

# The Kolmogorov Inequalities for Multiply Monotone Functions Defined on a Half-line

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## Abstract

We shall present new Kolmogorov type inequalities for intermediate derivatives of multiply monotone functions defined on the negative half-line as well as their applications to the Kolmogorov problem consisting of finding necessary and sufficient conditions for the system of positive numbers  $M_{0,p_0}, M_{k,p_k}, M_{r,p_r}$  which guarantee the existence of an  $(r - 1)$  - monotone function  $x$  such that  $\|x\|_{p_0} = M_{0,p_0}$ ,  $\|x^{(k)}\|_{p_k} = M_{k,p_k}$  and  $\|x^{(r)}\|_{p_r} = M_{r,p_r}$ .

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## Inequalities for $(r - 1)$ -monotone functions

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# 1 Introduction.

Let  $G$  be the real line  $\mathbb{R} = (-\infty, \infty)$  or the negative half-line  $\mathbb{R}_- = (-\infty, 0]$ . Let  $L_p(G)$ ,  $1 \leq p \leq \infty$ , be the space of functions  $x : G \rightarrow \mathbb{R}$ , measurable and integrable in the power  $p$  (essentially bounded when  $p = \infty$ ), with the usual norm

$$(1) \quad \|x\|_p = \|x\|_{L_p(G)} = \begin{cases} \left( \int_G |x(\tau)|^p d\tau \right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty; \\ \text{esssup}\{|x(\tau)| : \tau \in G\}, & \text{if } p = \infty. \end{cases}$$

We shall use sometimes notation  $L_p(G)$  and  $\|x\|_p$  also in the case  $p \in (0, 1)$ , although in this case  $\|x\|_p$  is not a norm.

For  $r \in \mathbb{N}$  and  $s \geq 1$  we shall denote by  $L_s^r(G)$ ,  $r \in \mathbb{N}$ , the space of functions  $x : G \rightarrow \mathbb{R}$  that have derivatives up to and including the order  $r - 1$  (in the case  $G = \mathbb{R}_-$  we take, as usual, the one-sided derivative at the point  $x = 0$ ), such that derivatives  $x^{(r-1)}$  are locally absolutely continuous and  $x^{(r)} \in L_s(G)$ . For  $0 < p \leq \infty$  and  $1 \leq s \leq \infty$  let  $L_{p,s}^r(G) = L_s^r(G) \cap L_p(G)$ .

It is known that inequalities estimating the norms of an intermediate derivative of a function in terms of the norms of the function itself and its highest derivative of the type

$$(2) \quad \|x^{(k)}\|_q \leq K \|x\|_p^\alpha \|x^{(r)}\|_s^{1-\alpha}$$

play an important role in many questions of analysis and its applications. Sharp inequalities or inequalities with best possible constants are especially important and interesting.

The first sharp inequality of type (??) was obtained by Landau [?] for the case  $x \in L_{\infty,\infty}^2(G)$ ,  $k = 1$ . One of the first complete results in this area was obtained by Kolmogorov [?, ?]: for any  $k, r \in \mathbb{N}$ ,  $k < r$ , and any function  $x \in L_{\infty,\infty}^r(\mathbb{R})$

$$(3) \quad \|x^{(k)}\|_\infty \leq \frac{\|\phi_{r-k}\|_\infty}{\|\phi_r\|_\infty^{1-k/r}} \|x\|_\infty^{1-k/r} \|x^{(r)}\|_\infty^{k/r},$$

where  $\phi_r(t)$  is the  $r^{\text{th}}$  periodic integral with zero mean value on a period of the function  $\phi_0(t) = \text{sgn} \sin t$  (such functions are called Euler perfect splines). Note that Shilov [?] proved this inequality in the cases  $r = 3, 4$ ,  $k = 1, \dots, r - 1$ , and  $r = 5, k = 2$ .

The Kolmogorov inequality becomes an equality for any function of the form  $a\phi_r(\lambda t)$ ,  $a, \lambda \in \mathbb{R}$ ,  $\lambda > 0$ . After this result inequalities of type (??) are often called Kolmogorov type inequalities.

In [?] Gabushin gave necessary and sufficient conditions for the existence of inequalities of the type (??) for functions  $x \in L_{p,s}^r(G)$ .

**Theorem 1.** (Gabushin, [?]) *Let  $1 \leq q, p, s \leq \infty$ ;  $\alpha \in (0, 1]$ ;  $k, r \in \mathbb{N}$ ,  $k < r$ . For the function  $x \in L_{p,s}^r(G)$  inequality (??) holds with a constant  $K$  which does not depend on function  $x$  if and only if*

$$\frac{r-k}{p} + \frac{k}{s} \geq \frac{r}{q}.$$

*This condition uniquely defines  $\alpha$  as*

$$\alpha = \frac{r-k-s^{-1}+q^{-1}}{r-s^{-1}+p^{-1}}.$$

Beside the Kolmogorov inequality mentioned above, general (i.e. for all  $k, r \in \mathbb{N}, k < r$ ) sharp inequalities of type (??) for  $G = \mathbb{R}$  are known in the following cases:

- 1)  $p = q = s = 2$  (Hardy, Littlewood, Polya [?]);
- 2)  $p = q = s = 1$  (Stein [?]);
- 3)  $q = \infty, p = s = 2$  (Taikov [?]).

When domain is the half-line  $\mathbb{R}_-$  general results are known in the following cases:

- 1)  $p = q = s = \infty$  (Landau [?], Matorin [?], Schöenberg and Cavaretta [?, ?]);
- 2)  $p = q = s = 2$  (Lubich [?], Kuptsov [?]);
- 3)  $q = \infty, p = s = 2$  (Gabushin [?]).

There is also a number of results for both  $G = \mathbb{R}$  and  $G = \mathbb{R}_-$  in the case of low smoothness of a function. A number of interesting results were obtained in the periodic case  $G = \mathbb{T}$ . For review and bibliography we refer the reader to [?, ?, ?, ?, ?].

Regarding the case  $G = \mathbb{R}_-$  we would like to remark here that the constant in the result of Schöenberg and Cavaretta is not explicit. Earlier, in 1951, Olovyanishnikov [?] considered a more narrow class of functions, namely the class of multiply monotone functions, and obtained Kolmogorov type inequality for this class with a simple and elegant constant. We shall discuss his result in Section 2. In this section we shall also give generalizations of his result to the case of arbitrary integral norms of the intermediate

derivative and a function itself. Similar result will be obtained in the case of  $L_1$ -norm of the highest derivative. The main results of this paper are in Theorem 3 and Theorem 5 of Section 2.

Section 3 is devoted to one of the applications of inequalities of type (??), namely to the problem about existence of a multiply monotone function having prescribed  $L_p$ -norm of a function itself,  $L_q$ -norm of its intermediate derivative and  $L_\infty$ -norm of its highest derivative.

## 2 Generalization of the Olovyanishnikov inequality for norms of the derivative on a half-line.

Given  $r, m \in \mathbb{N}$ ,  $m \leq r$ , and  $0 < p \leq \infty$ ,  $1 \leq q \leq \infty$ , denote by  $L_{p,s}^{r,m}(\mathbb{R}_-)$  the class of functions  $x \in L_{p,s}^r(\mathbb{R}_-)$  that are nonnegative along with all their derivatives up to and including order  $m$  (derivative of order  $m$  must be nonnegative almost everywhere if  $m = r$ ). We shall call these functions  $(m - 1)$ -monotone functions.

Observe that by these definitions every function  $x \in L_{p,s}^{r,m}(\mathbb{R}_-)$  is bounded and

$$\|x\|_\infty = x(0).$$

As we already mentioned, Olovyanishnikov [?] in 1951 considered the class  $L_{\infty,\infty}^{r,r-1}(\mathbb{R}_-)$ , and for this class he obtained an exact inequality of Kolmogorov type with a simple and elegant constant. To introduce his result we need the following definition. For positive parameters  $a$  and  $l$ , set

$$(4) \quad \varphi_r(a, l; t) = \begin{cases} 0, & -\infty < t \leq -a; \\ \frac{l^{(a+t)r}}{r!}, & -a \leq t \leq 0. \end{cases}$$

**Theorem 2.** (Olovyanishnikov, [?]) *For any  $k, r \in \mathbb{N}$ ,  $k < r$ , and for any function  $x \in L_{\infty,\infty}^{r,r-1}(\mathbb{R}_-)$ , the following inequality holds:*

$$(5) \quad \|x^{(k)}\|_\infty \leq C_{kr} \|x\|_\infty^{1-k/r} \|x^{(r)}\|_\infty^{k/r},$$

where

$$(6) \quad C_{kr} = \frac{\|\varphi_r^{(k)}(1, 1; \cdot)\|_\infty}{\|\varphi_r(1, 1; \cdot)\|_\infty^{1-k/r}} = \frac{r!^{1-k/r}}{(r-k)!}.$$

*This inequality becomes an equality for any function of type (??).*

Let  $k, r \in \mathbb{N}$ ,  $p \in (0, \infty]$ ,  $q \in [1, \infty]$ . In this section we shall show that for functions from the class  $L_{p,\infty}^{r,r-2}(\mathbb{R}_-)$ ,  $r \geq 2$ , with any  $k < r$  it is possible to obtain a sharp inequality of the form (??) which estimates  $\|x^{(k)}\|_q$  using  $\|x\|_p$  and  $\|x^{(r)}\|_\infty$  (see Theorem 3). For a function of the class  $L_{p,1}^{r,r-3}(\mathbb{R}_-)$ ,  $r \geq 3$ , with any  $k < r - 1$ , it is possible to obtain a sharp inequality of the form (??) which estimates  $\|x^{(k)}\|_q$  using  $\|x\|_p$  and  $\|x^{(r)}\|_1$  (see Theorem 5).

**Remark.** Observe that our assumptions on the class of functions are weaker than assumptions in Olovyanishnikov theorem.

One of the main results of this section is the following theorem.

**Theorem 3.** *Let  $k, r \in \mathbb{N}$ ,  $k < r$ . For all  $p \in (0, \infty]$ ,  $q \in [1, \infty]$  and for any function  $x \in L_{p,\infty}^{r,r-2}(\mathbb{R}_-)$ ,*

$$(7) \quad \|x^{(k)}\|_q \leq C_{kr} \|x\|_p^\alpha \|x^{(r)}\|_\infty^{1-\alpha},$$

where

$$C_{kr} = \frac{\|\varphi_r^{(k)}(1, 1; \cdot)\|_q}{\|\varphi_r(1, 1; \cdot)\|_p^\alpha} = \frac{(r!)^\alpha (rp + 1)^{\frac{\alpha}{p}}}{(r - k)!((r - k)q + 1)^{\frac{1}{q}}},$$

$$\alpha = \frac{r - k + 1/q}{r + 1/p}.$$

Inequality (??) becomes an equality for any function of type (??).

**Remark.** Observe that inequality (??) for the multiply monotone functions is possible without any restrictions on  $k$ ,  $r$ ,  $q$  and  $p$ . This is not true in general, see Theorem 1 of Gabushin.

In order to prove Theorem 3 we shall prove the following theorem which is an analog of the Kolmogorov comparison theorem for the class  $L_{p,\infty}^{r,r-2}(\mathbb{R}_-)$ .

**Theorem 4.** *Let  $\varphi_r(a, l; t)$ ,  $r \geq 2$ , be as in (??) and let  $x \in L_{\infty,\infty}^{r,r-2}(\mathbb{R}_-)$ . If the values of parameters  $a$  and  $l$  of  $\varphi_r(a, l; t)$  are chosen so that*

$$(8) \quad \|x\|_\infty \leq \|\varphi_r(a, l; \cdot)\|_\infty,$$

and

$$(9) \quad \|x^{(r)}\|_\infty \leq \|\varphi_r^{(r)}(a, l; \cdot)\|_\infty = l,$$

then  $\forall \xi, \eta \in (-\infty, 0]$  such that  $x(\xi) = \varphi_r(a, l; \eta)$ , we have

$$(10) \quad x'(\xi) \leq \varphi_r'(a, l; \eta).$$

**Proof.** For shortness denote by

$$\varphi_r(t) := \varphi_r(a, l; t)$$

for chosen values of parameters.

Note that the statement is obvious if  $\varphi_r(\eta) = 0$ . Hence, assume now that  $\varphi_r(\eta) \neq 0$ .

Assume that contrary to the statement of the theorem there exist  $\xi, \eta \in (-\infty, 0]$  such that  $x(\xi) = \varphi_r(\eta)$ , but

$$(11) \quad x'(\xi) > \varphi_r'(\eta).$$

Let us consider the shifts of the functions  $\varphi_r(t)$  and  $x(t)$  defined by  $\tilde{\varphi}_r(t) := \varphi_r(t - \eta)$  and  $\tilde{x}(t) := x(t - \xi)$ . Observe that clearly

$$(12) \quad \|\tilde{x}\|_\infty = \|\tilde{\varphi}_r\|_\infty,$$

and

$$(13) \quad \|\tilde{x}^{(r)}\|_\infty \leq \|\tilde{\varphi}_r^{(r)}\|_\infty = l.$$

Let us consider the difference  $\delta(t) := \tilde{x}(t) - \tilde{\varphi}_r(t)$ . From the construction of  $\tilde{\varphi}_r(t)$  and  $\tilde{x}(t)$  it follows that  $\delta(0) = 0$ . Due to assumption (??), there exists a point  $t_1 \in (-a - \eta, 0)$  such that  $\delta(t_1) < 0$ . Because of the condition  $\tilde{x} \in L_{p,\infty}^{r,r-2}(\mathbb{R}_-)$  and the fact that  $\tilde{\varphi}_r(t) = 0$  for  $t \leq -a - \eta$ , it follows that  $\delta(-a - \eta) \geq 0$ . From this, by Rolle's theorem, it follows that for the derivative  $\delta'(t)$  there exist points

$$-a - \eta < t_1^1 < t_2^1 < 0$$

such that

$$\delta'(t_1^1) < 0, \quad \delta'(t_2^1) > 0.$$

From the condition  $\tilde{x} \in L_{p,\infty}^{r,r-2}(\mathbb{R}_-)$  and the fact that  $\tilde{\varphi}_r'(a, l; t) = 0$  for  $t \leq -a - \eta$ , it follows that  $\delta'(-a - \eta) \geq 0$ .

Applying Rolle's theorem again we obtain that for the derivative  $\delta''(t)$  there exist points

$$-a - \eta < t_1^2 < t_2^2 < 0$$

such that

$$\delta''(t_1^2) < 0, \quad \delta''(t_2^2) > 0.$$

From the condition  $\tilde{x} \in L_{p,\infty}^{r,r-2}(\mathbb{R}_-)$  and the fact that  $\tilde{\varphi}_r''(a, l; t) = 0$  for  $t \leq -a - \eta$ , it follows that  $\delta''(-a - \eta) \geq 0$ .

Continuing similarly up to and including the derivative of order  $r - 2$ , we obtain that for the derivative  $\delta^{(r-2)}(t)$  there exist points

$$-a - \eta < t_1^{r-2} < t_2^{r-2} < 0$$

such that

$$\delta^{(r-2)}(t_1^{r-2}) < 0, \quad \delta^{(r-2)}(t_2^{r-2}) > 0.$$

As before,  $\delta^{(r-2)}(-a - \eta) \geq 0$ . It implies that there exist points

$$-a - \eta < t_1^{r-1} < t_2^{r-1} < 0$$

such that

$$\delta^{(r-1)}(t_1^{r-1}) < 0, \quad \delta^{(r-1)}(t_2^{r-1}) > 0.$$

Recalling the definition of  $\delta(t)$  observe that it means that

$$\tilde{x}^{(r-1)}(t_1^{r-1}) < \tilde{\varphi}_r^{(r-1)}(t_1^{r-1}) \quad \text{and} \quad \tilde{x}^{(r-1)}(t_2^{r-1}) > \tilde{\varphi}_r^{(r-1)}(t_2^{r-1}).$$

Subtracting first inequality from the second one we obtain that

$$\begin{aligned} & \int_{t_1^{r-1}}^{t_2^{r-1}} \tilde{x}^{(r)}(t) dt = \tilde{x}^{(r-1)}(t_2^{r-1}) - \tilde{x}^{(r-1)}(t_1^{r-1}) \\ (14) \quad & > \tilde{\varphi}_r^{(r-1)}(t_2^{r-1}) - \tilde{\varphi}_r^{(r-1)}(t_1^{r-1}) = \int_{t_1^{r-1}}^{t_2^{r-1}} \tilde{\varphi}_r^{(r)}(t) dt. \end{aligned}$$

Therefore,  $\tilde{x}^{(r)}(t) > \tilde{\varphi}_r^{(r)}(t)$  on some set of positive measure which is a subset of the interval  $(-a - \eta, 0)$  where  $\tilde{x}^{(r)}(t) \leq l$  and  $\tilde{\varphi}_r^{(r)}(t) \equiv l$ . We obtain a contradiction to (??).  $\square$

**Corollary 1.** *Under the assumptions of Theorem 4,*

$$(15) \quad \|x^{(k)}\|_\infty \leq \|\varphi_r^{(k)}(a, l; \cdot)\|_\infty, \quad k = 1, \dots, r - 1.$$

**Proof.** For every  $\xi \in \mathbb{R}_-$  there exists  $\eta \in \mathbb{R}_-$  such that

$$x(\xi) = \varphi_r(a, l; \eta).$$

Hence, for every  $\xi$

$$x'(\xi) \leq \varphi_r'(a, l; \eta) \leq \|\varphi_r'(a, l; \cdot)\|_\infty,$$

and, therefore,

$$\|x'\|_\infty \leq \|\varphi'_r(a, l; \cdot)\|_\infty.$$

Hence, for  $r = 2$  the corollary is already proved. For  $r > 2$  the last inequality means that  $x'(t)$  and  $\varphi'_r(t)$  satisfy conditions of Theorem 4. Hence, applying it we obtain

$$\|x''\|_\infty \leq \|\varphi''_r(a, l; \cdot)\|_\infty.$$

Proceeding by induction, we obtain for all  $k = 1, \dots, r - 1$

$$\|x^{(k)}\|_\infty \leq \|\varphi_r^{(k)}(a, l; \cdot)\|_\infty.$$

□

**Corollary 2.** *Under the assumptions of Theorem 4,*

$$(16) \quad \nu(x^{(k)} - \varphi_r^{(k)}) \leq 1, \quad k = 1, \dots, r - 1,$$

where  $\nu(x^{(k)} - \varphi_r^{(k)})$  is the number of sign changes of the difference  $x^{(k)} - \varphi_r^{(k)}$ .

**Proof.** Assume to the contrary that for some  $k$

$$\nu(x^{(k)} - \varphi_r^{(k)}) \geq 2.$$

Then there exist points  $t_1, t_2 \in (-a, 0)$ ,  $t_1 < t_2$ , such that  $x^{(k)}(t_1) < \varphi_r^{(k)}(t_1)$  and  $x^{(k)}(t_2) > \varphi_r^{(k)}(t_2)$ . Using Lemma 5.6.1 from [?] we obtain that there exist points  $\xi, \eta \in (t_1, t_2)$  such that  $x^{(k)}(\xi) = \varphi_r^{(k)}(\eta)$  and  $x^{(k+1)}(\xi) > \varphi_r^{(k+1)}(\eta)$ . It leads to a contradiction because functions  $x^{(k)}$  and  $\varphi_r^{(k)}$  satisfy assumptions of Theorem 4. □

**Corollary 3.** *If in the assumptions of Theorem 4 condition (??) holds with an equality sign, i.e.  $x(0) = \varphi_r(a, l; 0)$ , then*

$$(17) \quad x(t) \geq \varphi_r(a, l; t), t \in \mathbb{R}_-,$$

and, consequently, for all  $p > 0$

$$(18) \quad \|x\|_p \geq \|\varphi_r(a, l; \cdot)\|_p.$$

**Proof.** For the proof see Lemma 1 in [?]. □

We shall need the following lemma, variations of which have often been used to prove integral inequalities for rearrangements (see, for example, [?], proofs of Theorems 6.7.2 and 6.8.4).

**Lemma 1.** *Let  $f$  and  $g$  be non-decreasing, nonnegative functions from  $L_1(\mathbb{R}_-)$ .*

*If*

- 1)  $\|g\|_\infty \leq \|f\|_\infty$ ,
- 2)  $\|g\|_1 \leq \|f\|_1$ ,
- 3)  $\nu(f - g) \leq 1$ , ( $\nu(f - g)$  is the number of sign changes of  $f - g$ ),

*then*

$$\int_t^0 g(u)du \leq \int_t^0 f(u)du, \quad \forall t \in \mathbb{R}_-.$$

We shall also need the following statement which is a special case of the famous theorem of Hardy-Littlewood-Polya (see, for example, [?], lemma 1.3.11).

**Lemma 2.** *Let  $f$  and  $g$  be nonnegative, non-decreasing functions from  $L_1(\mathbb{R}_-)$ .*

*If for all  $t \in \mathbb{R}_-$*

$$\int_t^0 g(u)du \leq \int_t^0 f(u)du,$$

*then for every  $q \geq 1$*

$$\|g\|_q \leq \|f\|_q.$$

**Proof of Theorem 3.** Let  $x \in L_{p,\infty}^{r,r-2}(\mathbb{R}_-)$ . Assume first that

$$(19) \quad \|x^{(r)}\|_\infty = 1$$

and choose parameter  $l$  of the function  $\varphi_r(a, l; t)$  to be equal to 1, i.e.  $l = 1$ . Then we choose parameter  $a$  in such a way that

$$(20) \quad \|x\|_\infty = \|\varphi_r(a, 1; \cdot)\|_\infty.$$

Observe that it is always possible because

$$\|\varphi_r(a, 1; \cdot)\|_\infty = \varphi_r(a, 1; 0) = \frac{a^r}{r!}$$

and, hence, we can simply take

$$a = (r!\|x\|_\infty)^{1/r}.$$

From Corollary 3 of Theorem 4 and (??) it follows that for all  $p > 0$  (for  $p \geq 1$  norm was defined in (??), for  $p < 1$  norm is defined in the same way)

$$(21) \quad \|x\|_p \geq \|\varphi_r(a, 1; \cdot)\|_p.$$

We now show that for all  $k = 1, \dots, r - 1$

$$(22) \quad \|x^{(k)}\|_1 \leq \|\varphi_r^{(k)}(a, 1; \cdot)\|_1.$$

By elementary calculations,

$$\begin{aligned} \|x^{(k)}\|_1 &= \int_{-\infty}^0 x^{(k)}(t) dt = \lim_{b \rightarrow +\infty} \int_{-b}^0 x^{(k)}(t) dt \\ &= \lim_{b \rightarrow +\infty} (x^{(k-1)}(0) - x^{(k-1)}(-b)) \leq x^{(k-1)}(0) = \|x^{(k-1)}\|_\infty \\ &\leq \|\varphi_r^{(k-1)}(a, 1; \cdot)\|_\infty = \varphi_r^{(k-1)}(a, 1; 0) = \int_{-\infty}^0 \varphi_r^{(k)}(a, 1; t) dt \\ &= \|\varphi_r^{(k)}(a, 1; \cdot)\|_1. \end{aligned}$$

This, together with Corollaries 1 and 2, means that all conditions of Lemma 1 are satisfied for functions  $x^{(k)}(t)$  and  $\varphi_r^{(k)}(a, 1; t)$ . Applying this lemma, we obtain that for all  $t \in (-\infty, 0)$

$$\int_t^0 x^{(k)}(u) du \leq \int_t^0 \varphi_r^{(k)}(a, 1; u) du.$$

Therefore, by Lemma 2, it follows that for all  $q \geq 1$

$$\|x^{(k)}\|_q \leq \|\varphi_r^{(k)}(a, 1; \cdot)\|_q.$$

Thus, in view of (??),

$$\|x^{(k)}\|_q \leq \|\varphi_r^{(k)}(a, 1; \cdot)\|_q \frac{\|\varphi_r(a, 1; \cdot)\|_p^\alpha}{\|\varphi_r(a, 1; \cdot)\|_p^\alpha} \leq \frac{\|\varphi_r^{(k)}(a, 1; \cdot)\|_q}{\|\varphi_r(a, 1; \cdot)\|_p^\alpha} \|x\|_p^\alpha,$$

where

$$\alpha = \frac{r - k + \frac{1}{q}}{r + \frac{1}{p}}.$$

Changing variables and using the expression for  $\alpha$  we obtain the following estimate

$$\|x^{(k)}\|_q \leq \frac{\|\varphi_r^{(k)}(1, 1; \cdot)\|_q a^{r-k+\frac{1}{q}}}{\|\varphi_r(1, 1; \cdot)\|_p^\alpha a^{\alpha(r+\frac{1}{p})}} \|x\|_p^\alpha = \frac{\|\varphi_r^{(k)}(1, 1; \cdot)\|_q}{\|\varphi_r(1, 1; \cdot)\|_p^\alpha} \|x\|_p^\alpha.$$

Thus, for all  $x$  such that  $\|x^{(r)}\|_\infty = 1$ , and for all  $q \geq 1$  and  $p > 0$

$$(23) \quad \|x^{(k)}\|_q \leq \frac{\left\| \varphi_r^{(k)}(1, 1; \cdot) \right\|_q}{\left\| \varphi_r(1, 1; \cdot) \right\|_p^\alpha} \|x\|_p^\alpha.$$

Let  $\|x^{(r)}\|_\infty \neq 1$ . Applying (??) to the function  $x/\|x^{(r)}\|_\infty$ , we obtain

$$(24) \quad \frac{\|x^{(k)}\|_q}{\|x^{(r)}\|_\infty} \leq \frac{\left\| \varphi_r^{(k)}(1, 1; \cdot) \right\|_q}{\left\| \varphi_r(1, 1; \cdot) \right\|_p^\alpha} \frac{\|x\|_p^\alpha}{\|x^{(r)}\|_\infty^\alpha}$$

which is equivalent to (??). Inequality (??) is sharp, it clearly becomes an equality for any function of type (??). This concludes the proof of Theorem 3.  $\square$

**Theorem 5.** For all  $0 < p \leq \infty$ ,  $1 \leq q \leq \infty$ ,  $k, r \in \mathbb{N}$ ,  $k \leq r - 2$ , and for any function  $x \in L_{p,1}^{r,r-3}(\mathbb{R}_-)$ , the following sharp inequality holds:

$$(25) \quad \|x^{(k)}\|_q \leq C_{k,r-1} \|x\|_p^\alpha \|x^{(r)}\|_1^{1-\alpha},$$

where

$$C_{k,r-1} = \frac{\left\| \varphi_{r-1}^{(k)}(1, 1; \cdot) \right\|_q}{\left\| \varphi_{r-1}(1, 1; \cdot) \right\|_p^\alpha} = \frac{(r-1)!^\alpha ((r-1)rp+1)^{\frac{\alpha}{p}}}{(r-k-1)!((r-k-1)q+1)^{\frac{1}{q}}},$$

and

$$\alpha = \frac{r-1-k+1/q}{r-1+1/p}.$$

**Proof.** Let  $x \in L_{p,1}^{r,r-3}(\mathbb{R}_-)$ . Since  $x^{(r)} \in L_1(\mathbb{R}_-)$  then

$$\lim_{t \rightarrow -\infty} x^{(r-1)}(t) = 0,$$

and, hence,

$$\|x^{(r-1)}\|_\infty \leq \bigvee_{-\infty}^0 x^{(r-1)} = \|x^{(r)}\|_1.$$

Therefore, if  $x \in L_{p,1}^{r,r-3}(\mathbb{R}_-)$  then  $x \in L_{p,\infty}^{r-1,r-3}(\mathbb{R}_-)$ . Hence, taking into account (??)

$$\begin{aligned} \|x^{(k)}\|_q &\leq \frac{\|\varphi_{r-1}^{(k)}(1, 1; \cdot)\|_q}{\|\varphi_{r-1}(1, 1; \cdot)\|_p^\alpha} \|x\|_p^\alpha \|x^{(r-1)}\|_\infty^{1-\alpha} \\ &\leq \frac{\|\varphi_{r-1}^{(k)}(1, 1; \cdot)\|_q}{\|\varphi_{r-1}(1, 1; \cdot)\|_p^\alpha} \|x\|_p^\alpha \|x^{(r)}\|_1^{1-\alpha}, \end{aligned}$$

and, hence, inequality (??) is proved. Let us show that the constant in this inequality can not be improved. Indeed, let us consider the following family of functions, depending on parameter  $h > 0$ :

$$\Phi_r(h; t) := \frac{1}{h} \int_{t-h}^t \varphi_{r-1}(1, 1; s) ds.$$

Note that for  $h \rightarrow 0$

$$\begin{aligned} \|\Phi_r^{(r)}(h; \cdot)\|_1 &\rightarrow 1, \\ \|\Phi_r(h; \cdot)\|_p &\rightarrow \|\varphi_{r-1}(1, 1; \cdot)\|_p, \end{aligned}$$

and

$$\|\Phi_r^{(k)}(h; \cdot)\|_q \rightarrow \left\| \varphi_{r-1}^{(k)}(1, 1; \cdot) \right\|_q.$$

Therefore,

$$\lim_{h \rightarrow 0} \frac{\|\Phi_r^{(k)}(h; \cdot)\|_q}{\|\Phi_r(h; \cdot)\|_p^\alpha \|\Phi_r^{(r)}(h; \cdot)\|_1^{1-\alpha}} = \frac{\|\varphi_{r-1}^{(k)}(1, 1; \cdot)\|_q}{\|\varphi_{r-1}(1, 1; \cdot)\|_p^\alpha}.$$

Hence, inequality (??) is sharp.  $\square$

**Remark.** Observe that inequality (??) for the multiply monotone functions is possible without any restrictions on  $k$ ,  $r$ ,  $q$  and  $p$ . This is not true in general, see Theorem 1 of Gabushin.

### 3 The Kolmogorov problem for three numbers.

One of the applications of Kolmogorov type inequalities is the problem about existence of a function with prescribed norms of function itself and its

derivatives. This problem was posed by Kolmogorov in 1926 (see [?]). The problem can be stated as follows.

Let arbitrary systems of integers

$$k_0 = 0 < k_1 < \dots < k_d = r$$

and reals

$$1 \leq p_0, p_1, \dots, p_d \leq \infty$$

be given. The problem is to find necessary and sufficient conditions for the system of positive numbers

$$M_{k_0, p_0}, M_{k_1, p_1}, \dots, M_{k_d, p_d}$$

to guarantee the existence of a function  $x \in X = L_{p_0, p_d}^r(G)$  such that

$$\|x^{(k_i)}\|_{p_i} = M_{k_i, p_i}, \quad i = 0, \dots, d.$$

The complete solution of the problem for three numbers and functions from the class  $L_{\infty, \infty}^r(\mathbb{R})$  (in the case of all sup-norms) was given by Kolmogorov in 1938 [?, ?]. He showed that for any three positive numbers  $M_{0, \infty}, M_{k, \infty}, M_{r, \infty}$ ,  $0 < k < r$ , there exists a function  $x \in L_{\infty, \infty}^r(\mathbb{R})$  which has these numbers as values of the norm of the function itself, its  $k^{\text{th}}$  and its  $r^{\text{th}}$  derivatives, respectively, if and only if

$$M_{k, \infty} \leq C_{kr} M_{0, \infty}^{1-k/r} M_{r, \infty}^{k/r},$$

where

$$C_{kr} = \frac{\|\phi_{r-k}\|_{\infty}}{\|\phi_r\|_{\infty}^{1-k/r}},$$

and  $\phi_r(t)$  is the perfect Euler spline defined above.

Solution of the Kolmogorov problem for three numbers in the case when the function is defined on the half-line and in the case of all sup-norms follows from the results of Landau [?], Matorin [?], Schönberg and Cavaretta [?, ?]. However, as we already mentioned, the result of Schönberg and Cavaretta is not explicit.

As we already mentioned, in 1951 Olovyanishnikov considered the class of multiply monotone functions and for this class he obtained simple solution

of the Kolmogorov problem in the case of three numbers. He proved the following theorem.

**Theorem 6.** (Olovyanishnikov, [?]) *For any three positive numbers  $M_{0,\infty}$ ,  $M_{k,\infty}$ ,  $M_{r,\infty}$ , where  $k, r \in \mathbb{N}$ ,  $k < r$ , there exists a function  $x \in L_{\infty,\infty}^{r,r-1}(\mathbb{R}_-)$  such that*

$$\|x\|_{\infty} = M_{0,\infty}, \quad \|x^{(k)}\|_{\infty} = M_{k,\infty}, \quad \|x^{(r)}\|_{\infty} = M_{r,\infty},$$

if and only if the following inequality holds

$$(26) \quad M_{k,\infty} \leq C_{kr} M_{0,\infty}^{1-k/r} M_{r,\infty}^{k/r},$$

where

$$C_{kr} = \frac{\left\| \varphi_r^{(k)}(1, 1; \cdot) \right\|_{\infty}}{\left\| \varphi_r(1, 1; \cdot) \right\|_{\infty}^{1-k/r}} = \frac{r!^{1-k/r}}{(r-k)!},$$

and  $\varphi_r(t)$  is as in (??).

In this section we shall present the solution of Kolmogorov problem in the case  $X = L_{\infty,\infty}^{r,r-2}(\mathbb{R}_-)$ ;  $k_0 = 0 < k < r$  and  $p_0 = p$ ,  $p_1 = q$ ,  $p_2 = \infty$ .

Let  $a > b \geq 0$ ,  $l > 0$ . Define a family of functions  $\varphi_r(a, b, l; t)$  by

$$(27) \quad \varphi_r(a, b, l; t) := \varphi_r(a, l; t) - \varphi_r(b, l; t).$$

**Theorem 7.** *For any three positive numbers  $M_{0,p}$ ,  $M_{k,q}$ ,  $M_{r,\infty}$  with arbitrary  $q, p \in [1, \infty)$  and  $k, r \in \mathbb{N}$ ,  $k < r$ , there is a function  $x \in L_{\infty,\infty}^{r,r-2}(\mathbb{R}_-)$  such that*

$$\|x\|_p = M_{0,p}, \quad \|x^{(k)}\|_q = M_{k,q}, \quad \|x^{(r)}\|_{\infty} = M_{r,\infty}$$

if and only if

$$(28) \quad M_{k,q} \leq C_{kr} M_{0,p}^{\alpha} M_{r,\infty}^{1-\alpha},$$

where

$$\alpha = \frac{r-k+1/q}{r}, \quad C_{kr} = \frac{r!}{(r-k)!((r-k)q+1)^{1/q}}.$$

**Proof.** First observe that the necessity of condition (??) follows from Theorem 3.

To show the sufficiency of condition (??) we shall find a function  $x \in L_{\infty,\infty}^{r,r-2}(\mathbb{R}_-)$  that has given three numbers  $M_{0,p}$ ,  $M_{k,q}$ ,  $M_{r,\infty}$  as values of the norms of function itself and its corresponding derivatives.

Let  $M_{0,p}$ ,  $M_{k,q}$ ,  $M_{r,\infty}$  be positive numbers satisfying condition (??). Let us show that there exist parameters  $a, b$  and  $l$  of the function  $\varphi_r(a, b, l; t)$  defined in (??) such that

$$\begin{aligned}\|\varphi_r(a, b, l; \cdot)\|_p &= M_{0,p}, \\ \|\varphi_r^{(k)}(a, b, l; \cdot)\|_q &= M_{k,q}, \\ \|\varphi_r^{(r)}(a, b, l; \cdot)\|_\infty &= M_{r,\infty}.\end{aligned}$$

Without loss of generality we may assume that

$$\|\varphi_r^{(r)}(a, b, l; \cdot)\|_\infty = l = 1 = M_{r,\infty}.$$

Thus, we need to prove the existence of a solution  $a > b \geq 0$  of the following system

$$(29) \quad \begin{cases} \|\varphi_r(a, b, 1; \cdot)\|_p = M_{0,p}, \\ \|\varphi_r^{(k)}(a, b, 1; \cdot)\|_q = M_{k,q}. \end{cases}$$

We can rewrite system (??) as follows

$$(30) \quad \begin{cases} \frac{1}{r!} \left\{ \int_0^{a-b} t^{rp} dt + \int_{a-b}^a [t^r - (t - (a - b))^r]^p dt \right\}^{1/p} = M_{0,p}, \\ \frac{1}{(r-k)!} \left\{ \int_0^{a-b} t^{(r-k)q} dt + \int_{a-b}^a [t^{r-k} - (t - (a - b))^{r-k}]^q dt \right\}^{1/q} = M_{k,q}. \end{cases}$$

Setting  $c = a - b$  (it is clear that  $0 < c \leq a$ ) we can rewrite system (??) as follows

$$(31) \quad \begin{cases} \frac{1}{r!} \left\{ \int_0^a [t^r - (t - c)_+^r]^p dt \right\}^{1/p} = M_{0,p}, \\ \frac{1}{(r-k)!} \left\{ \int_0^a [t^{r-k} - (t - c)_+^{r-k}]^q dt \right\}^{1/q} = M_{k,q}. \end{cases}$$

Let us consider the relation

$$(32) \quad R(a, c) = \frac{\frac{1}{(r-k)!} \left\{ \int_0^a [t^{r-k} - (t - c)_+^{r-k}]^q dt \right\}^{1/q}}{\left(\frac{1}{r!}\right)^\alpha \left\{ \int_0^a [t^r - (t - c)_+^r]^p dt \right\}^{\alpha/p}},$$

where

$$\alpha = \frac{r - k + 1/q}{r + 1/p}.$$

Note that  $(t - c)_+^r = c^r (\frac{t}{c} - 1)_+^r$  for  $c > 0$ . Taking into account this fact as well as the fact that  $r - k + 1/q = \alpha(r + 1/p)$  we can rewrite the last expression as follows

$$\begin{aligned} R(a, c) &= \frac{(r!)^\alpha}{(r - k)!} \frac{c^{r-k} \left\{ \int_0^a \left[ \left(\frac{t}{c}\right)^{r-k} - \left(\frac{t}{c} - 1\right)_+^{r-k} \right]^q dt \right\}^{1/q}}{c^{\alpha r} \left\{ \int_0^a \left[ \left(\frac{t}{c}\right)^r - \left(\frac{t}{c} - 1\right)_+^r \right]^p dt \right\}^{\alpha/p}} \\ &= \frac{(r!)^\alpha}{(r - k)!} \frac{c^{r-k+1/q} \left\{ \int_0^{a/c} \left[ u^{r-k} - (u - 1)_+^{r-k} \right]^q du \right\}^{1/q}}{c^{\alpha(r+1/p)} \left\{ \int_0^{a/c} \left[ u^r - (u - 1)_+^r \right]^p du \right\}^{\alpha/p}} \\ &= \frac{(r!)^\alpha}{(r - k)!} \frac{\left\{ \int_0^{a/c} \left[ u^{r-k} - (u - 1)_+^{r-k} \right]^q du \right\}^{1/q}}{\left\{ \int_0^{a/c} \left[ u^r - (u - 1)_+^r \right]^p du \right\}^{\alpha/p}}. \end{aligned}$$

Hence,  $R(a, c)$  is a function of  $a/c$ , i.e. setting

$$F(z) = \frac{(r!)^\alpha}{(r - k)!} \frac{\left\{ \int_0^z \left[ u^{r-k} - (u - 1)_+^{r-k} \right]^q du \right\}^{1/q}}{\left\{ \int_0^z \left[ u^r - (u - 1)_+^r \right]^p du \right\}^{\alpha/p}}, \quad z \in [1, \infty),$$

we obtain

$$R(a, c) = F\left(\frac{a}{c}\right).$$

Note that there exist constants  $C_1$ ,  $C_2$  and  $C_3$  such that for big enough  $z$

$$\left\{ \int_0^z \left[ u^{r-k} - (u - 1)_+^{r-k} \right]^q du \right\}^{1/q} \leq C_1 \left\{ \int_1^z \left[ u^{r-k} - (u - 1)_+^{r-k} \right]^q du \right\}^{1/q}$$

$$\leq C_2 \left\{ \int_1^z u^{(r-k-1)q} du \right\}^{1/q} \leq C_3 z^{r-k-1+1/q},$$

and there exist constants  $C_4$ ,  $C_5$  and  $C_6$  such that for big enough  $z$

$$\begin{aligned} \left\{ \int_0^z [u^r - (u-1)_+]^p du \right\}^{1/p} &\geq C_4 \left\{ \int_1^z [u^r - (u-1)_+]^p du \right\}^{1/p} \\ &\geq C_5 \left\{ \int_1^z u^{(r-1)p} du \right\}^{1/p} \geq C_6 z^{r-1+1/q}. \end{aligned}$$

From here it follows that

$$(33) \quad F(z) \leq \frac{(r!)^\alpha C_3 z^{r-k-1+1/q}}{(r-k)! C_6 z^{\alpha(r-1+1/q)}}.$$

It is easy to show that the exponent of the denominator in (??) is greater than the exponent of the numerator. Indeed, recall that  $\alpha = \frac{r-k+1/q}{r+1/p}$ . Hence, one can easily check that  $(r-k-1+1/q)(r+1/p) < (r-1+1/p)(r-k+1/q)$ . Therefore,

$$\lim_{z \rightarrow \infty} F(z) = 0.$$

On the other hand,

$$F(1) = \frac{(r!)^\alpha \left\{ \int_0^1 u^{(r-k)q} du \right\}^{1/q}}{(r-k)! \left\{ \int_0^1 u^{rp} du \right\}^{\alpha/p}} = \frac{(r!)^\alpha \left( \frac{1}{(r-k)q+1} \right)^{1/q}}{(r-k)! \left( \frac{1}{rp+1} \right)^{\alpha/q}} = C_{kr}.$$

Since  $F$  is a continuous function of  $z \in [1, \infty)$ , for any number  $A$ ,  $0 < A \leq C_{kr}$ , there exists  $z$  such that  $F(z) = A$ .

By assumptions of the theorem

$$0 < \frac{M_{k,q}}{M_{0,p}^\alpha} \leq C_{kr}.$$

Therefore, there exists  $z_0 \geq 1$  such that

$$F(z_0) = \frac{M_{k,q}}{M_{0,p}^\alpha}.$$

Substituting  $a = cz_0$  in the first equation of the system (??) we have

$$\left\{ \int_0^{cz_0} [t^r - (t - c)_+^r]^p dt \right\}^{1/p} = r!M_{0,p}$$

or

$$c^{r+1/p} \left\{ \int_0^{z_0} [u^r - (u - 1)_+^r]^p du \right\}^{1/p} = r!M_{0,p}.$$

Set

$$c_0 = (r!M_{0,p})^{\frac{1}{r+1/p}} \left\{ \int_0^{z_0} [u^r - (u - 1)_+^r]^p du \right\}^{-\frac{1}{r+1/p}},$$

$$a_0 = c_0 z_0, \quad b_0 = c_0 z_0 - c_0.$$

Clearly we obtained the solution of the system (??). This concludes the proof of the theorem.  $\square$

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