15. Multivariate Wavelet Bases

15.1. Tensor Product. The tensor product is a simplest and effective method to construct the wavelet bases of high dimension. Here we only introduce the wavelet bases of two variables. For the variables more than two, the method is same.

Definition 15.1. Let $U_1$ and $U_2$ be two subspace of $L^2(\mathbb{R})$. The space 
\[ \{ f(x, y) := f_1(x)f_2(y); \ f_1 \in U_1, f_2 \in U_2 \} \]
is called the tensor product of $U_1$ and $U_2$ and denoted by $U_1 \otimes U_2$.

Assume that \( \{V_j\}_{j \in \mathbb{Z}} \) is an MRA of $L^2(\mathbb{R})$. Then \( \{V_j\}_{j \in \mathbb{Z}} \) is an MRA of $L^2(\mathbb{R}^2)$, where \( V_j = V_j \otimes V_j \).

Let $\phi$ be the refinable function generating the MRA \( \{V_j\}_{j \in \mathbb{Z}} \).

It is easy to verify that \( \{\phi_{jn}(x)\phi_{jm}(y)\}_{n,m \in \mathbb{Z}} \) is an orthonormal bases of $V_j$, i.e.,
\[ \Phi(x, y) := \phi(x)\phi(y) \]
is a generator of the MRA \( \{V_j\}_{j \in \mathbb{Z}} \).

We now construct the wavelet subspace of $L^2(\mathbb{R}^2)$. Let \( \{W_j\}_{j \in \mathbb{Z}} \) be the wavelet subspace corresponding to the MRA \( \{V_j\}_{j \in \mathbb{Z}} \) of $L^2(\mathbb{R})$. Define
\[ W_j^h = V_j \otimes W_j \quad W_j^v = W_j \otimes V_j \quad W_j^d = W_j \otimes W_j. \]

We have 
\[ V_{j+1} = V_j \oplus W_j^h \oplus W_j^v \oplus W_j^d. \]
Let 
\[ \{\psi_{jn}(x)\}_{n \in \mathbb{Z}} \]
be the orthonormal basis of $W_j$. Then the orthonormal bases in $W_j^h$, $W_j^v$ and $W_j^d$ are
\[ \{\phi_{jn}(x)\psi_{jm}(y)\}_{n,m \in \mathbb{Z}}, \]
\[ \{ \psi_{jn}(x) \phi_{jm}(y) \}_{n,m \in \mathbb{Z}}, \]

and

\[ \{ \psi_{jn}(x) \psi_{jm}(y) \}_{n,m \in \mathbb{Z}} \]

respectively. Therefore these four subspaces are mutually orthogonal. It follows that

\[ L^2(\mathbb{R}^2) = \bigoplus_{j \in \mathbb{Z}} W^h_j \bigoplus_{j \in \mathbb{Z}} W^v_j \bigoplus_{j \in \mathbb{Z}} W^d_j. \]

15.2. Decompose and Recover Bivariate Functions. Let \( f \in L^2(\mathbb{R}^2) \) and \( P_j \) be the orthogonal project operator from \( L^2 \) to \( V_j \). Since \( \{ V_j \} \) is an MRA, there is a large \( j \) such that the \( f_j = P_j f \) is close to \( f \), i.e., \( \| f - f_j \| \approx 0 \). Hence we start from the decomposition of \( f_j \). Let \( f_j(x, y) \in V_j \) be expanded as

\[ f_j(x, y) = \sum a_{j,k,l} \phi_{j,k}(x) \phi_{j,l}(y). \]

We first decompose \( f_j \) into

\[ f_j(x, y) = f_{j-1}(x, y) + g^h_{j-1}(x, y) + g^v_{j-1}(x, y) + g^d_{j-1}(x, y), \]

where

\[ f_{j-1}(x) = \sum a_{j-1,k,l} \phi_{j-1,k}(x) \phi_{j-1,l}(y) \in V_{j-1}, \]

\[ g^h_{j-1}(x) = \sum b_{j-1,k,l} \phi_{j-1,k}(x) \psi_{j-1,l}(y) \in W^h_{j-1}, \]

\[ g^v_{j-1}(x) = \sum b_{j-1,k,l} \psi_{j-1,k}(x) \phi_{j-1,l}(y) \in W^v_{j-1}, \]

and

\[ g^d_{j-1}(x) = \sum b_{j-1,k,l} \psi_{j-1,k}(x) \psi_{j-1,l}(y) \in W^d_{j-1}. \]

Assume that

\begin{align*}
\phi(x) &= 2 \sum h(k) \phi(2x - k) \\
\psi(x) &= 2 \sum g(k) \phi(2x - k),
\end{align*}

are the pair of orthonormal scaling function and wavelet. Then \( g(k) = (-1)^k h(N - k) \) with an odd \( N \). From the result for one dimension, we have

\[ a_{j-1,k,l} = 2 \sum h(s - 2k) h(t - 2l) a_{j,s,t}, \]

\[ b^h_{j-1,k,l} = 2 \sum g(s - 2k) h(t - 2l) a_{j,s,t}, \]

\[ b^v_{j-1,k,l} = 2 \sum h(s - 2k) g(t - 2l) a_{j,s,t}, \]

and

\[ b^d_{j-1,k,l} = 2 \sum g(s - 2k) g(t - 2l) a_{j,s,t}. \]
This is *Fast 2-D Wavelet Transform*. We also can obtain the *Fast 2-D Inverse Wavelet Transform* which recovers a function from its wavelet coefficients.

\[ a_{j,k,l} = 2 \left( \sum_{s,t} h(s-2k)h(t-2l)a_{j-1,s,t} + \sum_{s,t} h(s-2k)g(t-2l)b_{j-1,s,t}^h \right. \]

\[ + \left. \sum_{s,t} g(s-2k)h(t-2l)b_{j-1,s,t}^v + \sum_{s,t} g(s-2k)g(t-2l)b_{j-1,s,t}^d \right). \]
16. Wavelet Packet

16.1. Construct Wavelet Packets. Assume that \( \phi \) is an orthonormal refinable function and its corresponding orthonormal wavelet is \( \psi \). They are satisfy (15.1) and (15.2). Note that the usual wavelet decomposition is not a tree structure. In order to obtain more orthonormal bases from the refinable function \( \phi \) and wavelet \( \psi \), we introduce the wavelet packets. Here we only discuss the 1-D wavelet packets. It is not difficulty to generalize it to high dimensions.

We first rewrite \( \phi = \mu_0 \) and \( \psi = \mu_1 \). Correspondingly, write \( p_0(z) = h(k)z^k \) and \( p_1(z) = g(k)z^k \). Then the Fourier transform of equations (15.1) and eqrefrefin2 are

\[
\hat{\mu}_0(\omega) = p_0(e^{-i\omega/2})\hat{\mu}_0(\omega/2)
\]
\[
\hat{\mu}_1(\omega) = p_1(e^{-i\omega/2})\hat{\mu}_0(\omega/2)
\]

respectively.

Definition 16.1. Let the system of functions \( \{\mu_l\}_{l=0}^\infty \) is defined inductively by

\[
\hat{\mu}_{2n}(\omega) = p_0(e^{-i\omega/2})\hat{\mu}_n(\omega/2)
\]
\[
\hat{\mu}_{2n+1}(\omega) = p_1(e^{-i\omega/2})\hat{\mu}_n(\omega/2) \quad n = 0, 1, \ldots.
\]

Then this system of functions is called a wavelet packet of \( L^2 \).

To find the Fourier transform of \( \mu_l \), \( 0 \leq l < \infty \), we assume the dyadic expansion of a nonnegative integer \( n \) is

\[
n = \sum_{j=1}^\infty e_j 2^{j-1}, \quad e_j = 0 \text{ or } 1.
\]

Then

\[
\hat{\mu}_n(\omega) = \prod_{j=1}^\infty p_{e_j}(e^{-i\omega/2^j}).
\]

We have the following.

Theorem 16.1. Let \( \mu_n \) be defined by (16.3) and (16.4). Then

\[
\int_{\mathbb{R}} \mu_n(x-j)\mu_n(x-k) \, dx = \delta_{jk}
\]

and

\[
\int_{\mathbb{R}} \mu_n(x-j)\mu_{n+1}(x-k) \, dx = 0.
\]

This theorem can be proved in the same way in the proof of the orthogonality of \( \phi \) and \( \psi \).
16. WAVELET PACKET

16.2. Construct Orthonormal Bases Using Wavelet Packets. Defining now

\[ U^n_j = \text{clos}_{L^2} \{ 2^{j/2} \mu_n(2^j x - k) ; k \in \mathbb{Z} \} , \]

we have

\[ U^{n+1}_j = U^{2n}_j + U^{2n+1}_j , \quad j \in \mathbb{Z}, \ n \in \mathbb{Z}^+ . \]

Using this decomposition, we can obtain many different orthonormal bases of \( L^2 \).