Global support of a scaling vector

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Abstract

Multiwavelet decompositions are based on scaling vectors satisfying matrix refinement equations. The support and linear independence of scaling vectors play an essential role in the study of multiwavelets. In this paper we relate these properties with the coefficients in the matrix refinement equation satisfied by the scaling vector.

By having more than one wavelet for a multi-resolution analysis, there is added flexibility in choosing desirable properties of the wavelets. When considering $r > 1$ wavelets, the usual two-scale refinement equation becomes a matrix refinement equation (MRE) with $r \times r$ matrix coefficients C_k and the scaling function becomes a scaling vector. To be precise, we say $\Phi(x) = [\phi_1(x), \phi_2(x), \ldots, \phi_r(x)]^T$, a complex vector-valued function on the reals $\mathbf{R}$, is a *scaling vector* if it satisfies a MRE with a finite number of terms:

$$\Phi(x) = \sum_{k=0}^{N} C_k \Phi(2x - k) \tag{1}$$

where we assume $N \geq 1$ to avoid trivialities. For the rest of the paper, Φ is assumed to be a compactly supported scaling vector satisfying the MRE (1), and we define the *support* of Φ to be the convex hull of $\{x \in \mathbf{R} : \Phi(x) \neq 0\}$. In the case of a single scaling function, $r = 1$, it is known [1] that

$$supp\,(\Phi) = [0, N] \quad \Leftrightarrow \quad C_0, C_N \neq 0 \tag{2}$$

for a solution Φ to (1). However, when $r > 1$, the situation is less clear.

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In this note we focus on the connection between the support of the scaling vector Φ, the linear independence of the components ϕ_j and their integer translates, and the matrices C_k. First we collect some existing results about the support of scaling vectors. An immediate result of a compactly supported scaling vector, which is observed in [2], is

Theorem 1. *supp*$(\Phi) \subset [0, N]$.

The next result is a direct generalization of the scaling function case in [1].

Theorem 2. If C_0 and C_N are invertible then *supp*$(\Phi) = [0, N]$.

However the converse of Theorem 2 is not true.

Example 3. Consider the MRE

$$\Phi(x) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Phi(2x) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Phi(2x-1) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Phi(2x-2).$$

$\Phi(x) = \begin{bmatrix} \chi_{(0,1]} \\ \chi_{(1,2]} \end{bmatrix}$ is a solution with *supp*$(\Phi) = [0, 2]$ and C_0, C_2 are both not invertible.

A partial converse of Theorem 2 is the following theorem from [3].

Theorem 4. If *supp*$(\Phi) = [0, N]$ then C_0 and C_N are not nilpotent.

The converse of Theorem 4 is not true.

Example 5. Consider the MRE

$$\Phi(x) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Phi(2x) + \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \Phi(2x-1) + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \Phi(2x-2)$$
$$+ \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \Phi(2x-3) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Phi(2x-4).$$

$\Phi(x) = \begin{bmatrix} \chi_{(0,1]} \\ \chi_{(1,2]} \end{bmatrix}$ is a solution with *supp*$(\Phi) = [0, 2] \neq [0, 4]$ and C_0, C_4 are both not nilpotent.

Now we state our main result as follows.

Theorem 6. Suppose $\Phi(x)$ is a globally linearly independent scaling vector. $Supp(\Phi) = [0, N]$ if and only if C_0 and C_N are not nilpotent.

Before we give the proof, we introduce some preliminaries. $\Phi(x)$ is said to be *globally linearly independent* if $\sum_{k=-\infty}^{\infty} a_k^T \Phi(x-k) = 0$ implies that $a_k^T = 0$ for all k.

Let $D_0, \ldots, D_N$ be matrices such that the first row of D_N is of the form $\lambda e^T = \begin{bmatrix} \lambda & 0 & \cdots & 0 \end{bmatrix}$ and $\lambda \neq 0$. For an integer $k \geq 0$ and an integer t such that $k < 2^t$, we define the row vector

$$a_k^T(t) = (\frac{1}{\lambda})^t e^T \left[\ \sum_{(k_1,\ldots,k_t) \in D(k,t)} D_{N-k_1} \cdots D_{N-k_t} \ \right]$$

where

$$D(k,t) = \left\{ (k_1, \ldots, k_t) : \sum_{i=1}^{t} k_i 2^{t-i} = k \ \text{ and } \ 0 \leq k_i \leq N \right\}.$$

Since $k < 2^t$ and $N \geq 1$, $D(k,t) \neq \emptyset$. Note that if $k < 2^{t_1} \leq 2^{t_2}$ then

$$D(k, t_2) = \{(0, \ldots, 0, k_1, \ldots, k_{t_1}) : (k_1, \ldots, k_{t_1}) \in D(k, t_1)\},$$

and

$$\begin{aligned}
& a_k^T(t_2) \\
= \ & (\frac{1}{\lambda})^{t_2} e^T \left[\ \sum_{(h_1,\ldots,h_{t_2}) \in D(k,t_2)} D_{N-h_1} \cdots D_{N-h_{t_2}} \ \right] \\
= \ & (\frac{1}{\lambda})^{t_2} e^T \left[\ \sum_{(k_1,\ldots,k_{t_1}) \in D(k,t_1)} D_N \cdots D_N D_{N-k_1} \cdots D_{N-k_{t_1}} \ \right] \\
= \ & (\frac{1}{\lambda})^{t_2} e^T [D_N]^{(t_2-t_1)} \left[\ \sum_{(k_1,\ldots,k_{t_1}) \in D(k,t_1)} D_{N-k_1} \cdots D_{N-k_{t_1}} \ \right] \\
= \ & (\frac{1}{\lambda})^{t_2} \lambda^{t_2-t_1} e^T \left[\ \sum_{(k_1,\ldots,k_{t_1}) \in D(k,t_1)} D_{N-k_1} \cdots D_{N-k_{t_1}} \ \right] \\
= \ & (\frac{1}{\lambda})^{t_1} e^T \left[\ \sum_{(k_1,\ldots,k_{t_1}) \in D(k,t_1)} D_{N-k_1} \cdots D_{N-k_{t_1}} \ \right] \\
= \ & a_k^T(t_1).
\end{aligned}$$

Hence $a_k^T(t)$ depends on k but not t. Moreover, for $k = 0$, we can take $t = 1$ and so $D(0, 1) = \{1\}$. Then

$$a_0^T = \frac{1}{\lambda} e^T D_N = e^T \neq 0.$$

Lemma 7. If Ψ satisfies

$$\Psi(x) = \sum_{k=0}^{N} D_k \Psi(2x - k) \tag{3}$$

and $\Psi(x) = 0$ on $[N - \delta, \infty)$ where $0 < \delta < 1$, then for integer $t \geq 1$

$$\sum_{k=0}^{2^t-1} \left[\textstyle\sum_{(k_1,\ldots,k_t)\in D(k,t)} D_{N-k_1} \cdots D_{N-k_t} \right] \Psi(u + k) = 0$$

on $[N - 2^t\delta, \infty)$.

Proof. By induction on t. Since $\Psi(x) = 0$ on $[N - \delta, \infty)$, by the equation (3), we have

$$\sum_{k=0}^{N} D_k \Psi(2x - k) = 0 \quad \text{on} \quad [N - \delta, \infty).$$

By the substitution $u = 2x - N$ and re-indexing, we obtain

$$\sum_{k=0}^{N} D_{N-k} \Psi(u + k) = 0 \quad \text{on} \quad [N - 2\delta, \infty).$$

Note that, for $k \geq 2$, $\Psi(u+k) = 0$ on $[N - 2\delta, \infty)$. Hence the statement is true for $t = 1$ because of $D(k, 1) = \{k\}$ for $k \leq N$. Assume the conclusion is true for t, i.e.

$$\sum_{h=0}^{2^t-1} \left[\textstyle\sum_{(k_1,\ldots,k_t)\in D(h,t)} D_{N-k_1} \cdots D_{N-k_t} \right] \Psi(w + h) = 0$$

on $[N - 2^t\delta, \infty)$. Replacing $\Psi(w + h)$ by the equation (3) and using the substitution $u = 2w - N$, we obtain

$$\sum_{h=0}^{2^t-1} \left\{ \left[\textstyle\sum_{(k_1,\ldots,k_t)\in D(h,t)} D_{N-k_1} \cdots D_{N-k_t} \right] \sum_{h_{t+1}=0}^{N} D_{h_{t+1}} \Psi(u + 2h + N - h_{t+1}) \right\} = 0$$

on $[N - 2^{t+1}\delta, \infty)$. Re-indexing by $k_{t+1} = N - h_{t+1}$,

$$\sum_{h=0}^{2^t-1} \left\{ \left[\textstyle\sum_{(k_1,\ldots,k_t)\in D(h,t)} D_{N-k_1} \cdots D_{N-k_t} \right] \sum_{k_{t+1}=0}^{N} D_{N-k_{t+1}} \Psi(u + 2h + k_{t+1}) \right\} = 0$$

on $[N-2^{t+1}\delta,\infty)$. Re-indexing by $k=2h+k_{t+1}$, we have

$$\sum_{k=0}^{2^{t+1}-1}\left[\ \textstyle\sum_{(k_1,\ldots,k_{t+1})\in D(k,t+1)} D_{N-k_1}\cdots D_{N-k_t}D_{N-k_{t+1}}\ \right]\Psi(u+k)=0$$

on $[N-2^{t+1}\delta,\infty)$. Here we use the fact that $\Psi(u+k)=0$ for $k\geq 2^{t+1}$ on $[N-2^{t+1}\delta,\infty)$, and $(k_1,\ldots,k_t)\in D(h,t)$ if and only if $(k_1,\ldots,k_{t+1})\in D(k,t+1)$, which is true because $k=2h+k_{t+1}$. $\square$

Proof of Theorem 6. Because of Theorem 4, we only need to prove the sufficiency part. Suppose that $supp(\Phi)=[L,R]\neq[0,N]$. Then, by Theorem 1, $L>0$ or $R<N$. We will use the "propogation" idea of Lemma 7 to show that $R<N$ contradicts the global linear independence of Φ. (The proof that $L>0$ contradicts the global linear independence of Φ is similar.) Since C_N is not nilpotent, there exists invertible matrix S such that the first row of SC_NS^{-1} is of the form $\lambda e^T=\left[\begin{array}{cccc}\lambda & 0 & \cdots & 0\end{array}\right]$ where $\lambda\neq 0$. Let $\Psi(x)=S\Phi(x)$. Then $supp(\Psi)=supp(S\Phi)=supp(\Phi)=[L,R]$ and

$$\Psi(x)=\sum_{k=0}^{N}D_k\Psi(2x-k)$$

where $D_k=SC_kS^{-1}$. Because $R<N$, there exists $0<\delta<1$ such that $supp(\Psi)\subset[0,N-\delta]$. Hence

$$\Psi(x)=0 \quad\text{on}\quad [N-\delta,\infty).$$

By Lemma 7, for integer $t\geq 1$,

$$\sum_{k=0}^{2^t-1}\left[\ \textstyle\sum_{(k_1,\ldots,k_t)\in D(k,t)} D_{N-k_1}\cdots D_{N-k_t}\ \right]\Psi(u+k)=0$$

on $[N-2^t\delta,\infty)$. Multiplying both sides by $(\frac{1}{\lambda})^te^T$ yields

$$\sum_{k=0}^{2^t-1}(\frac{1}{\lambda})^te^T\left[\ \textstyle\sum_{(k_1,\ldots,k_t)\in D(k,t)} D_{N-k_1}\cdots D_{N-k_t}\ \right]\Psi(u+k)=0$$

on $[N-2^t\delta,\infty)$. Thus, by the definition of $a_k^T(t)$,

$$\sum_{k=0}^{2^t-1}a_k^T(t)\Psi(u+k)=0 \quad\text{on}\quad [N-2^t\delta,\infty).$$

Recall that $a_k^T(t)$ is independent of t and indeed depends only on k. Consequently, when t goes to infinity, we have

$$\sum_{k=0}^{\infty} a_k^T \Psi(u+k) = 0 \quad \text{on} \quad (-\infty, \infty).$$

It follows that

$$\sum_{k=0}^{\infty} (a_k^T S)\Phi(u+k) = 0 \quad \text{on} \quad (-\infty, \infty).$$

Recall that $a_0^T = e^T \neq 0$ and S is invertible, so $a_0^T S \neq 0$. This contradicts the fact that Φ is globally linearly independent. $\square$

A related unsolved problem is to investigate the possibility of weakening the hypothesis of global linear independence in Theorem 6 to stability or finite linear independence.

References

[1] C. K. Chui and J. Wang, A study of compactly supported scaling functions and wavelets, In *Wavelets, Images, Surface Fitting*, P. J. Laurant, A. Le Mehaute, and L. L. Schumaker (eds.), 121–140 (1994) Wellesley.

[2] C. Heil and D. Colella, Matrix refinement equations: existence and uniqueness, *J. Fourier Anal. Appl.*, 2 (1996) 363–377.

[3] P. R. Massopust, D. K. Ruch, and P. J. Van Fleet, On the support properties of scaling vectors, *Applied and Computational Harmonic Analysis*, 3(1996) 229–238.