On Spline Wavelets

Dedicated to Professor Charles Chui on the occasion of his 65th birthday

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Outline

A survey of my joint work with Charles on spline wavelets

- Duality principle.
- Construction of spline wavelets.
- Truncate errors of spline wavelet IIR filters.
- Time-frequency localization of spline wavelets.
- Sub-band decomposition of signals.

Wavelet basis

A wavelet basis of $L^2(\mathbb{R})$ is an `unconditional basis` (also called a `Riesz basis`) of $L^2(\mathbb{R})$, generated by the dilations and translates of a wavelet function ψ :

$$\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}, \quad \psi_{j,k} = 2^{j/2}\psi(2^j x - k).$$

Multiresolution analysis

[Meyer and Mallat (1988)]
A `multiresolution analysis` of L^2 is a nest of subspaces of L^2

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$$

that satisfies the following conditions.

- $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$,
- $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2$,
- $f(\cdot) \in V_j$ if and only if $f(2\cdot) \in V_{j+1}$,
- there exists a function $\phi \in V_0$ such that $\{\phi(x-n)\}_{n \in \mathbb{Z}}$ is an unconditional basis of V_0.

Scaling function

An MRA generator satisfies a scaling equation:

$$\phi(x) = 2\sum_{m\in\mathbb{Z}} h(m)\phi(2x-m), \quad (h(m))_{m\in\mathbb{Z}} \in l^2.$$

In frequency Domain,

$$\hat{\phi}(\omega) = H(e^{-i\omega/2})\hat{\phi}(\omega/2),$$

where $\{h(m)\}$ is the mask of ϕ and $H(z)$ is the symbol of ϕ.

Wavelet subspaces

Wavelet subspace W_j is the supplement of V_j with respect to V_{j+1}:

$$W_j \oplus V_j = V_{j+1}, \quad j \in \mathbb{Z}.$$

Then we have the space decomposition

$$L^2(\mathbb{R}) = \oplus_{j\in\mathbb{Z}} W_j, \quad W_j \cap W_k = \{0\}, j \neq k,$$
$$g \in W_j \Longleftrightarrow g(2\cdot) \in W_{j+1}, \quad j \in \mathbb{Z}$$

or

$$L^2(\mathbb{R}) = V_j \oplus_{k\geq j} W_k.$$

Wavelet function

The wavelet ψ is a function in W_0 such that

- $\{\psi(\cdot - n)\}_{n\in\mathbb{Z}}$ forms a Riesz basis of W_0.
- Its mask $\{g(m)\}$ is defined by

$$\psi(t) = 2\sum_{k\in\mathbb{Z}} g(k)\phi(2x-k), \quad g \in l^2.$$

- Symbol $G(z)$ is

$$\hat{\psi}(\omega) = G(e^{-i\omega/2})\hat{\phi}(\omega/2).$$

Duality principle in time domain

Let ϕ and $\tilde{\phi}$ be the dual scaling functions with masks $\mathbf{h} = (h_k)$ and $\tilde{\mathbf{h}} = (\tilde{h}_k)$ respectively. Let ψ and $\tilde{\psi}$ be their dual wavelets with masks $\mathbf{g} = (g_k)$ and $\tilde{\mathbf{g}} = (\tilde{g}_k)$ respectively. Then

$$\begin{aligned} 2\sum_k h(k)\tilde{h}(k-2l) &= \delta_{0l}, \\ 2\sum_k g(k)\tilde{g}(k-2l) &= \delta_{0l}, \\ \sum_k h(k)\tilde{g}(k-2l) &= 0, \\ \sum_k \tilde{h}(k)g(k-2l) &= 0. \end{aligned}$$

Duality principle in frequency domain

Duality principle in frequency domain derives equations in C^*-algebra.

$$\overline{H(z)}\tilde{H}(z) + \overline{H(-z)}\tilde{H}(-z) = 1,$$
$$\overline{G(z)}\tilde{G}(z) + \overline{G(-z)}\tilde{G}(-z) = 1,$$
$$\overline{\tilde{G}(z)}H(z) + \overline{\tilde{G}(-z)}H(-z) = 0,$$
$$\overline{G(z)}\tilde{H}(z) + \overline{G(-z)}\tilde{H}(-z) = 0.$$

Solutions of duality equations (I)

The symbol of an orthonormal scaling function Satisfies

$$|H(z)|^2 + |H(-z)|^2 = 1.$$

By the duality principle, we have

$$G(z) = z^{2l-1}\overline{H(-z)},$$

which is the symbol of the corresponding orthonormal wavelet. Their dual functions are themselves, i.e.,

$$\tilde{H}(z) = H(z), \quad \tilde{G}(z) = G(z).$$

Solutions of duality equations II

For semi-orthogonal scaling function and wavelet, we have,

$$V_j = \tilde{V}_j \quad W_j = \tilde{W}_j.$$

Let $\Pi(e^{-i\omega}) = \sum_{k\in\mathbb{Z}} |\hat{\phi}(\omega + 2k\pi|^2$. Then

$$\tilde{H}(z) = \frac{H(z)\Pi(z)}{\Pi(z^2)},$$
$$G(z) = z^{2l-1}\overline{H(-z)\Pi(-z)}K(z^2),$$
$$\tilde{G}(z) = \frac{z^{2l-1}\overline{H(-z)}}{K(z^2)\Pi(z^2)},$$

where K is in the Wiener class $\mathcal{W}$ and $K(z) \neq 0$ on Γ.

Solutions of duality equations III

For biorthogonal compactly supported scaling function and wavelet, we have that H and $\tilde{H}$ both are finite Laurent polynomials. It yields

$$\overline{H(z)}\tilde{H}(z) + \overline{H(-z)}\tilde{H}(-z) = 1,$$
$$G(z) = z^{2l-1}\overline{\tilde{H}(-z)},$$
$$\tilde{G}(z) = z^{2l-1}\overline{H(-z)}.$$

Cardinal B-splines

[Schonberg, deBoor, 1973]
The mth order B-spline with integer knots is defined by

$$N_m = N_{m-1} * N_1, \qquad N_1 = \chi_{[0,1)}.$$

The Fourier transform of N_m is

$$\hat{N}_m(\omega) = \left(\frac{1 - e^{-i\omega}}{i\omega} \right)^m .$$

Properties of B-splines

The cardinal B-spline of order m satisfies the following:

- supp $N_m = [0, m]$, and $\{N_{m;k}\}_{k \in \mathbb{Z}}$ is locally linearly independent.

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- $N_m(x) > 0$, for all $x \in (0, m)$, and its support is $[0, m]$.
- Its symbol is $\left(\frac{1+z}{2}\right)^m$.

Euler-Frobenius polynomials

The Euler-Frobenius polynomial (of order $m-1$)

$$E_{m-1}(z) = \sum_{j=0}^{m-2} N_m(j+1) z^j, \quad m \geq 2.$$

When $m = 2n$, we write

$$\Pi_n(z) = z^{-n+1} E_{2n-1}(z), \quad n \geq 1.$$

Properties of Π_n

The Laurent polynomial $\Pi_n(z)$ has the following properties.

- $\Pi_n(z) > 0, \forall z \in \Gamma$, and all of its $2n-2$ roots are

$$0 < r_{n,1} < \cdots < r_{n,n-1} < 1 < r_{n,n} < \cdots < r_{n,2n-2},$$

  where $r_{n,j} r_{n,2n-1-j} = 1$.

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- Let $P_n(z) = \left(\frac{1+z}{2}\right)^n \left(\frac{1+z^{-1}}{2}\right)^n$. Then Π_n satisfies the following identity.

$$P_n(z)\Pi_n(z) + P_n(-z)\Pi_n(-z) = \Pi_n(z^2).$$

Orthonormal spline wavelets

[Lemarié (1988), Mallat (1989)] The symbol of orthonormal scaling spline of order n is

$$H_n^{\perp}(z) = \left(\frac{1+z}{2}\right)^n \frac{\sqrt{\Pi_n(z)}}{\sqrt{\Pi_n(z^2)}}.$$

The symbol of the orthonormal spline-wavelet is

$$G_n^{\perp}(z) = -z\overline{H_n^{\perp}(-z)}.$$

Semi-orthogonal spline wavelets

[Chui and Wang (1992)] Let ψ_n be the semi-orthogonal, compactly supported wavelet corresponding to N_n. Then the symbol of ψ_n is

$$G_n(z) = z^{2n-1}\left(\frac{1-z^{-1}}{2}\right)^n \Pi_n(-z),$$

$\operatorname{supp}(\psi_n) = [0, 2n-1]$. The symbols of the dual scaling function and wavelet of N_n and ψ_n are

$$\tilde{H}_n(z) = \frac{\left(\frac{1+z}{2}\right)^n \Pi(z)}{\Pi(z^2)}, \qquad \tilde{G}_n(z) = \frac{z^{2n-1}\left(\frac{1-z^{-1}}{2}\right)^n}{\Pi(z^2)}.$$

Interpolating spline-wavelets

[Chui and Wang 1991] An interpolating spline of order $2n$, $L_{2n} \in V_{2n,0}$, is defined by $L_{2n}(k) = \delta_{0,k}\ \forall k \in \mathbb{Z}$. It satisfies

$$\hat{L}_{2n}(\omega) = \frac{e^{in\omega}\hat{N}_{2n}(\omega)}{\Pi_n(e^{-i\omega})}.$$

We call $L_{2n}^{(n)}(2x-1-n) \in W_{n,0}$ an interpolating spline-wavelet. It is a semi-orthogonal spline-wavelet with respect to N_n with the symbol

$$G(z) = z\left(\frac{1-z^{-1}}{2}\right)^n \Pi_n(-z)\frac{(-2)^n}{\Pi_n(z)\Pi_n(-z)}.$$

Truncate error

The FWT algorithm based on spline-wavelets will be involved in filters of infinite impulse responses (IIR). In computation, an IIR filters have to be truncated.
Let $H = \{h_m\}_{m\in\mathbb{Z}}$ be an IIR. Assume we truncate H to $H_M = \{h_k\}_{k=-M}^{M}$.
The M-truncated error of H is

$$E_M(H) := \|(H - H_M)\|_{l^2}.$$

For any $\mathbf{c} \in l^2$, we have

$$\|H * c - H_M * c\|_{l^2} \le E_M(H)\|c\|_{l^2}.$$

Truncate error for semi-orthogonal spline-wavelet

[Chui and Wang,(1992)]
We obtained the following estimate.

Let $\tilde{H}_m$ be the symbol of the semi-orthogonal spline dual to N_m. Let $\{r_{m,k}\}_{k=1}^{2m-2}$ be the root set of $\Pi_m(z)$. For any $m > 0$, there is an integer M_m such that for all $M \geq M_m$,

$$||(\tilde{H}_m - \tilde{H}_{m,M})||_{l^2} \leq \left(2 \sum_{j=1}^{m-1} \frac{r_{m,k}^{m+M}}{\Pi_m(r_{m,k})} \right).$$

Particularly, when $m = 2, 3, 4$, the estimate above is true for all truncations (i.e., $M_m = 0$).

Examples of error estimations

- For linear spline-wavelets,

$$||(\tilde{H}_2 - \tilde{H}_{2,M})||_{l^2} \leq 0.73205 \times (0.26795)^M.$$

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$$||(\tilde{H}_2 - \tilde{H}_{2,M})||_{l^2} \leq 0.73205 \times (0.26795)^M.$$

- For cubic spline wavelets

$$||(\tilde{H}_4 - \tilde{H}_{4,M})||_{l^2} \leq 4.1952 \times (0.53528)^M.$$

Asymptotic characteristic of time-frequency localization

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- The measure of time-frequency localization.
- Uncertainty principle.
- Semi-orthogonal spline wavelets have asymptotically optimal time-frequency localization.
- Orthonormal wavelets asymptotically lose time-frequency localization.

Centers of time-frequency window

Let $\phi \in L^2 \cap L^1$ (called a window function).

- The time center of ϕ is

$$t_\phi = \lim_{N\to\infty} \int_{-N}^{N} x\, |\phi(x)|^2\, dx \Bigg/ \int_{-\infty}^{\infty} |\phi(x)|^2\, dx.$$

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- The frequency center of ϕ is

$$w_\phi = \lim_{N\to\infty} \int_{-N}^{N} \omega\,\left|\widehat{\phi}(\omega)\right|^2\,d\omega \Big/ \int_{-\infty}^{\infty} \left|\widehat{\phi}(\omega)\right|^2\,d\omega.$$

Radii of time/frequency localization

- The time localization radius.

$$\Delta_\phi = \frac{\left(\int_{-\infty}^{\infty} (x - t_\phi)^2 \, |\phi(x)|^2 \, dx\right)^{\frac{1}{2}}}{\left(\int_{-\infty}^{\infty} |\phi(x)|^2 \, dx\right)^{\frac{1}{2}}}.$$

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- The frequency localization radius.

$$\Delta_{\widehat{\phi}} = \frac{\left(\int_{-\infty}^{\infty} (\omega - \omega_\phi)^2 \left|\widehat{\phi}(\omega)\right|^2 d\omega\right)^{\frac{1}{2}}}{\left(\int_{-\infty}^{\infty} \left|\widehat{\phi}(\omega)\right|^2 d\omega\right)^{\frac{1}{2}}}.$$

Size of time-frequency window of ϕ

The size of time-frequency window of ϕ is measured by

$$M(\phi) = \Delta_\phi \Delta_{\widehat{\phi}}.$$

Uncertain Principle

If ϕ is a window function, then

$$M(\phi) \geq \frac{1}{2},$$

and the equality holds if and only if

$$\phi = kG_\sigma(x-\mu), \quad k \neq 0, \quad \sigma > 0, \quad \mu \in \mathbb{R},$$

where

$$G_\sigma(x) = \frac{1}{2\pi} e^{-\frac{(\sigma x)^2}{2}}$$

is a Gaussian function.

Frequency window of a wavelet

[Charles and Wang (1994)] We modify frequency window for wavelets because a wavelet is a bandpass function. Hence, it usually has two peaks in the frequency domain.
The positive frequency center.

$$\omega^{+}_{\widehat{\psi}} = \frac{\int_0^\infty \omega \left|\widehat{\psi}(\omega)\right|^2 d\omega}{\int_0^\infty \left|\widehat{\psi}(\omega)\right|^2 d\omega}.$$

Frequency window of a wavelet

The positive frequency localization radius.

$$\Delta^{+}_{\widehat{\psi}} = \frac{\left(\int_0^\infty (\omega - \omega^{+}_{\widehat{\psi}})^2 \left|\widehat{\psi}(\omega)\right|^2 d\omega\right)^{\frac{1}{2}}}{\left(\int_0^\infty \left|\widehat{\psi}(\omega)\right|^2 d\omega\right)^{\frac{1}{2}}}.$$

Size of time-frequency window of ψ

When ψ is a real-valued function, we can ignore its negative frequency since it can be derived from the positive one. Hence, we measure its time-frequency localization by

$$M^+(\psi) := \Delta_\psi \Delta^+_{\widehat{\psi}}.$$

Uncertain principle for wavelets

[Chui and Wang (1994)]
If $\psi \in L^2 \cap L^1$ is a real-valued symmetric or anti-symmetric function that satisfies $t\psi(t) \in L^2, \psi' \in L^2$, and $\widehat{\psi}(0) = 0$, then

$$M^+(\psi) > \frac{1}{2}.$$

Furthermore, the lower bound $\frac{1}{2}$ cannot be improved and cannot be attained.

Asymptotic time-frequency localization of orthogonal wavelets (1)

[Chui and Wang (1996)]
Let ϕ_n be the orthonormal scaling function with the symbol

$$P(z) = \left(\frac{1+z}{2}\right)^n S_n(z),$$

where

$$|S_n(e^{-i\omega})| \leq C2^n |\sin^n\left(\frac{\omega}{2}\right)|, \quad \frac{\pi}{2} \leq |\omega| \leq \pi.$$

Let ψ_n be the corresponding orthonormal wavelet.

Asymptotic time-frequency localization of orthogonal wavelets (2)

Then

- $\lim_{n\to\infty} || \, |\hat{\phi}_n| - \chi_{[-\pi,\pi]} ||_{L^2} = 0$, and

$$\lim_{n\to\infty} \Delta_{\hat{\phi}_n} = \frac{\pi}{\sqrt{3}}, \qquad \lim_{n\to\infty} \Delta_{\phi_n} = \infty.$$

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- $\lim_{n\to\infty} || \, |\hat{\psi}_n| - \chi_{[-2\pi,-\pi]\cup[\pi,2\pi]} ||_{L^2} = 0$ and

$$\lim_{n\to\infty} \Delta^{+}_{\hat{\psi}_n} = \frac{\pi}{2\sqrt{3}}, \quad \lim_{n\to\infty} \Delta_{\psi_n} = \infty.$$

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Then

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$$\lim_{n\to\infty} \Delta_{\hat{\phi}_n} = \frac{\pi}{\sqrt{3}}, \quad \lim_{n\to\infty} \Delta_{\phi_n} = \infty.$$
- $\lim_{n\to\infty} \| \, |\hat{\psi}_n| - \chi_{[-2\pi,-\pi]\cup[\pi,2\pi]} \|_{L^2} = 0$ and
$$\lim_{n\to\infty} \Delta^+_{\hat{\psi}_n} = \frac{\pi}{2\sqrt{3}}, \quad \lim_{n\to\infty} \Delta_{\psi_n} = \infty.$$
- $\lim_{n\to\infty} M(\phi_n) = \infty, \quad \lim_{n\to\infty} M^+(\psi_n) = \infty.$

Time-frequency localization of semi-orthogonal spline-wavelets (1)

[Chui and Wang (1997)]

For each n, let $\mathbf{a}^n = \{a_j^n\}_{j=0}^{k_n}$ be a finite symmetric Pólya frequency sequence with

$$a_n(z) = \left(\frac{1+z}{2}\right)^n p_n(z)$$

and

$$\deg p_n \leq Cn.$$

Let ϕ_n be the scaling function with the mask a^n and σ_n be its standard variation.

Time-frequency localization of semi-orthogonal spline-wavelets(2)

- For $1 \leq p < \infty$,

$$\lim_{n\to\infty} \left\| \widehat{\phi}_n \left(\frac{\omega}{\sigma_n} \right) e^{i\frac{k_n \omega}{2\sigma_n}} - e^{-\frac{\omega^2}{2}} \right\|_{L^p} = 0.$$

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- For $2 \le q < \infty$,

$$\lim_{n\to\infty} \left\| \sigma_n \phi_n \left(\sigma_n x + \frac{k_n}{2} \right) - \frac{1}{2\pi} e^{-\frac{x^2}{2}} \right\|_{L^q} = 0.$$

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- $\lim_{n\to\infty} M(\phi_n) = \frac{1}{2}$.

Optimal time-frequency localization of semi-orthogonal spline-wavelets(3)

Let ψ_n be the wavelet corresponding to ϕ_n. Then

- For $1 \le p < \infty$,

$$\lim_{n\to\infty} \left\| e^{\frac{i\omega}{2\tau_n}} \widehat{\psi}_n\left(\frac{\omega}{\tau_n}\right) - e^{-\frac{(\omega-\omega_n\tau_n)^2}{2}} \right\|_{L^p_{[0,\infty)}} = 0.$$

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- For $2 \le q < \infty$,

$$\lim_{n\to\infty} \left\| \tau_n\psi_n\left(\tau_n x + \frac{1}{2}\right) - \frac{1}{2\pi}\cos\left(\tau_n\omega_n x\right) e^{-\frac{x^2}{2}} \right\|_{L^q} = 0$$

Optimal time-frequency localization of semi-orthogonal spline-wavelets(4)

- For odd n and $2 \le q < \infty$,

$$\lim_{n\to\infty} \left\| \tau_n \psi_n \left(\tau_n x + \frac{1}{2} \right) - \frac{1}{2\pi} \sin(\tau_n \omega_n x) e^{-\frac{x^2}{2}} \right\|_{L^q} = 0,$$

Optimal time-frequency localization of semi-orthogonal spline-wavelets(4)

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- $$\lim_{n\to\infty} \frac{1}{\tau_n} \Delta_{\psi_n} = \lim_{n\to\infty} \tau_n \Delta^+_{\widehat{\psi_n}} = \frac{1}{\sqrt{2}}$$

so that

$$\lim_{n\to\infty} M^+(\psi_n) = \frac{1}{2}.$$

Sub-band code and general sample theory

[Chui & Wang, 2002]

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- Sub-band code is the mail tool in multiple-channel synchronized transmission.
- Find the lowest bound of bit rate for sub-band code.
- Find the method for sub-band coding that achieves the lowest bit rate.

Shannon Sampling Theorem

A continuous-time bandlimited signal $f(t)$ with

$$\operatorname{supp}^{+} \hat{f} \subset [0, 2\pi\sigma] \tag{1}$$

has the infinite series representation

$$f(t) = \sum_{k=-\infty}^{\infty} f(kT) \operatorname{sinc}\left(2\sigma(t - kT)\right),$$

where $T = \frac{1}{2\sigma}$.

Nyquist frequency

We always assume $2\pi\sigma$ is the least upper bound (lub) of $\text{supp}^+ \hat{f}$. Under the assumption,

- the highest band of f is σ;
- 2σ is called the Nyquist frequency (or rate) of $f(t)$;

The sampling theorem asserts that, for a lossless sampling, the sampling rate must be larger than the Nyquist rate.

Examples

- To get a satisfied speech signal, the highest band usually is set to 4 kHz. Then the sampling rate used is 8 kHz to avoid distortion. (Computer speech quality).
- The standard highest band of a high-quality music signals is 22.05 kHz. The usually sample rate for music signals is 44.1 kHz.

However, the speech signals **do not cover all bands in** $[0, \sigma]$. There is a room for transmitting signals in a lower bit rate.

Generalized Nyquist rate

We give the following definitions.

- The (generalized) Nyquist rate for a signal is defined as the smallest lossless sample rate for the signal.
- The bit rate is the ratio of half of the generalized Nyquist rate of a signal and the highest frequency of the signal.

The bit rate for a single band-signals

Assume a signal f has a lowest band σ_1 and the highest band σ_2. Then $\sigma := \sigma_2 - \sigma_1$ is called the bandwidth of $f(t)$. In this case, the Nyquist rate of $f(t)$ is given by

$$\sigma_m = \sigma \frac{1 + \sigma_1/\sigma}{1 + \lfloor \sigma_1/\sigma \rfloor}.$$

- If σ_1/σ is an integer, then $\sigma_m = \sigma$ and the bit rate of $f(t)$ is σ/σ_2.
- If σ_1/σ is not an integer, then $\sigma_m \neq \sigma$ and the bit rate is a little larger.

Multiple-channel synchronized transmission

It transmits a signal by dividing it into multiple channels in such a way that the signal in each channel can be coded from the original sampling code.
It can be done by sub-band decomposition.

Sub-band decomposition

Let $f(t)$ be a speech signal.
Decompose $f(t)$ in to $f(t) = \sum_{k=1}^{n} f_k(t)$, where
$\text{supp}^+ \hat{f}_k \subset [2\pi\mu_k, 2\pi\nu_k]$,

$$0 \leq \mu_1 < \nu_1 \leq \mu_2 < \nu_2 \leq \cdots \leq \mu_n < \nu_n \ ,$$

and μ_k and ν_k satisfy the sub-band decomposition conditions:

- $\mu_k/\sigma_k, k = 1, \ldots, n$, are integers;
- $\sigma_k/\sigma_\ell, k, \ell = 1, \ldots, n$, are rational numbers.

Theoretical Nyquist frequency

Let

$$\sigma_f = \frac{\operatorname{mes}(\operatorname{supp}^+ f)}{\pi}.$$

where the notation “mes”stands for the Lebesgue measure. Then $2\sigma_f$ is called the theoretical Nyquist frequency of a bandlimited signal $f(t)$.

Main results

Let f have the theoretical Nyquist frequency $2\sigma_f$ and the highest band σ. Then

- for any $\lambda_f > \sigma_f$, there is a sub-band decomposition of f, with sub-band coding bit rate no greater than λ_f/σ. and the sub-band coding bit rate of f is at least σ_f/σ;

Main results

Let f have the theoretical Nyquist frequency $2\sigma_f$ and the highest band σ. Then

- for any $\lambda_f > \sigma_f$, there is a sub-band decomposition of f, with sub-band coding bit rate no greater than λ_f/σ. and the sub-band coding bit rate of f is at least σ_f/σ;
- for any $\tilde{\sigma} > \sigma_f$, the sub-band coding with the bit rate $\tilde{\sigma}/\sigma$ can be realized by a Shannon wavelet packet.

Shannon wavelet

The Shannon sampling function is $\phi(t) = \operatorname{sinc} t$ and the Shannon waveletis

$$\psi(t) := 2\operatorname{sinc}(2t) - \operatorname{sinc} t,$$

$$\hat{\psi}(\omega) = \chi_{[-2\pi,-\pi)\cup[\pi,2\pi)}(\omega)$$

They form an o.n. MRA. Let $p_0(\omega) = \chi_{[-\frac{\pi}{2},\frac{\pi}{2})}$, $p_1(\omega) = p_0(\omega + \pi)$. Then we have

$$\hat{\phi}(\omega) = p_0(\omega/2)\hat{\phi}(\omega/2),$$

$$\hat{\psi}(\omega) = p_1(\omega/2)\hat{\phi}(\omega/2).$$

Shannon wavelet packets

Write $\mu_0(t) = \phi(t), \quad \mu_1(t) = \psi(t)$. Define $\{\mu_l\}_{l=0}^{\infty}$ inductively as follows. For even n, set

$$\begin{cases} \hat{\mu}_{2n}(\omega) = p_0(e^{-i\omega/2})\hat{\mu}_n(\omega/2), \\ \hat{\mu}_{2n+1}(\omega) = p_1(e^{-i\omega/2})\hat{\mu}_n(\omega/2), \end{cases}$$

and for odd n, set

$$\begin{cases} \hat{\mu}_{2n}(\omega) = p_1(e^{-i\omega/2})\hat{\mu}_n(\omega/2), \\ \hat{\mu}_{2n+1}(\omega) = p_0(e^{-i\omega/2})\hat{\mu}_n(\omega/2), \end{cases}$$

Then the collection $\{\mu_l\}_{l=0}^{\infty}$ is the family of Shannon wavelet packets.

Properties

Write $\mu_{l,j,k}(t) = 2^{j/2}\mu_l(2^j t - k)$. Define

$$U_j^l = \text{clos}_{L^2} \ \text{span} \ \{2^{j/2}\mu_l(2^j t - k) : \quad k \in \mathbb{Z}\}, \quad l \geq 0.$$

Then each function in U_j^l is a bandpass signal with lowest band $2^{j-1}l$ and highest band $2^{j-1}(l+1)$ and it satisfies the sub-band coding condition (i).

Proof

- **Lemma 1.** Let $\{I_{j,l} : j \in \Lambda, l \in \Gamma\}$ be a dyadic partition of R^+. Then the family $\{\psi_{j,l,k} : j \in \Lambda, l \in \Gamma, k \in Z\}$ is an orthonormal basis of L^2 and $L^2 = \oplus_{j\in\Lambda, l\in\Gamma} U_j^l$.
- **Lemma 2.** If $f \in U_j^n$, then

$$f(t) = \sum_{k\in Z} f(k)\mu_{n,j,k}(t).$$

Recall that the Nyquist frequency of all functions in U_j^n is 2^j. The proof of the result is based on the dyadic decomposition of the signal in each channel.

More discussion

The Shannon wavelet packet introduced above has a primary bandwidth $1/2$ (i.e., both ϕ and ψ have bandwidth $1/2$). Therefore, each function in the subspace U_j^l has the dyadic bandwidth 2^{j-1}. In application we need to construct the Shannon wavelet packets with primary band different from 2^j.

Wavelet packets with other primary band

For a positive number $\nu \notin 2^j$, define

$$\phi^\nu(t) = (2\nu)^{1/2}\phi(2\nu t), \quad \psi^\nu(t) = (2\nu)^{1/2}\psi(2\nu t).$$

Then both $\phi^\nu(t)$ and $\psi^\nu(t)$ have bandwidth ν. Let

$$\phi^\nu_{j,k}(t) = 2^{j/2}\phi^\nu(2^j t - \frac{k}{2\nu}) = \left(2^{(j+1)/2}\nu^{1/2}\phi(2^{j+1}\nu x - t)\right),$$

$$\psi^\nu_{j,k}(t) = 2^{j/2}\psi^\nu(2^j t - \frac{k}{2\nu}) = \left(2^{(j+1)/2}\nu^{1/2}\psi(2^{j+1}\nu t - k)\right).$$

Then $\{\psi^\nu_{j,k} : j, k \in Z\}$ creates the Shannon wavelet packet has primary band ν.

Shannon wavelet library

Denote the Shannon wavelet packets with the primary band ν by P^ν. Two positive numbers ν and μ are said to be binarily similar if there exists an integer j such that $\nu = 2^j\mu$. Let $B \subset R$ be the set of all numbers that are not binarily similar to each other. Then

$$\{\mathcal{P}^\nu : \quad \nu \in B\}$$

constitutes a Shannon wavelet library.

Use a single packet for sub-band decomposition

Decompose a signal into sub-band signals from a single wavelet packet, say P^ν. In this case, the decomposition always satisfies the sub-band coding conditions (i) and (ii). In addition, all sub-band functions of a sub-band coding obtained in this way are synchronic, and therefore, no additional code is needed for synchronized transmission.

Use the library for sub-band decomposition

We may decompose a signal $f(t)$ into sub-band signals using several packets:

$$f(t) = \sum_{l=1}^{m} \sum_{k=1}^{n_l} f_{lk}(t),$$

where $f_{lk}(t)$ and $f_{l'k'}(t)$ have different primary bands if and only if $l \neq l'$. Thus, the signal f is decomposed by using m different wavelet packets $P^{\nu_l}, 1 \leq l \leq m$. The decomposition is a sub-band decomposition if all the ratios $\nu_k/\nu_{k'}, 1 \leq k, k' \leq m$, are rational numbers. In this case, additional code may be needed for synchronized transmission.

END

THANK YOU