0$ for any $\mathbf{u} \in S^{d-1}$. Thus, it suffices to estimate $D_{\mathbf{u}} h(\mathbf{x})$ for $\mathbf{x} \in \partial K_r \setminus \{\mathbf{0}\}$. Note that property (12) of the star-like domain K_r yeilds that $\mathrm{Int} K_r \ne \emptyset$. Thus, there exists a d-dimensional ball $B \subset K_r$, $\mathbf{0} \notin B$. Consider the quantity

$$\eta(K) := \inf_{\mathbf{u} \in S^{d-1}} \sup_{\mathbf{w} \in B} | < \mathbf{u}, \mathbf{w} > |.$$

We claim that $\eta(K) > 0$. Indeed, for any $\mathbf{u} \in S^{d-1}$ we clearly have

$$\tau(\mathbf{u}) := \sup_{\mathbf{w}\in B} | < \mathbf{u}, \mathbf{w} > | > 0.$$

Since $\tau(\mathbf{u})$ is continuous on compact set S^{d-1} the claim is obvious. Thus, for every $\mathbf{u} \in S^{d-1}$ there exists $\mathbf{b} \in B$ such that $| < \mathbf{u}, \mathbf{b} > | \geq \eta(K)$.

Let now $h \in H_n^d$ satisfy $\|h\|_{K_r} \leq 1$, and consider an arbitrary $\mathbf{x} \in \partial K_r \setminus \{\mathbf{0}\}$. Denote

$$\mathbf{u} := \frac{\nabla h(\mathbf{x})}{|\nabla h(\mathbf{x})|} \in S^{d-1},$$

where ∇h is the gradient of h. By the above observation we can choose $\mathbf{b} \in B$ so that $| < \mathbf{u}, \mathbf{b} > | \geq \eta(K)$. Set $\mathbf{w} := \mathbf{b}/|\mathbf{b}|$. Then, using that $\nabla h(\mathbf{x}) = |\nabla h(\mathbf{x})|\mathbf{u}$, we have

$$|D_{\mathbf{w}} h(\mathbf{x})| = | < \nabla h(\mathbf{x}), \mathbf{w} > | = \frac{|\nabla h(\mathbf{x})|}{|\mathbf{b}|} | < \mathbf{u}, \mathbf{b} > | \geq \frac{\eta(K)}{|\mathbf{b}|} |\nabla h(\mathbf{x})|.$$

In other words, with some constant $C_{21} > 0$ depending only on K_r

$$|\nabla h(\mathbf{x})| \leq \frac{|\mathbf{b}|}{\eta(K)} |D_{\mathbf{w}} h(\mathbf{x})| \leq C_{21} |D_{\mathbf{w}} h(\mathbf{x})|. \tag{37}$$

Thus, it suffices to estimate $|D_{\mathbf{w}} h(\mathbf{x})|$, where $\mathbf{w} = \mathbf{b}/|\mathbf{b}|$ and $\mathbf{b} \in B$. Consider now the 2-dimensional plane $L_{\mathbf{b}} := \text{span}\{\mathbf{x}, \mathbf{b}\}$. Then $\tilde{K}_r := K_r \cap L_{\mathbf{b}}$ is a 2-dimensional star-like domain with $\mathbf{x} \in \tilde{K}_r$. Moreover, by (36), if we estimate all directional derivatives of $p|_{L_{\mathbf{b}}} \in H_n^2$ at $\mathbf{x}$ then this will yield an upper bound for $|\nabla h(\mathbf{x})|$. Hence, our considerations can be reduced to the 2-dimensional plane $L_{\mathbf{b}}$ and the star-like domain $\tilde{K}_r$. Let $\tilde{r} := r|_{L_{\mathbf{b}} \cap S^{d-1}}$ be the corresponding radial function associated with $\tilde{K}_r$. In order to complete the proof we need to show that $\tilde{r}$ satisfies condition (12) with some constants independent of $\mathbf{x}$. Let

$$l := L_{\mathbf{b}} \cap \left\{(0, x_2, \ldots, x_d) : (x_2, \ldots, x_d) \in \mathbb{R}^{d-1}\right\}$$

be the line in $L_{\mathbf{b}}$ supporting $\tilde{K}_r$ at $\mathbf{0}$, and denote by $\mathbf{y}^{\perp}$ the orthogonal projection of $\mathbf{y} = (y_1, \ldots, y_d) \in L_{\mathbf{b}}$ to the line l. Clearly,

$$|\mathbf{y} - \mathbf{y}^{\perp}| = \text{dist}(\mathbf{y}, l) \geq y_1. \tag{38}$$

Furthermore, set $\tilde{\mathbf{y}} = (0, y_2, \dots, y_d)$, where $\mathbf{y} = (y_1, y_2, \dots, y_d)$. Obviously, $\mathbf{y} - \mathbf{y}^\perp \in L_\mathbf{b}$ is a normal direction to l in $L_\mathbf{b}$, $\mathbf{y} - \tilde{\mathbf{y}} = t\mathbf{e}_1$, where $t \in \mathbb{R}$, $\mathbf{e}_1 = (1, 0, \dots, 0)$. This means that the angle between $\mathbf{y} - \mathbf{y}^\perp$ and $\mathbf{y} - \tilde{\mathbf{y}}$ is invariant of the choice of $\mathbf{y} \in L_\mathbf{b}$, and so is the angle between $\tilde{\mathbf{y}} - \mathbf{y}^\perp$ and $\mathbf{y} - \mathbf{y}^\perp$. Denote by $\alpha(\mathbf{u}, \mathbf{v})$ the angle between $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$. Then by the above observation using that $\mathbf{b} \in L_\mathbf{b}$ we have

$$\sin \alpha(\tilde{\mathbf{y}} - \mathbf{y}^\perp, \mathbf{y} - \mathbf{y}^\perp) = \sin \alpha(\tilde{\mathbf{b}} - \mathbf{b}^\perp, \mathbf{b} - \mathbf{b}^\perp)$$

$$= \frac{|\mathbf{b} - \tilde{\mathbf{b}}|}{|\mathbf{b} - \mathbf{b}^\perp|} \geq \frac{b_1}{|\mathbf{b}|} \geq \frac{\min\limits_{\mathbf{x} \in B} x_1}{\max\limits_{\mathbf{x} \in B} |\mathbf{x}|} = C_{22} > 0.$$

(It was used above that $B \subset K_r \subset \mathbb{R}^d_+$ and $\mathbf{0} \notin B$, i.e. $\min\limits_{\mathbf{x} \in B} x_1 > 0$.)

Thus, for every $y \in L_\mathbf{b}$

$$|\mathbf{y} - \mathbf{y}^\perp| = \frac{|\mathbf{y} - \tilde{\mathbf{y}}|}{\sin \alpha(\tilde{\mathbf{y}} - \mathbf{y}^\perp, \mathbf{y} - \mathbf{y}^\perp)} \leq \frac{y_1}{C_{22}}.$$

This together with (38) implies that $|\mathbf{y} - \mathbf{y}^\perp| \asymp |y_1|$ whenever $\mathbf{y} \in L_\mathbf{b}$ with constants involved being independent of $\mathbf{x}$. Thus, in conditions (8) we can replace $|y_1|$ by $|\mathbf{y} - \mathbf{y}^\perp| = \operatorname{dist}(\mathbf{y}, l)$, where l is the supporting line of $\tilde{K}_r$ at $\mathbf{0}$. Without loss of generality we can assume that l is a coordinate axis (this can be achieved by a rotation), i.e. it remains to refer to the already verified case when $d = 2$. This completes the proof of the Main Theorem.

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