Research Statement

My current research interests are in computational commutative algebra and algebraic geometry, algebraic combinatorics and dimension theory of partially ordered sets (posets). This statement contains an outline of my results on particular problems in these areas and highlights of related future research projects.

The impetus for my research in these areas stems not only from my attraction to these topics, but also from the accessibility of the topics by undergraduate and graduate students. One of my personal missions is to increase the number of students obtaining advanced degrees in mathematics. In my experience, these areas have served that purpose with great success.

For example, thus far, I had the fortune to work intimately with nine (9) undergraduate students on research questions in the areas of computational algebra and algebraic geometry. I am directing two master's students in their research for their theses at Sam Houston State University. All of these undergraduate students have enrolled in graduate programs around the country. Three have finished with master's degrees in mathematics. The remaining six are in the research phase of their respective doctoral programs.

In my teaching statement, I include a discussion of my undergraduate research program in the making called PURE Mathematics (Pacific Undergraduate Research Experience in Mathematics). It is a very large part of my long-term research goals. Participants in this summer program will come from universities around the country, including the territories of Puerto Rico and Guam. They will work closely with researchers and graduate students in the mathematical sciences on the accessible parts of their current research problems. The goal of this program is to expose the students to very interesting and new topics in the mathematical sciences, life as a researcher and a graduate student and the experiences that will empower them to pursue a career in graduate school.

1 Dimension Theory of Partially Ordered Sets

In the part of my doctoral dissertation on dimension theory [Ga], I develop an operation called coadunation defined over certain classes of posets. The result of the coadunation of two posets $\mathbb{P}'$ and $\mathbb{P}$ is a larger poset denoted $\mathbb{P}' \rtimes \mathbb{P}$. The objective for this construction is to determine the order dimension of $\mathbb{P}' \rtimes \mathbb{P}$ from the order dimensions of its coadunants $\mathbb{P}'$ and $\mathbb{P}$. It is well-known that computing order dimension of a poset in general is very difficult. However, for a small class of posets known as generalized crowns, Trotter shows that the order dimension is simply a formula based on its defining parameters.

For the generalized crowns $\mathbb{S}_n^k$, I proved that the coadunation $\mathbb{S}_n^k \rtimes \mathbb{S}_n^k$ has order dimension $\dim(\mathbb{S}_n^k) = \lceil \frac{2(n+k)}{k+2} \rceil$, whenever $n \geq k+3$. When $3 \leq n < k+3$, I proved that $\mathbb{S} = \mathbb{S}_n^k \rtimes \mathbb{S}_n^k$ has order dimension $\dim(\mathbb{S}_{k-n+2}^{2n-2}) = \lceil \frac{2(n+k)}{k-n+4} \rceil$. The most useful observation in tackling this second case is in seeing that the distinguished subposet generated by $\max(\mathbb{S}) \cup \min(\mathbb{S})$ is itself a crown. This provided the skeletal framework for constructing minimal realizers for $\mathbb{S}$ out of the minimal realizers for its coadunants $\mathbb{S}_n^k$ [GS]. The techniques in this proof have paved the way for the following future considerations.

Currently, my master's student Darrel Silva and I are working on finding the order dimension for posets of the form $\mathbb{P} = \mathbb{S}_n^k \rtimes \mathbb{S}_m^j$. We are in the stages of determining the minimal realizers of the distinguished subposet $\mathcal{E}(\mathbb{P})$ generated by the maximal elements of the $\mathbb{S}_m^j$ and the minimal elements of $\mathbb{S}_n^k$.

A biposet $\mathbb{B}$ is a poset of height two such that the ground set $X = \max \mathbb{B} \cup \min \mathbb{B}$. It is homogeneous if the downset of every maximal element has cardinality $n < \infty$ and the upset of every minimal element has the cardinality $m < \infty$. I am interested in applying Trotter's method to capturing the order dimension of homogeneous biposets.

2 Computational Commutative Algebra

2.1 Gröbner Basis Technique for Padé Approximations

My first taste of research in computational commutative algebra was in directing three undergraduate students in a summer project spearheaded by John Little of The College of the Holy Cross. Together we extended the work of Fitzpatrick and Flynn in [FF] on a technique using Gröbner bases for computing Padé approximations [LP]. Like rational approximations to irrational numbers, a Padé approximation is a rational function in one or more variables that approximates a given function h in the same variables. These approximations are used in both numerical and symbolic computation and have a number of applications in numerical analysis, coding theory and multidimensional signal processing.

A restatement of the Padé approximation problem in terms of polynomials follows. Let $R = k[x_1, \ldots, x_n]$. Let the polynomial $h \in R$ represent the initial segment of the Taylor series of the function f to be approximated. Let $I \subset R$ be the ideal encoding the desired agreement conditions between an approximating rational function $a(x_1, \ldots, x_n)/b(x_1, \ldots, x_n)$ and the polynomial h at $x = 0$. Consider the module M of solutions $(a, b) \in R^2$ of the congruence

$$a \equiv bh \mod I. \tag{1}$$

Each pair (a, b) with $b(0) \neq 0$ yields a rational function a/b that approximates h modulo I.

The main idea developed in [FF] involves a guarantee that a particular solution of the Padé approximation problem appears in any Gröbner basis for M endowed with a special term ordering called the *weak term order*. However, in many instances, such results do not apply as the weak term order condition is highly restrictive. Our results loosen these restrictions by considering a total degree τ condition of a specific form:

$$\tau(a) < \tau(b) \leq m. \tag{2}$$

We show that if a solution of the form (2) exists, then an element of that form must still appear in a suitable Gröbner basis for M. In addition to extending the range of applications, these results further the theoretical understanding of the Padé approximation problem.

2.2 Catalecticant Ideals

The commutative algebra topic of my dissertation [Ga] is the study of the structure of catalecticant ideals. Catalecticant matrices arose as tools for the pursuit of a classical problem with a long history:

that of representing a homogeneous polynomial as a sum of powers of linear forms. Throughout what follows, $R = k[x_1, \ldots, x_n]$ is the polynomial ring in n variables over an algebraically closed field k. As a module over k, it is well-known that $R = \oplus_i R_i$, where each component R_i is generated by the monomials of degree i.

We begin by imposing an order on the monomials in R by using the graded lexicographic order. Then, the set $\{m_1, m_2, \ldots, m_p\}$ of monomials in R of degree d is an ordered basis for the finite dimensional vector space R_d with dimension $p = \binom{n+d-1}{n-1}$. Define

$$F_d = z_1 \cdot m_1 + z_2 \cdot m_2 + \cdots + z_p \cdot m_p,$$

where the z_j are considered variables over the ring R. By a generic catalecticant matrix, we mean a matrix $\text{Cat}(s, d; n) = (c_{ij})$, where $c_{ij} = z_k$, the coefficient in F_d of the term obtained as the product of the i^{th} monomial in the ordered basis of R_s and the j^{th} monomial in the ordered basis of R_{d-s}. The catalecticant ideal is the ideal generated by the maximal minors of $\text{Cat}(s, d; n)$. For example,

$$\text{Cat}(1,3;3) = \begin{bmatrix} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \\ z_2 & z_4 & z_5 & z_7 & z_8 & z_9 \\ z_3 & z_5 & z_6 & z_8 & z_9 & z_{10} \end{bmatrix}.$$

In [Ge], Geramita presents the current state of catalecticant ideals. The well-studied determinantal ideals of Hankel matrices are a special case of catalecticant matrices, namely, $\text{Cat}(s, d; 2)$. These ideals are known to be prime. The conditions under which catalecticant ideals are prime or radical are unknown in general. There is also great interest in understanding their associated Hilbert functions. Thus the computation of the primary decomposition of a catalecticant ideal would prove extremely useful in attacking these questions. In [C], Conca develops techniques in commutative algebra to yield the symbolic powers and primary decomposition of determinantal ideals of Hankel matrices.

I extended Conca's work to include a large family of generic catalecticant matrices. In [Ga] I developed a standard monomial theory for generic catalecticant matrices. I show that the ideal of minors of concatenations and stackings of Hankel matrices stabilizes and that this fact does not depend on the "shape" of the matrices. I also showed that these concatenations and stackings of Hankel matrices are in fact a generalization of catalecticant matrices. Currently, I am working on extending Gröbner basis theory to the setting where the maximal minors of concatenations and stackings of Hankel matrices are the "new" monomials. This is expected to provide access to the primary decomposition and symbolic powers of the ideals of their maximal minors.

2.3 Algebraic Combinatorics

Algebraic combinatorics involves using techniques from geometry, algebra and topology and applies them to problems in combinatorics. I am interested in the problem of enumerating and constructing magic circles. Like magic squares, magic circles have a rich history, with origins in twelfth-century Chinese manuscripts. However, the definition of magic circles is far from standardized. I am interested in a particular type of magic n-circle. These are defined by placing $2n^2$ non-negative integers in a circular grid formed by n concentric circles cut by n diameters in such a way that the sum of the numbers going around any circle equals the sum of the numbers across any diameter,

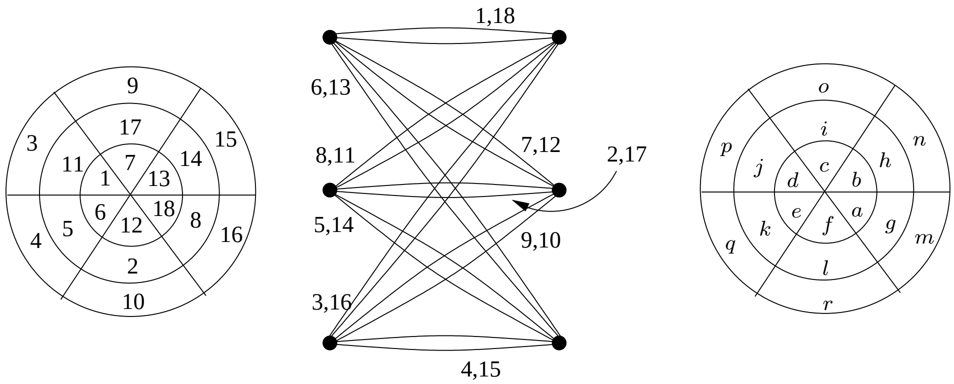

Figure 1: Natural Magic 3-circle, Corresponding Graph and a generic labeling of a 3-circle

as in Figure 1. The relations that define a magic n-circle give rise to a certain polynomial ideal. For example, using the generic labeling in Figure 1, the magic 3-circle would have the following relations:

$$
\begin{aligned}
a+b+c+d+e+f &= g+h+i+j+k+l \\
a+b+c+d+e+f &= m+n+o+p+q+r \\
a+b+c+d+e+f &= p+j+d+a+g+m \\
a+b+c+d+e+f &= q+k+e+b+h+n \\
a+b+c+d+e+f &= r+l+f+c+i+o
\end{aligned}
$$

These relations give rise to a matrix with entries in $\{0, 1, -1\}$, which in turn produces a map between polynomial rings. The Gröbner basis for the kernel of this map will produce among its elements the minimal Hilbert basis, which are the building blocks for any magic 3-circle.

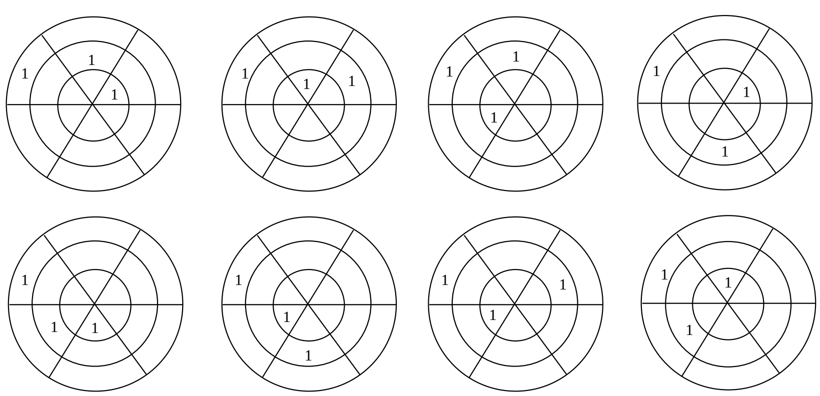

Figure 2: The minimal Hilbert basis for the Magic 3 circle

To obtain a magic n-circle with magic sum S, one need only sum S Hilbert basis elements. My master's student Mark Lane and I developed a way of producing the minimal Hilbert basis of the magic n-circle for any n without the need for computation. Figure 2 gives eight elements in the minimal Hilbert basis for the magic 3-circles. The remaining forty elements of the Hilbert basis for the magic 3-circle are obtained by rotating each of these magic circles. In [GL], we show that there is a one-to-one correspondence between the set of magic n-circles and the set of magic labelings of a complete double-edged bipartite graph on $2n$ nodes. Figure 1 also shows the corresponding magic labeling of the complete double-edged bipartite graph on 6 nodes to the magic 3-circle in Figure 1.

We have also developed methods of producing at least $2^n(n!)n$ natural magic circles of magic sum $n^2(2n^2+1)$. Currently, we are developing a function $H_n(S)$ that will count the number of distinct magic n-circles with magic sum S and extending our results to a classic variation on magic circles discovered by Benjamin Franklin [P] and to the other types of magic circles [J].

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