Lecture Notes on Filter Banks and Wavelets

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1. Introduction

1.1. Functions and Digital Signals (Images).

1.1.1. $L^2$ functions and their Fourier transforms. $L^2$ space is often used in the wavelet course. It is defined as follows.

$$L^2 = \left\{ f ; \quad \| f \|_2 := \left( \int_{\mathbb{R}} |f(x)|^2 \, dx \right)^{1/2} < \infty \right\},$$

where $\| f \|_2$ is called the norm (or the energy) of $f \in L^2$.

Sometimes other spaces, such as Lebesgue spaces $L^p$, $1 \leq p \leq \infty$, Sobolev spaces $W^{s,p}$, and distribution space $\mathcal{D}$, are also in use.

The Fourier Transform of a function $f \in L^2$ is defined by

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-i\omega t} \, dt$$

The inverse Fourier transform is

$$f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\omega) e^{i\omega t} \, d\omega$$

Example 1. Let

$$f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$

Then its Fourier transform is

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(x) e^{-i\omega x} \, dx = \frac{2 \sin \omega}{\omega}$$

Let $< f, g >$ be the inner product of $f, g \in L^2$: $\int_{\mathbb{R}} f(x)g(x) \, dx$. We have the following Parseval Formula,

$$< f, \hat{g} > = < \hat{f}, g >.$$

By the inverse Fourier transform formula, we also have

$$< f, g > = 2\pi < \hat{f}, \hat{g} >.$$

Using Parseval Formula, we can extend Fourier transform on other spaces. For example, let $S$ be the space of rapidly decreasing functions and $S^*$ is the dual space of $S$, that is the space containing all slowly increasing distributions. Then for $f \in S^*$, we define $\hat{f}$ by the following way. for each $g \in S$,

$$< \hat{f}, g > = < f, \hat{g} >.$$

Example 2. Delta function $\delta(x)$ is a linear and bounded functional on $C(\mathbb{R})$ defined by

$$\int_{\mathbb{R}} \delta(x)f(x) \, dx = f(0)$$

Then the Fourier transform of $\delta(x)$ is

$$\int_{\mathbb{R}} \delta(x)e^{-i\omega x} \, dx = 1.$$
1. INTRODUCTION

Generally, we have

\[(1.1) \int_{\mathbb{R}} \delta(x - x_0)e^{-ix\omega}dx = e^{-ix_0\omega}.\]

From (1.1), we get

\[
\frac{1}{2\pi} \int_{\mathbb{R}} \delta(\omega - \omega_0)e^{ix\omega}d\omega = \frac{1}{2\pi}e^{ix\omega_0}.
\]

Using the inverse Fourier transform, we get the Fourier transform of \(e^{ix\omega_0}\),

\[
\int_{\mathbb{R}} e^{ix\omega_0}e^{-ix\omega}dx = 2\pi\delta(\omega - \omega_0).
\]

1.1.2. Digital signals (Image). Sequence \(u = (u(n))_{-\infty}^{\infty}\) is called a signal. An image usually is represented by a two-dimensional sequence \(q = (q(n,m))_{n,m \in \mathbb{Z}}\). (or a vector of two-dimensional sequences for a color image.) For signals we introduce the sequence space

\[
l^2 = \{u; \|u\|_2 := \left(\sum |u(n)|^2\right)^{1/2} < \infty\},
\]

where \(\|u\|_2\) is the \(l^2\) norm (or energy) of signal \(u\).

Similarly, we can define spaces \(l^p\), \(1 \leq p \leq \infty\) by

\[
l^p = \{u; \|u\|_p := \left(\sum |u(n)|^p\right)^{1/p} < \infty\}.
\]

1.2. Transforms. A way to obtain new representations of functions or signals

1.2.1. Fourier series. Fourier series of a periodic function is a transform from periodic functions to sequences. Let \(\tilde{L}^2\) be the space of all square integrable \(2\pi\)-periodic functions. The norm of \(f \in \tilde{L}^2\) is

\[
\left(\frac{1}{2}\int_{-\pi}^{\pi} |f(x)|^2dx\right)^{1/2}.
\]

The (complex) Fourier series of \(f(x) \in \tilde{L}^2\) is

\[(1.2) f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx},\]

where \(c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-ikx}dx\). Note that the equality in (1.2) holds in the \(\tilde{L}^2\) sense, i.e.,

\[
\lim_{n,m \to \infty} \|f(x) - \sum_{k=-n}^{m} c_k e^{ikx}\|_{\tilde{L}^2} = 0.
\]

An important result for \(f(x) \in \tilde{L}^2\) is that \(f(x) \in \tilde{L}^2\) if and only if the coefficient sequence \((c_n) \in l^2\). Moreover, we have

\[
\|f\|_{\tilde{L}^2} = \|c\|_2.
\]
The Fourier series of a function defined on a finite interval can be obtained in a similar way.

**Example 3.** The Fourier series of function \( y = x, -l < x < l \) is

\[
y = \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n \pi x}{n} \sin \frac{n \pi x}{l}
\]

1.2.2. **Filter Bank Transform.** Filter bank transform is a transform from signals to signals. Haar Transform is the simplest 2-channel filter bank transform, which changes a sequence into a pair of sequences. For a signal \( x \), its Haar transform is

\[
\begin{align*}
y_0 &= H_0 x, & y_0(n) &= \frac{x((2n-1)+x(2n))}{2}, \\
y_1 &= H_1 x, & y_1(n) &= \frac{x(2n)-x(2n-1)}{2}.
\end{align*}
\]

**Example 4.** The Haar transform of a finite sequence

\[
x = (x(1), x(2), x(3), x(4))
\]

is

\[
\begin{align*}
y_0 &= (y_0(1), y_0(2)), & y_0(1) &= \frac{x(1)+x(2)}{2}, y_0(2) &= \frac{x(3)+x(4)}{2}, \\
y_1 &= (y_1(1), y_1(2)), & y_1(1) &= \frac{x(1)-x(2)}{2}, y_1(2) &= \frac{x(3)-x(4)}{2}.
\end{align*}
\]

There are two kinds of transforms. (a) Invertible (lossless) transforms. (b) Non-invertible (lossy) transforms. The purposes of transforms are the followings.

- To obtain good structure of functions or signals. (To find patterns.)
- To delete the redundancy in the original representation. (Image compression.)
- To create new signals and functions, that model the objects more precisely. (Signal processing, de-noising.)

1.3. **Wavelet Transform and Wavelet Basis.**

1.3.1. **Continuous wavelet transform.** Wavelet is used to represent a function in time-frequency domain locally.

**Definition 1.1.** A function \( h \in L^1 \cap L^2 \) is said to be a wavelet function if it satisfies the zero-moment condition

\[
\int_{\mathbb{R}} h(x) \, dx = 0.
\]
We can verify that the zero-moment condition implies

\[ C_h = \int \frac{|\hat{h}(\omega)|^2}{|\omega|} d\omega < \infty. \]

A wavelet is expected to be “local” in both time domain and frequency domain. The locality can be defined in the following way.

**Definition 1.2.** Let \( h \) be a wavelet.

\[ t_0 = \frac{\int t|h(t)|^2 dt}{\|h\|_2^2} \]

is called the time center of \( h \), while

\[ \omega_0^\pm = \frac{\int_{0 \leq \omega \leq \infty} \omega |\hat{h}(\omega)|^2 d\omega}{\|\hat{h}\|_2^2} \]

is called the (positive and negative respectively) frequency center of \( h \); and

\[ \triangle_h = \left( \frac{\int (t - t_0)^2|h(t)|^2 dt}{\|h\|_2^2} \right)^{1/2} \]

is called the time width of \( h \) while

\[ \triangle_{\hat{h}}^\pm = \left( \frac{\int_{0 \leq \omega \leq \infty} (\omega - \omega_0^\pm)^2|\hat{h}(\omega)|^2 d\omega}{\|\hat{h}\|_2^2} \right)^{1/2} \]

is called the (positive and negative respectively) frequency width of \( h \).

**Remark 1.1.** For a function \( h(t) \in L^2 \) (not necessarily to be a wavelet), we usually assign it a single frequency center and therefore a single frequency width as follows.

\[ \omega_0 = \frac{\int_{\mathbb{R}} \omega |\hat{h}(\omega)|^2 d\omega}{\|\hat{h}\|_2^2} \]

and

\[ \triangle_{\hat{h}} = \left( \frac{\int_{\mathbb{R}} (\omega - \omega_0)^2|\hat{h}(\omega)|^2 d\omega}{\|\hat{h}\|_2^2} \right)^{1/2} \]

If \( h(t) \) and \( th(t) \) both are in \( L^2 \), then the time width and the frequency width both are finite. We call \( h \) a window function. A window wavelet function has finite time width and (positive and negative respectively) frequency width.

The time width and frequency width measure the locality of a function in time domain as well in frequency domain. The rectangle centered in \((t_0, \omega_0)\) and with the length and width \( \triangle_h, \triangle_{\hat{h}} \) is called the time-frequency window of \( h \). The size of any time-frequency window is restricted by Heisenberg Uncertainty Principle.
Theorem 1.1 (Heisenberg Uncertainty Principle). If \( g \) is a window function, then

\[
\Delta_g \Delta \hat{g} \leq \frac{1}{2},
\]

where the equality holds if and only if \( g \) is a Gaussian window, i.e.,

\[
g(x) = \frac{1}{(2\pi)\sigma^{1/2}} e^{-\frac{(t-t_0)^2}{4\sigma^2}}, \quad t_0 \in \mathbb{R}, \sigma \in \mathbb{R}^+.
\]

We now go back to discuss continuous wavelet transform. Write

\[
h_{ab}(x) = |a|^{-1/2} h\left(\frac{x-b}{a}\right).
\]

Definition 1.3. Let \( h \) be a wavelet function. The continuous wavelet transform of \( f \) with respect to wavelet \( h \) is defined by

\[
W_f(a, b) = \langle h_{ab}, f \rangle = |a|^{-1/2} \int f(x) h\left(\frac{x-b}{a}\right) dx.
\]

Continuous wavelet transform is a lossless transform. Its inverse wavelet transform is given by the following theorem.

Theorem 1.2. Let \( W_f(a, b) \) be the continuous wavelet transform of function \( f \in L^2 \) with respect to \( h \) and \( C_h \) is defined by (1.7). Then the inverse wavelet transform is given by

\[
f(x) = C_h^{-1} \int_{\mathbb{R}^2} W_f(a, b) h_{ab}(x) \frac{dadb}{a^2}.
\]

1.3.2. Wavelet basis.

Definition 1.4. Let \( \psi \) be a wavelet. If the sequence of functions

\[
\{\psi_{kj} := 2^{k/2}\psi(2^k x - j); \quad k, j \in \mathbb{Z}\}.
\]

forms an unconditional basis of \( L^2 \), then it is called a wavelet basis of \( L^2 \).

If we assume that \( \{\psi_{kj}\}_{k,j \in \mathbb{Z}} \) is an orthonormal basis of \( L^2 \), then for any function \( f \in L^2 \), we have

\[
f(x) = \sum_{k,j \in \mathbb{Z}} d_{kj} \psi_{kj}(x),
\]

where

\[
d_{kj} = 2^{k/2} \int f(x) \overline{\psi(2^k x - j)} dx.
\]
1.4. Purposes of The Course. In this course we shall discuss the following problems.

- How to transform a function to a signal (or to a image if the function is a two variable function).
- How to decompose a signal (image) using filter banks.
- The relation of the filter banks and the wavelets.
- How to expand a function into a wavelet series.
- How to predict the properties of a function from its wavelet expansion.
- How to judge a wavelet basis good or not good.
- How to construct a good wavelet basis.
- How to apply wavelets in signal and image procession.