

## 11. Infinite Product Formula

The Fourier transform of a refinable function can be represented as an infinite product of its dilated symbols. The infinite product representation of the refinable function is useful for studying the properties of a refinable function. Let  $\phi$  be a refinable function satisfying the following refinable equation.

$$(11.1) \quad \phi(x) = 2 \sum_{k=0}^N h_k \phi(2x - k),$$

where  $\sum h_k = 1$ . Then the Fourier transform of  $\phi$  satisfies the following equation

$$(11.2) \quad \hat{\phi}(\omega) = p(z) \hat{\phi}(\omega/2), \quad z = e^{-i\omega/2},$$

where  $p(z) = \sum_{k=0}^N h_k z^k$  with  $p(1) = \sum h_k = 1$ . If the solution of (11.1) exists, then, under the normalization condition  $\hat{\phi}(0) = 1$ , we have

$$(11.3) \quad \hat{\phi}(\omega) = \prod_{k=1}^{\infty} p(e^{-i\omega/2^k}).$$

We now prove that the infinite product in (11.3) is always convergent.

**11.1. The Convergence of Infinite Product.** Let  $\Pi_n$  denote the set of polynomial of degree no more than  $n$ . For a polynomial  $p(z)$ , we write

$$p_k^*(z) = \prod_{j=1}^k p(e^{-iz/2^j})$$

and

$$p^k(z) = \prod_{j=0}^{k-1} p(z^{2^j}).$$

Hence we have

$$p^k(e^{-i2^{-k}z}) = p_k^*(z).$$

We also write

$$B_k(p) = \sup_{|z|=1} |p^k(z)|.$$

**THEOREM 11.1.** *Let  $p(z)$  be a polynomial with  $p(1) = 1$ . Then for any given  $R > 0$ , the infinite product  $\prod_{k=1}^{\infty} p(e^{-i\omega/2^k})$  is uniformly convergent on  $w \leq R$  in the complex plane. Furthermore, let  $1 \leq p \leq \infty$  and  $s$  be the real number such that*

$$(11.4) \quad \inf \lim_{l \rightarrow \infty} 2^{sl} \left( \int_0^{2^l} |p_l^*(2\pi\theta)|^p d\theta \right)^{1/p} \leq \infty.$$

Then

$$(11.5) \quad \left( \int_{\theta \in \mathbb{R}} |\hat{\phi}(\theta)|^p (1 + \theta^p)^s d\theta \right)^{1/p} < \infty.$$

PROOF. There is a  $\delta > 0$  such that for  $|w| \leq \delta$ ,

$$|1 - p(e^{-iw})| = \left| \sum_{k=0}^N h_k (1 - e^{-ikw}) \right| \leq c|w|.$$

Hence, if  $|w| \leq R$  and  $j \geq \log_2(1 + \frac{R}{\delta})$ , then

$$|1 - p(e^{-i2^{-j}w})| \leq c2^{-j}R.$$

It follows that The infinite product (11.3) is uniformly convergent on  $|w| \leq R$ . Then

$$\hat{\phi}(\omega) = \prod_{k=1}^{\infty} p(e^{-i\omega/2^k}) = \lim_{l \rightarrow \infty} p_l^*(\omega), \quad \omega \in \mathbb{R}.$$

To get the (11.5), we assume that  $\max_{|w|=1} |p(w)| = 2^r$ , where  $r \geq 0$ . Hence,

$$|p_l^*(w)| = \sup_{|z|=1} \left| \prod_{j=1}^l p(w^{2^{-j}}) \right| \leq 2^{jl}.$$

It follow that there is an  $s$  such that (11.4) holds. Recalling that  $p_l^*(\theta)$  is a  $2^l\pi$ - periodic function, we have

$$\begin{aligned} & \left( \int_{\theta \in \mathbb{R}} |\hat{\phi}(\theta)|^p (1 + \theta^2)^s d\theta \right)^{1/2} \\ &= \inf \lim_{l \rightarrow \infty} \left( \int_{\theta=-2^l}^{2^l} |\hat{\phi}(2^l\theta) p_l^*(\theta)|^p (1 + \theta^p)^s d\theta \right)^{1/p} \\ &\leq 2^r \inf \lim_{l \rightarrow \infty} (1 + 2^l)^s \left( \int_{\theta=-2^l\pi}^{2^l\pi} |p_l^*(\theta)|^p d\theta \right)^{1/p} \\ &\leq C \inf \lim_{l \rightarrow \infty} 2^{ls} \left( \int_{\theta=0}^{2^l} |p_l^*(2\pi\theta)|^p d\theta \right)^{1/p} \leq \infty. \end{aligned}$$

□

We now define a space of functions  $L_{(s)}^p$  by

$$L_{(s)}^p = \left\{ f; \left( \int_{\mathbb{R}} |\hat{f}(\theta)|^p (1 + \theta^p)^s d\theta \right)^{1/2} < \infty \right\}.$$

Note that  $L_{(0)}^p = L^p$ , and for  $s \in \mathbb{N}$ ,  $L^p(s)$  contains the functions whose  $s$ -th derivative in  $L^p$ . For  $s = n + \beta$ ,  $n \in \mathbb{N}$ ,  $0 \leq \beta < 1$ , we define  $C^s$  to be the set of functions  $f$  which are  $n$  times differentiable and such that the  $n$ th derivative  $f^{(n)}$  is Hölder continuous with exponent  $\beta$ , i.e.,

$$|f^{(n)}(x) - f^{(n)}(x+t)| \leq C|t|^\beta, \quad \forall x, t.$$

It is also well-known that for  $s > 0$ ,  $f \in L_{(s)}^1$  implies  $f \in C^s$ .

**11.2. The Support of Refinable Functions.** We can use (11.3) to determine the support length of  $\phi$ . Here we say that an interval  $[a, b]$  is an interval support of  $\phi$  if  $[a, b]$  is the smallest interval containing  $\text{supp } \phi$ .

**THEOREM 11.2.** *If the refinable function  $\phi$  satisfies the refinable equation (11.1), then its interval support is  $[0, N]$ .*

**PROOF.** To prove this theorem we need Palay–Wiener–Schwarz Theorem.

**Theorem** Assume that the entire function  $F(z)$  is the Fourier transform of a distribution  $f(x)$  :

$$F(z) = \int_{x \in \mathbb{R}} f(x) e^{-ixz} dx$$

and

$$|F(z)| \leq C(1 + |z|)^r e^{N \max(\text{Im } z, 0)}.$$

Then

$$\text{supp } f \subset [0, N].$$

We now start to prove the theorem. Note that  $p^l(w)$  is a polynomial of  $w$  with degree  $N(2^l - 1)$  and

$$\sup_{|w|=1} |p^l(w)| = \sup_{\omega \in \mathbb{R}} |p_i^*(2^j w)| = \sup_{\omega \in \mathbb{R}} |p_i^*(w)| \leq 2^{lr}.$$

By Phragmén–Lindelöf Theorem, we know that on the complex plane,

$$(11.6) \quad |p_i^*(w)| \leq 2^{rl} e^{N(1-2^{-l-1}) \max(\text{Im } w, 0)}.$$

Let

$$\Phi(w) = \hat{\phi}(w) = \prod_{k=1}^{\infty} p(e^{-iw/2^k}), \quad w \in \mathbb{C}.$$

Then we have

$$|\Phi(w)| \leq 2^r |p_i^*(w)|, \quad |w| \leq 2^l.$$

Thus for  $2^{l-1} \leq |w| \leq 2^l$ , we have

$$\begin{aligned} |\Phi(w)| &\leq 2^r 2^{rl} e^{N(1-2^{-l-1}) \max(\text{Im } w, 0)} \\ &\leq 2^{r+1} (1 + |w|)^r e^{N \max(\text{Im } w, 0)}. \end{aligned}$$

By Palay–Wiener–Schwarz Theorem,  $\text{supp } \phi \subset [0, N]$ . The theorem is proved.  $\square$

We can also prove that if the function  $\phi$  is linearly independent, then the interval support of  $\phi$  is same as the support of  $\phi$  that is also  $[0, N]$ .

**11.3. The Symmetric of Refinable Functions.** If a compactly supported refinable function  $\phi$  is symmetric, then so is its mask. (i.e.  $h_k = h_{N-k}$ ). More precisely, we have the following.

**THEOREM 11.3.** *Let  $\phi$  be the refinable function satisfying (11.1). Then  $\phi$  is symmetric if and only if the mask  $\mathbf{h}$  is symmetric.*

**PROOF.** If  $\mathbf{h}$  is symmetric, then on  $\Gamma$ ,  $\overline{P(z)} = z^{-N}P(z)$ . Hence,

$$P(e^{-i\omega}) = e^{-iN\omega/2}R(\omega),$$

where  $R(\omega) = e^{iN\omega/2}P(e^{-i\omega})$  is a real function. It follows that

$$\hat{\phi}(\omega) = e^{-iN\omega/2}\Phi_r(\omega),$$

where  $\Phi_r(\omega) = e^{iN\omega/2}\hat{\phi}(\omega)$  is a real function. Hence,  $\phi$  is symmetric. On the other hand, if  $\phi$  is symmetric with respect to  $a$ , i.e.,  $\phi(x) = \phi(a-x)$ , then

$$\hat{\phi}(\omega) = e^{-ia\omega/2}R(\omega),$$

where  $R(\omega) = e^{ia\omega/2}\hat{\phi}(\omega)$  is a real function. It follows that

$$p(e^{-i\omega}) = e^{-ia\omega/2}\frac{R(2\omega)}{R(\omega)},$$

where  $\frac{R(2\omega)}{R(\omega)}$  is a real function. Hence,  $a = N$  and  $h_k = h_{n-k}$ .  $\square$

## 12. Linear Independence and Stability

In this section, we still assume that any refinable function is compactly supported. We already know the concept of stability of a refinable function in  $L^2$ . Recall that a function  $\phi \in L^2$  is called stable, if (7.5) holds. Theorem 7.3 gives a necessary and sufficient condition for  $\phi$  being stable. That is  $0 < \sum_{k \in \mathbb{Z}} |\hat{\phi}(\omega + 2k\pi)|^2 < \infty$ . If  $\phi$  is compactly supported, then the right inequality is always true. Since, in this case,  $\hat{\phi}(\omega)$  is an entire function, the condition is equivalent to that  $\hat{\phi}(\omega)$  has no  $2\pi$ -periodic zero on the real line. A corollary of this result is the following.

**THEOREM 12.1.** *A compact supported function  $\phi$  is stable, if and only if there is no complex number  $w \in \Gamma$  such that*

$$\sum_{k \in \mathbb{Z}} w^k \phi(x - k) = 0.$$

Another important concept is the linear independence of  $\phi$ .

**DEFINITION 12.1.** *A compactly supported function  $\phi$  is said to have linearly independent integer translates if there is a complex sequence  $(a_k)_{k \in \mathbb{Z}}$  ( $\neq 0$ ) such that*

$$(12.1) \quad \sum_{k \in \mathbb{Z}} a_k \phi(x - k) = 0.$$

For convenience, we shall simply say that  $\phi$  is linearly independent instead of that  $\phi$  has linearly independent integer translates. An important result for the linear independence of a function is the following.

**THEOREM 12.2.**  *$\phi$  is linearly dependent if and only if there is a complex number  $\theta \neq 0$  such that*

$$(12.2) \quad \sum_{k \in \mathbb{Z}} \theta^k \phi(x - k) = 0,$$

*which is equivalent to that  $\hat{\phi}(w)$  has a  $2\pi$  periodic zero on  $\mathbb{C} \setminus \{0\}$ .*

Note that the difference between stability and linear independence. If  $\phi$  is linearly independent then it is stable. But the inverse is not true. In (12.2) if we set

$$\psi(x) = \sum_{k=0}^{\infty} w^k \phi(x - k) = 0,$$

then  $\psi$  is compactly supported and  $\phi(x) = \psi(x) - w\psi(x - 1)$ . Hence we have

**COROLLARY 12.3.**  *$\phi$  is linearly dependent (stable) if and only if there is a compactly supported function  $\psi$  and a complex number  $w \neq 0$  ( $w \in \Gamma$ ) such that  $\phi(x) = \psi(x) - w\psi(x - 1)$ , which is equivalent to that*

$$(12.3) \quad \hat{\phi}(\omega) = (e^{-i\omega} - w)\hat{\psi}(\omega).$$

**12.1. Linear Independence of a Refinable Function.** We now try to derive the necessary and sufficient condition for a refinable function being linearly independent (stable) via its symbol. Here we assume that  $\phi$  satisfies the normalization condition  $\int_{\mathbb{R}} \phi(x) dx = 1$ , which is equivalent to  $\hat{\phi}(0) = 1$ . From Corollary 12.3, we know that if  $\phi$  is linearly independent, then  $w \neq 1$ . Otherwise it contradicts to  $\hat{\phi}(0) = 1$ . Let  $p(z)$  and  $q(z)$  be the symbols of  $\phi$  and  $\psi$  respectively. Then from (12.3) we have

$$(12.4) \quad p(z) = \frac{z^2 - w}{z - w} q(z),$$

where  $w \neq 0, 1$ . We call a polynomial is trivial if it is a monomial. On the other hand, if (12.4) holds for some  $w \in \mathbb{C} \setminus \{0, 1\}$ , then (12.3) holds, i.e.,  $\phi$  is linearly dependent. Thus we reach

**THEOREM 12.4.**  *$\phi$  is linearly dependent if and only if its symbol  $p(z)$  has the following decomposition:*

$$(12.5) \quad p(z) = \frac{t(z^2)}{t(z)} q(z),$$

where  $t(z)$  is a non-trivial polynomial and  $t(1) \neq 0$ .

The polynomial  $t(z)$  in (12.5) is called the two-scale factor of  $p(z)$ . Similarly, we can prove

**THEOREM 12.5.**  *$\phi$  is stable if and only if its symbol  $p(z)$  has the decomposition:*

$$(12.6) \quad p(z) = \frac{t(z^2)}{t(z)} q(z),$$

where  $t(z)$  is a non-trivial polynomial,  $t(1) \neq 0$ , and all the zeros of  $t(z)$  are on  $\Gamma$ .

**12.2. The Root Criterion of Linear Independence.** We now use Theorem 12.4 to obtain a criterion for linear independence of a refinable function via the roots of its symbol. We first introduce some notions of polynomial roots.

**DEFINITION 12.2.** *Let  $m > 1$  be an odd integer, let  $z_0$  be a primitive  $m$ -th root of unit, and let  $h_m$  be the smallest positive integer such that  $2^{h_m} \equiv 1 \pmod{m}$ . Then we call*

$$c_m(z) = (z_0 - z)(z_0^2 - z) \cdots (z_0^{2^{h_m-1}} - z)$$

an irreducible cycle polynomial.

**EXAMPLE 12.1.** *The followings are examples of irreducible cycle polynomials.*

- (1)  $m = 3, h_3 = 2$ , and  $c_3(z) = 1 + z + z^2$ .
- (2)  $m = 5, h_5 = 4$ , and  $c_5(z) = 1 + z + z^2 + z^3 + z^4$ .

- (3)  $m = 7, h_7 = 6$ , and  $c_7(z) = 1 + z + z^2 + z^3 + z^4 + z^5 + z^6$ .  
 However,  
 (4)  $m = 9, h_9 = 6$ , and  $c_9(z) = 1 + z^3 + z^6$ .

Since  $z_0^{2^{hm}} = z_0$ , it can be verified that

$$(12.7) \quad \frac{c_m(z^2)}{c_m(z)} = c_m(-z).$$

The polynomial  $c_m(z)$  is called a irreducible co-cycle polynomial. It can be verified that if a polynomial  $r(z)$  satisfies  $r(z^2)/r(z) = R(-z)$  then  $r(z)$  is a finite product of irreducible polynomials. We call  $r(z)$  a *cycle polynomial* while  $r(-z)$  is *co-cycle polynomial*. We introduce two special kinds of roots of a polynomial.

**DEFINITION 12.3.**  $p(z)$  is said to have co-cycle zeros if there is at least one irreducible co-cycle polynomial  $c_m(-z)$  is a factor of  $p(z)$ , and  $p(z)$  is said to have symmetric zeros if there is an  $\alpha \in \mathbb{C} \setminus \{0\}$  such that  $p(\alpha) = p(-\alpha) = 0$ , or equivalently,  $z^2 - \alpha^2$  is a factor of  $p(z)$

We now can derive the following criterion from the Theorem 12.4

**THEOREM 12.6.** Let  $\phi$  be a refinable function with  $\hat{\phi}(0) = 1$ . Then  $\phi$  is linearly independent if and only if  $p(z)$  has neither cycle zeros nor symmetric zeros.

**PROOF.** The proof of “ONLY IF” is trivial. we now prove “IF”. Assume that  $\phi$  is linearly dependent. By Theorem 12.4,  $p(z)$  has a nontrivial two-scale factor  $t(z)$  that satisfies (12.5), in which the symbol  $q(z)$  satisfies  $q(1) = 1$ . We decompose  $t(z)$  into a product of linear factors  $t(z) = \prod_{i=1}^m t_i(z)$ , where  $t_i(z) = (z_i - z)$ ,  $z_i \neq 0$ . If  $t_1(z)$  is not a factor of  $t(z^2)$ , then it is a factor of  $q(z)$ . Hence,  $t_1(t^2)$  is a factor of  $p(z)$  which implies that  $p(z)$  has symmetric zeros. If  $t_1(z)$  is a factor of  $t(z^2)$  then it must be a factor of  $t_j(z^2)$  for some  $j$ . Since  $t(1) \neq 1$ ,  $j \neq 1$ . Without loss of generality, we can assume that  $t_1(z)$  is a factor of  $t_2(z^2)$ , which implies  $z_2 = z_1^2$ . Do not stop this procedure, until we meet an index  $J$  such that  $t_J(z)$  is a factor of  $q(z)$  or is a factor of  $t_l(z^2)$ ,  $1 \leq l < J$ . In the first case,  $t_1(z^2)$  is a factor of  $p(z)$ . In the second case, If  $l = 1$ ,  $\prod_{i=1}^s t_i(z)$  is a two-scale factor of  $p(z)$  which implies that  $p(z)$  has cycle zeros. Otherwise,  $z_1(z)$  and  $\prod_{i=l}^s t_i(z)$  both are two scale factors of  $p(z)$ , and therefore  $p(z)$  has both symmetric zeros and cycle zeros.  $\square$

Similarly we can prove

**THEOREM 12.7.** Let  $\phi$  be a refinable function with  $\hat{\phi}(0) = 1$ . Then  $\phi$  is stable if and only if  $p(z)$  has neither cycle zeros nor symmetric zeros on  $\Gamma$ .

### 12.3. Linearly Independent Functions in Refinable Space.

Using Theorem 12.6, we can find the linearly independent refinable function in the space spanned by  $\{\phi(x - k)\}$  whenever  $\phi$  is not linearly independent. Let  $p(z)$  be the symbol of  $\phi$ . First, find all co-cycle factors of  $p(z)$  and remove them from  $p(z)$ . Denote the reduced polynomial by  $p_1(z)$ . Then find a pair of symmetric zeros, say  $\pm\alpha$ , of  $p_1(z)$  if they exist. Then reduce  $p_1(z)$  to  $p_2(z) = p_1(z)(z - \alpha^2)/(z^2 - \alpha^2)$ . Note that  $\deg(p_2) \leq \deg(p_1)$ . For  $p_2(z)$ , doing the same procedure, after finite steps, we shall get a polynomial  $q(z)$  that has neither co-cycle zeros nor symmetric zeros, and  $q(1) = 1$ . Solving the refinable equation with the mask  $q(z)$ , we obtain the solution  $\psi$ , that is the linearly independent refinable function we required.

### 13. Accuracy and Vanish Moments of Wavelets

Let

$$S(\phi) := \left\{ \sum_k a_k \phi(x - k); \quad \mathbf{a} \in l \right\}.$$

Write

$$S^2 = S(\phi) \cap L^2,$$

and

$$S_h^2 = \{g(\cdot/h), \quad g \in S^2\}.$$

Let  $\Pi_k$  be the set of all polynomials of degree no more than  $k$ .

**DEFINITION 13.1.** *A compactly supported function  $\phi \in L^2$  is said to have accuracy  $k$  if  $\Pi_{k-1} \subset S(\phi)$ , and  $\phi$  is said to provide  $L^2$ -approximation order  $k$  if for any function  $f \in L^2$  with  $f^{(k)} \in L^2$ ,*

$$\text{dist}_{L^2}(f, S_h^2) \leq Ch^k,$$

where  $C$  is a positive constant independent of  $h$ .

Let  $\{V_j\}_{j \in \mathbb{Z}}$  be an MRA generated by a refinable function  $\phi$ . Denote the orthogonal project from  $L^2$  to  $V_j$  by  $P_j$ . Then we have  $\|f - P_j f\|_{L^2} \leq C2^{-jk}$  provided that  $\phi$  has approximation order  $k$ . It can be verified that if  $\phi$  has accuracy  $n$  (and  $\hat{\phi}(0) = 1$ ), then there is a function  $\psi$  that is a finite linear combination of  $\{\phi(x - k)\}$  such that

$$(13.1) \quad \sum_{k \in \mathbb{Z}} k^l \psi(x - k) = x^l, \quad l = 0, 1, \dots, n - 1.$$

(Please prove it using the mathematical induction.) From a result of finite elements, we can prove

**THEOREM 13.1.** *If  $\phi \in L^2$  is compactly supported, then  $\phi$  has approximation order  $k$  if and only if  $\phi$  has accuracy  $k$ .*

**13.1. A Criterion of Accuracy.** In order to obtain a condition of the accuracy of  $\phi$  in frequency domain, we introduce the Poisson Summation Formula.

**THEOREM 13.2.** *if  $f$  is a compactly supported function, then*

$$(13.2) \quad \sum_{k \in \mathbb{Z}} f(k) = \sum_{n \in \mathbb{Z}} \hat{f}(2n\pi)$$

From Poisson Summation Formula we have

**THEOREM 13.3.** *If  $\phi \in L^2$  is compactly supported and  $\hat{\phi}(0) = 1$ , then  $\phi$  has accuracy  $n$  if and only if*

$$(13.3) \quad \hat{\phi}^{(l)}(2k\pi) = 0, \quad k \in \mathbb{C} \setminus \{0\}, \quad l = 0, 1, \dots, n - 1.$$

PROOF. First we assume that  $\psi$  is such a finite linear combination of the integer translate of  $\phi$  that satisfies (13.1). We have

$$\sum k^l \psi(x - k) = x^l, \quad l = 0, 1, \dots, n - 1.$$

It follows that

$$\sum (x - k)^l \psi(x - k) = 0, \quad l = 1, \dots, n - 1.$$

Let  $\lambda(t) = (x - t)^l \psi(x - t)$ . Then  $\lambda(t)$  is compactly supported and

$$\hat{\lambda}(\omega) = (-i)^l e^{-ix\omega} \hat{\psi}^{(l)}(-\omega).$$

By the Poisson Summation Formula, we have

$$0 = \sum (x - k)^l \psi(x - k) = (-i)^l \sum e^{-i2k\pi x} \hat{\psi}^{(l)}(-2k\pi).$$

Therefore, (13.3) holds for  $\psi$ . Since  $\psi$  is the finite linear combination of  $\phi$ ,  $\phi$  also satisfies (13.1).  $\square$

**13.2. Accuracy of a Refinable Function.** We now develop the condition for a refinable function having accuracy  $k$ .

**THEOREM 13.4.** *Let  $\phi$  be linearly independent. Then  $\phi$  has accuracy  $n$  if and only if one of the following conditions is satisfied.*

(1) *Its symbol  $p(z) = (\frac{1+z}{2})^n q(z)$ , with  $q(-1) \neq 0$ .*

(2)  $\sum_{k \in \mathbb{Z}} (-1)^k k^j h_k = 0$ ,  $j = 0, 1, \dots, n - 1$ .

(3) *Let  $\mathbf{M} = (\downarrow 2)\mathbf{H} = (h_{2k-i})_{i,k \in \mathbb{Z}}$ . Then  $1, 1/2, \dots, (\frac{1}{2})^{n-1}$  are the eigenvalues of  $\mathbf{M}$ .*

PROOF. Using the Fourier transform of the two-scale function of  $\phi$  and Theorem 13.3, we can from (13.3) derive (1). If (1) is true, then by Leibniz rule for the derivative of product, we have

$$\hat{\phi}^{(l)}(\omega) = \left( \frac{1 + e^{-i\omega/2}}{2} \right) r(\omega), \quad 0 \leq l \leq n - 1,$$

where  $r(\omega)$  is an entire function. Hence,

$$\hat{\phi}^{(l)}(2k\pi) = 0, \quad k \in \mathbb{Z}, l = 0, 1, \dots, n - 1.$$

The equivalence of (1) and (2) is straightforward. To prove that (3) is equivalent to that  $\phi$  has accuracy  $n$ , we assume that

$$\sum a_{l,k} \phi(x - k) = x^l, \quad l = 0, 1, \dots, n - 1.$$

Then we have

$$x^l = \sum a_{l,k} \phi(x - k) = \sum_i \sum_k h_{i-2k} a_{l,k} \phi(2x - i).$$

Recall that

$$2^{-l}(2x)^l = \sum a_{l,k} \phi(2x - k).$$

Since  $\phi$  is linearly independent,

$$\sum_i h_{k-2i} a_{l,i} = 2^{-l} a_{l,k}.$$

Hence,  $2^{-l}, 0 \leq l \leq n-1$ , is the eigenvalue of  $\mathbf{M}^T$ , so is of  $\mathbf{M}$ . On the other hand, if  $1, 1/2, \dots, (1/2)^{n-1}$  are the eigenvalues of  $\mathbb{M}$ . Let  $(a_{l,k})_{k \in \mathbb{Z}}^T$  be the left eigenvector of  $\mathbf{M}$  corresponding to  $(1/2)^l$  and

$$f(x) = \sum_k a_{l,k} \phi(x-k).$$

Then we have  $f(2x) = 2^l f(x)$ . Without loss of generality, we can assume that  $f(x)$  is continuous (otherwise, we can use the indefinite integral of  $f(x)$  instead). It follows that  $f(x) = x^l$ .  $\square$

### 13.3. Vanish Moments of Wavelets.

DEFINITION 13.2. *A function  $\psi$  is said to have  $n$  vanish moments, if*

$$\int_{\mathbb{R}} \psi(x) x^l dx = 0, \quad l = 0, 1, \dots, n-1.$$

THEOREM 13.5. *If the refinable function  $\phi$  has accuracy  $n$  and it generates an MRA*

$$\dots \subset V_{-1} \subset V_0 \subset V_1 \subset \dots,$$

*then any compactly supported function in the corresponding wavelet subspaces has vanish moment  $n$ .*

PROOF. Without loss of generality, we assume that the compactly supported wavelet function  $g \in W_0$ . Then for any  $l, 0 \leq l \leq n-1$ ,

$$\begin{aligned} & \int_{\mathbb{R}} g(x) x^l dx \\ &= \int_{\mathbb{R}} g(x) \sum_k a_{l,k} \phi(x-k) dx \\ &= \sum_k a_{l,k} \left( \int_{\mathbb{R}} g(x) \phi(x-k) dx \right) \\ &= 0. \end{aligned}$$

$\square$

## 14. Smoothness of Wavelets

There are two approaches to the study of smoothness (or regularity) of wavelets. One is the infinite product form we used in §11. Another one is joint spectrum of the matrix associated with the mask of refinable function. We met those matrices first in §9, where we discussed the Cascade algorithm for finding the solution of refinable equation. However we did not discuss the convergence of the algorithm in that section. We now solve this question in the following subsection.

### 14.1. Smoothness of Refinable Functions.

14.1.1. *Infinite Product Method.* A special kind of refinable functions called cardinal B-splines play an important role in the discussion of the properties of wavelets. The cardinal B-spline of order  $n$ ,  $B_n(x)$ , is defined as following  $B_1(x) = \chi_{[0,1)}(x)$  and  $B_n(x) = B_{n-1} * B_1(x)$ ,  $n = 2, 3, \dots$ . We can verify that the symbol of  $B_n(x)$  is  $(\frac{1+z}{2})^n$ , and the Fourier transform of  $B_n(x)$  is

$$\hat{B}_n(\omega) = e^{-in\omega/2} \left( \frac{\sin \omega/2}{\omega/2} \right)^n.$$

$B_n(x)$  is the piecewise polynomial of degree  $n - 1$  with the continuous  $(n - 2)$ th derivative while  $(n - 1)$ th derivative of  $B_n(x)$  is the piecewise constant function. By the Fourier transform of  $B_n(x)$  we can find that if the function  $f \in L^2_{(s)}$ , then the convolution

$$f * B_n(x) = \int_{\mathbb{R}} f(x - t) B_n(t) dt$$

is in  $L^2_{(s+n)}$ . Hence, by Theorem 11.1, we have

**THEOREM 14.1.** *If the symbol of the refinable function  $\phi$ ,  $p(z)$  can be write as*

$$p(z) = \left( \frac{1+z}{2} \right)^n q(z),$$

where  $q(z)$  satisfies the condition

$$\inf \lim_{l \rightarrow \infty} 2^{sl} \left( \int_0^{2^l} |q_l^*(2\pi\theta)|^p d\theta \right)^{1/p} \leq \infty,$$

then  $f \in L^2_{(s+n)}$ .

From Theorem 14.1, we derive

**COROLLARY 14.2.** *If the symbol of the refinable function  $\phi$ ,  $p(z)$  can be write as*

$$p(z) = \left( \frac{1+z}{2} \right)^n q(z),$$

where  $q(z)$  satisfies the condition  $\max_{z \in \Gamma} |q(z)| \leq 2^{n-1/2}$ , then  $\phi \in L^2$ . Furthermore, if  $\max_{z \in \Gamma} |q(z)| < 2^{n-1}$ , then  $\phi \in C$ .

PROOF. From  $\max_{z \in \Gamma} |q(z)| \leq 2^r$ , we have

$$\inf \lim_{l \rightarrow \infty} \left( \int_0^{2^l} |q_l^*(2\pi\theta)|^p d\theta \right)^{1/p} \leq 2^{rl+l/p}.$$

Hence,  $\phi \in L^p_{(n-r-1/p)}$ . It follows that if  $\max_{z \in \Gamma} |q(z)| \leq 2^{n-1/2}$ , then  $\phi \in L^2$ , and if  $\max_{z \in \Gamma} |q(z)| < 2^{n-1}$ , then there is a  $\alpha$  such that  $\max_{z \in \Gamma} |q(z)| = 2^{n-1-\alpha}$ . It follows that  $\phi \in L^1_\alpha$  and therefore  $\phi \in C^\alpha \subset C$ .  $\square$

It is also easy to obtain the conditions for  $\phi$  having more smoothness.

14.1.2. *Joint Spectrum Method.* We now turn to the joint spectrum method. As we did in §9, for the two-scale equation

$$(14.1) \quad \phi(x) = \sum_{k=0}^N h_k \phi(2x - k),$$

we define

$$T_0 = (h_{2m-n-1})_{1 \leq n, m, \leq N} \quad T_1 = (h_{2m-n})_{1 \leq n, m, \leq N}.$$

(Note that the notation here is slightly different from that in §9, where we denote  $T_0$  and  $T_1$  by  $\mathbf{m}_0$  and  $\mathbf{m}_1$  respectively.)

THEOREM 14.3. *Assume that In the refinable equation (14.1)  $p(z)$  can be write as*

$$p(z) = \left( \frac{1+z}{2} \right)^n q(z),$$

where  $q(-1) \neq 0$ . Let  $T_0$  and  $T_1$  be two matrices defined above and  $\dim T_0 = K$ . Let  $E_K$  be the subspace of  $\mathbb{R}^K$  orthogonal to the vectors  $e_j = (1^{j-1}, 2^{j-1}, \dots, K^{j-1})^T$ ,  $1 \leq j \leq n$ . Assume that there is a real number  $0 \leq \lambda < 1$  and an integer  $0 \leq l \leq N - 1$  and  $C > 0$  such that, for all binary sequences  $(d_j)_{j \in \mathbb{N}}$ , and all  $m \in \mathbb{N}$ ,

$$(14.2) \quad \|T_{d_1} T_{d_2} \cdots T_{d_m} |_{E_K}\| \leq C 2^{-m(l+\lambda)}.$$

Then (1) there exists a non-trivial continuous  $L^1$  solution  $\phi$  for the equation (14.1). (2) If  $\lambda > 0$  then the  $l$ th derivative of  $\phi$  is Hölder continuous with the Hölder exponent  $\lambda$ , if  $\lambda = 0$ , then the  $(l - 1)$ th derivative of  $\phi$  satisfies

$$|\phi^{(l-1)}(x+t) - \phi^{(l-1)}(x)| \leq C |t \ln t|.$$

PROOF. At first we need some notations. Let  $T$  be the Cascade iterative operator

$$(T\phi)(x) = \sum_{k=0}^N h_k \phi(2x - k),$$

and  $\phi^0(x)$  be the initial function for the iterative algorithm. Then the successive iterative solution is

$$\phi^{j+1}(x) = \sum_{k=0}^N h_k \phi^j(2x - k), \quad j = 0, 1, \dots.$$

We already know that the values of the solution  $\phi$  (if  $\phi$  exists) at the integer points can be calculated from the equation

$$T_0\Phi(0) = \Phi(0),$$

where  $\Phi(0) = (\phi(0), \phi(1), \dots, \phi(N-1))^T$ . Thus we set the initial function

$$\phi^0(x) = \phi(k)(k+1-x) + \phi(x-k), \quad k \leq x \leq k_1, \quad k = 0, 1, \dots, N-1.$$

Note that then each  $\phi^i(x)$  is a piecewise linear function that is linear on  $[k/2^i, (k+1)/2^i]$ , and

$$\phi^{i+1}(k/2^i) = \phi^i(k/2^i), \quad k \in \mathbb{Z}, i = 0, 1, \dots.$$

We also need the following vector-valued function on  $[0, 1]$ .

$$v^i(x) = (\phi^i(x), \phi^i(x+1), \dots, \phi^i(x+N-1))^T, \quad x \in [0, 1].$$

Then, on  $[0, 1/2]$  we have

$$v^{i+1}(x) = T_0 v^i(2x)$$

and on  $[1/2, 1]$  we have

$$v^{i+1}(x) = T_1 v^i(2x-1).$$

In order to reduce them into a uniform equation, we define

$$\tau x = \begin{cases} 2x & x \in [0, 1/2] \\ 2x-1 & x \in [1/2, 1]. \end{cases}$$

For a real  $0 \leq x < 1$ , we define a binary sequence  $d(x) := (d_n(x))_{n \in \mathbb{N}}$  by

$$x = \sum_{n=1}^{\infty} d_n(x) 2^{-n}.$$

We can verify that  $(\tau d(x))_n = d(x)_{n+1}$ . Using the function  $\tau$ , we obtain

$$v^{i+1}(x) = T_{d_1(x)} v^i(\tau x).$$

Generally, we have

$$v^j(x) = T_{d_1(x)} T_{d_2(x)} \cdots T_{d_j(x)} v^0(\tau^j x),$$

which implies

$$v^j(x) - v^{j+l}(x) = T_{d_1(x)} T_{d_2(x)} \cdots T_{d_j(x)} [v^0(\tau^j x) - v^l(\tau^j x)].$$

It can be verified by  $\sum (-1)^s k_k^h = 0$ ,  $0 \leq s \leq n-1$ , that

$$e_s[v^j(x) - v^{j+l}(x)] = 0, \quad s = 1, 2, \dots, n.$$

Hence

$$\|T_{d_1}T_{d_2}\cdots T_{d_m}\| = \|T_{d_1}T_{d_2}\cdots T_{d_m}|_{E_K}\|.$$

Now by the condition (14.2), we get

$$\|v^i(x) - v^{i+l}(x)\| \leq C2^{-m(l+\lambda)}, \quad \forall j, l.$$

Hence  $v^i(x)$  is uniformly convergent on  $[0, 1]$ . It follows that  $\phi^i$  is uniformly convergent to a function  $\phi$  with the support  $[0, N]$ . We now also have

$$\|\phi - \phi^i\| \leq C2^{-m(l+\lambda)}.$$

Recall that  $\phi^i$  is a piecewise linear function that linear at each interval  $[k/2^i, (k+1)/2^i]$  and  $\phi(k/2^i) = \phi^i(k/2^i)$ . Hence, we have the result (2). the proof is completed.  $\square$

#### 14.2. Smoothness of Daubechies Orthonormal Wavelets.

We now use Theorem 14.3 to obtain the smoothness of Daubechies orthonormal wavelets. We only consider the bd2 scaling function  $\phi$  which is defined by

$$\phi(x) = \sum_{k=0}^3 c_k \phi(2x - k),$$

where

$$c_0 = \frac{1 + \sqrt{3}}{4}, \quad c_1 = \frac{3 + \sqrt{3}}{4}, \quad c_2 = \frac{3 - \sqrt{3}}{4}, \quad c_3 = \frac{1 - \sqrt{3}}{4}.$$

Then

$$T_0 = \begin{pmatrix} c_0 & 0 & 0 \\ c_2 & c_1 & c_0 \\ 0 & c_3 & c_2 \end{pmatrix}, \quad T_1 = \begin{pmatrix} c_1 & c_0 & 0 \\ c_3 & c_2 & c_1 \\ 0 & 0 & c_3 \end{pmatrix}.$$

Since the symbol of  $\phi$  has factor  $(\frac{1+z}{2})^2$ , we only need to calculate the norm of  $T_{d_1}T_{d_2}\cdots T_{d_m}$  on  $E_2$ . We find that

$$\|T_0|_{E_2}\| = \frac{1 + \sqrt{3}}{4}, \quad \|T_1|_{E_2}\| = \frac{1 - \sqrt{3}}{4}.$$

Hence

$$\begin{aligned} \|T_{d_1}T_{d_2}\cdots T_{d_m}|_{E_2}\| &\leq C \prod_{j=1}^m \|T_{d_j}|_{E_2}\| \\ &\leq C \left(\frac{1 + \sqrt{3}}{4}\right)^m \left(\frac{1 - \sqrt{3}}{1 + \sqrt{3}}\right)^{\sum_{n=1}^m d_m(x)}. \end{aligned}$$

Note that  $\frac{1-\sqrt{3}}{1+\sqrt{3}} < 1$  and  $\log_2 \frac{1+\sqrt{3}}{4} \approx 0.55$ , which implies that  $\phi \in C^{0.55}$ .