Shannon Wavelet Approach to Sub-Band Coding

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Abstract

It is well-known that the Shannon Sampling Theorem allows us to fully recover a continuous-time bandlimited signal from its digital samples, as long as the sampling rate to be chosen is not smaller than the Nyquist frequency. This theory applies to all bandlimited signals, which may or may not occupy the entire frequency band. Hence, it is intuitively convincing that for continuous-time signals, such as those in speech, that do not fully utilize the entire frequency intervals, less digital samples are required for their full recovery. Current techniques in sub-band coding are used for achieving this goal. The objective of this paper is to present a wavelet theory for establishing the mathematical foundation of this sub-band coding approach. A wavelet packet decomposition of the signal provides the optimal sub-band coding bit-rate by using the Shannon wavelet library introduced in this paper.

1. Introduction

All continuous-time signals $f(t)$ to be considered in this paper are real-valued functions in $L^2 := L^2(-\infty, \infty)$, with Fourier transform $\hat{f}(\omega)$ defined by the L^2-limit

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt := L^2\text{-}\lim_{N\to\infty} \int_{-N}^{N} f(t)e^{-i\omega t}dt. \qquad (1)$$

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Therefore, it follows that $\hat{f}(-\omega) = \overline{\hat{f}(\omega)}$, where the "bar" notation stands for complex conjugation; and hence, $|\hat{f}(\omega)|$ is an even function. The inverse Fourier transform $g^{\vee}(t)$ of $g(\omega)$ is defined, again as an L^2-limit, namely

$$g^{\vee}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)e^{it\omega}d\omega = L^2\text{-lim}_{N\to\infty}\int_{-N}^{N} g(\omega)e^{it\omega}d\omega.$$

In this paper, the support of a function $g(\omega)$ is defined by the set

$$\text{supp } g = \text{clos}\{\omega : g(\omega) \neq 0\}, \tag{2}$$

where the "clos" notation stands for the closure of the set, using the usual Eucledian metric.

A continuous-time signal $f(t)$ is said to be bandlimited, if the set supp$f$ is bounded. For such a bandlimited signal $f(t)$, since $|\hat{f}(\omega)|$ is even, the size of its support is uniquely determined by the bounded set

$$\text{supp}^+ \hat{f} := \left(\text{supp}\hat{f}\right) \cap [\,0, \infty) \tag{3}$$

of non-negative reals. The well-known Sampling Theorem allows us to recover the continuous-time bandlimited signal $f(t)$ from certain digital samples $f(kT), t > 0$, by using the sampling function

$$\phi(t) = \text{sinc } t := \frac{\sin \pi t}{\pi t}, \tag{4}$$

which is usually called the "sinc" function. A precise statement of this theorem is the following.

Shannon Sampling Theorem. *A continuous-time bandlimited signal $f(t)$ with*

$$\text{supp}^+ \hat{f} \subset [0, \pi\sigma] \tag{5}$$

has the infinite series representation

$$f(t) = \sum_{k=-\infty}^{\infty} f(kT)\text{sinc}\,(\sigma(t - kT)) \tag{6}$$

where $T = \frac{1}{\sigma}$.

In signal processing, frequency is measured in Hz, meaning "number of cycles per second." If a bandlimited signal $f(t)$ satisfies (5), with 0 and $\pi\sigma$

being the greatest lower bound (glb) and least upper bound (lub) of $\mathrm{supp}^+\hat{f}$, then $f(t)$ is called a lowpass signal, with "lowest band" 0 and "highest band" $\pi\sigma/2\pi = \sigma/2$, and σ is called the "Nyquist frequency" of $f(t)$. Hence, for this signal, the Sampling Theorem asserts that $f(t)$ can be recovered from its digital samples $\{f(k/\mu)\}$, where the "sampling rate" μ must satisfy $\mu \geq \sigma$.

As an example, to sample a speech signal with highest band $4kHz$, the sampling rate must be at least $8kHz$ to avoid distortion. Another example is that the sampling rate of high-quality music signals (with highest band 22.05 kHz) is at least 44.1 kHz.

On the other hand, there are signals with positive lowest band. These are called bandpass signals. To be specific, if a bandpass signal $f(t)$ satisfies

$$\mathrm{supp}^+\hat{f} \subset [2\pi\sigma_1, 2\pi\sigma_2] \tag{7}$$

where $0 < \sigma_1 < \sigma_2$, and $2\pi\sigma_1$ and $2\pi\sigma_2$ are the glb and lub of $\mathrm{supp}^+ f$, respectively, we call σ_1 and σ_2 the lowest band and highest band, respectively, of the bandpass signal $f(t)$. In addition, $\sigma := \sigma_2 - \sigma_1$ is called the bandwidth of $f(t)$. For such bandpass signals, if σ_1/σ is an integer, the sampling rate can be reduced from the Nyquist frequency $2\sigma_2$ to 2σ (see [7]). For this reason, 2σ is also called the Nyquist frequency for bandpass signals when σ_1/σ is an integer.

When σ_1/σ is not an integer, it was shown in [5] that the smallest sampling rate for perfect recovery of $f(t)$ is given by

$$\sigma_m = \sigma \frac{1 + \sigma_1/\sigma}{1 + \lfloor \sigma_1/\sigma \rfloor} \tag{8}$$

where the "floor" notation $\lfloor x \rfloor$ stands for the integer part of x.

Finally, we remark that a signal $f(t)$ that satisfies (7) with $\sigma_1 > 0$ can be considered either as a highpass signal or lowpass signal according to the way it is sampled and coded. If it is considered as a highpass signal, then $2(\sigma_2 - \sigma_1)$ could be chosen as the sampling rate; otherwise $2\sigma_2$ could be chosen as the sampling rate when $f(t)$ is considered as a lowpass signal.

The sampling theorem for bandpass signals has been applied in the study of sub-band coding (see [2], [3], [4], [10]), and is often considered a fundamental result for multiple-channel synchronized transmission. On the other hand, even as early as the late 1980's, it has been clear to the signal processing community that the introduction and development of the emerging wavelet field has a lot to offer to the theoretic and practical approaches of signal processing, and particularly to the area of sub-band coding. The objective of this paper is to introduce a wavelet approach to sub-band coding, using wavelet packets associated with the orthonormal scaling function $\phi(t)$ in (4) to build a wavelet library for achieving the theoretically smallest bit-rate for perfect recovery of bandlimited signals.

2. Discussion of Main Results

We first introduce the notion of theoretical Nyquist frequency σ_f of a bandlimited signal $f(t)$ that satisfies (5), defined by

$$\sigma_f = \frac{\operatorname{mes}(\operatorname{supp}^+ f)}{\pi}, \tag{9}$$

where the notation "mes" stands for the Lebesgue measure. If a bandlimited signal $f(t)$ can be written as

$$f(t) = \sum_{k=1}^{n} f_k(t) \tag{10}$$

with $\operatorname{supp}^+ \hat{f}_k \subset [2\pi\mu_k, 2\pi\nu_k]$, where

$$0 \le \mu_1 < \nu_1 \le \mu_2 < \nu_2 \le \cdots \le \mu_n < \nu_n \ , \tag{11}$$

we say that the decomposition in (10) is a sub-band decomposition of $f(t)$. If, in addition, μ_k and ν_k are the lowest and highest bands of $f_k(t)$, then the bandwidth of $f_k(t)$ is given by

$$\sigma_k := \nu_k - \mu_k, \tag{12}$$

$k = 1, \ldots, n$.

In certain application areas, such as multiple-channel synchronized transmission, a sub-band decomposition is required to satisfy the following conditions:

(i) $\mu_k/\sigma_k, k = 1, \ldots, n$, are integers;
(ii) $\sigma_k/\sigma_\ell, k, \ell = 1, \ldots, n$, are rationals.

In this paper, all sub-band decompositions are required to satisfy both (i) and (ii). Condition (i) ensures that the smallest sampling rate of each sub-band is equal to its Nyquist frequency. Condition (ii) ensures the existence of some positive integer N and some $\sigma > \sigma_k, k = 1, \ldots, n$, such that $N\sigma_k/\sigma, k =$

$1, \ldots, n$, are integers. This property allows for the feasibility of bit allocation for each sub-band (see [6]) that is required for synchronzied transmission.

A bandlimited signal $f(t)$ with sub-band decomposition given in (10), where each $f_k(t)$ is sampled with sampling rate $\lambda_k, k = 1, \ldots, n$, is said to have sub-band sampling (or coding) rate

$$\lambda := \sum_{k=1}^{n} \lambda_k. \tag{13}$$

Observe that since the Nyquist frequency of $f_k(t)$ is $2\sigma_k$, then under the assumptions (i), $f_k(t)$ can be sampled with sampling rate $\lambda_k = 2\sigma_k$, so that the sub-band coding rate is $\lambda = 2\sum_{k=1}^{n} \sigma_k$, which certainly depends on the sub-band decomposition (10).

In this paper, we are interested in finding a sub-band decomposition that achieves, as close as possible, the optimal sub-band coding rate, namely, the theoretical Nyquist frequency σ_f definded in (9). We have the following results.

Theorem 1. *Let $f(t)$ be a bandlimited signal with theoretical Nyquist frequency σ_f. Then for any $\lambda_f > \sigma_f, f(t)$ has a sub-band decomposition (10), with sub-band coding rate no greater than λ_f. Furthermore, the sub-band coding rate of any sub-band decomposition of $f(t)$ is at least σ_f.*

Theorem 2. *Let $f(t)$ be a bandlimited lowpass signal that satisfies (5) with highest band $\sigma/2$ and theoretical Nyquist frequency σ_f. Then for any $\tilde{\sigma} > \sigma_f$, there exists a sub-band decomposition that achieves bit-rate compression ratio larger than $\sigma/\tilde{\sigma}$.*

3. Proof of Main Results

The essential ingredient in our proof of Theorem 1 is the notion of wavelet packets associated with the sampling funciton $\phi(t)$ in (4). Let

$$\cdots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots,$$

be the multiresolution approximation (MRA) ([8], [9]) generated by the Shannon sampling function $\phi(t) = \text{sinc}\ t$. Thus,

$$V_n = \{f \in L^2 : \quad \text{supp}\hat{f} \subset [-2^n\pi, 2^n\pi]\}$$

and the corresponding wavelet subspaces $\{W_n\}_{n\in\mathbb{Z}}$, where $W_n \perp V_n, \quad W_n + V_n = V_{n+1}$, can be generated by the Shannon wavelet:

$$\psi(t) := 2\text{sinc}(2t) - \text{sinc}\ t, \tag{14}$$

whose Fourier transform is given by

$$\hat{\psi}(\omega) = \chi_{[-2\pi,-\pi)\cup[\pi,2\pi)}(\omega).$$

Let $p_0(\omega)$ be the 2π-periodic function

$$p_0(\omega) = \begin{cases} 1, & \omega \in [-\frac{\pi}{2}, \frac{\pi}{2}), \\ 0, & \omega \in [-\pi, -\frac{\pi}{2}) \cup [\frac{\pi}{2}, \pi), \end{cases} \tag{15}$$

and

$$p_1(\omega) = p_0(\omega + \pi).$$

Then, we have

$$\hat{\phi}(\omega) = p_0(\omega/2)\hat{\phi}(\omega/2),$$
$$\hat{\psi}(\omega) = p_1(\omega/2)\hat{\phi}(\omega/2).$$

It is easy to see that ϕ is an orthonormal scaling function and ψ is an orthonormal wavelet. (See for example, [11].)

Remark. *Although it is customary to use the two-scaling relation*

$$\hat{\psi}(\omega) = e^{i\frac{\omega}{2}}\overline{p_0(\omega/2 + \pi)}\hat{\phi}(\omega/2)$$

to construct the orthonormal wavelet from an scaling function ϕ that satisfies the two-scaling equation

$$\hat{\phi}(\omega) = p_0(\omega/2)\hat{\phi}(\omega/2)$$

(see [11]), we prefer to drop the factor $e^{i\frac{\omega}{2}}$ for $\hat{\psi}$, because of the particular structure of p_0 in (15).

Following [1], the Shannon wavelet packets can be constructed as follows. Write

$$\begin{cases} \mu_0(t) = \phi(t), \\ \mu_1(t) = \psi(t). \end{cases} \tag{16}$$

Then, we have

$$\begin{cases} \hat{\mu}_0(\omega) = p_0(e^{-i\omega/2})\hat{\mu}_0(\omega/2), \\ \hat{\mu}_1(\omega) = p_1(e^{-i\omega/2})\hat{\mu}_0(\omega/2). \end{cases} \tag{17}$$

Definition. Let the collection of functions $\{\mu_l\}_{l=0}^{\infty}$ be defined inductively as follows. For even n, set

$$\begin{cases} \hat{\mu}_{2n}(\omega) = p_0(e^{-i\omega/2})\hat{\mu}_n(\omega/2), \\ \hat{\mu}_{2n+1}(\omega) = p_1(e^{-i\omega/2})\hat{\mu}_n(\omega/2), \quad n = 0, 2, \cdots, \end{cases} \tag{18}$$

and for odd n, set

$$\begin{cases} \hat{\mu}_{2n}(\omega) = p_1(e^{-i\omega/2})\hat{\mu}_n(\omega/2) \\ \hat{\mu}_{2n+1}(\omega) = p_0(e^{-i\omega/2})\hat{\mu}_n(\omega/2), \quad n = 1, 3, \cdots. \end{cases} \tag{19}$$

Then the collection $\{\mu_l\}_{l=0}^{\infty}$ is called the "family of Shannon wavelet packets."

It can be easily verified that

$$\mu_l(t) = (l+1)\text{sinc}\left((l+1)t\right) - l\text{sinc}\left(lt\right),$$

or, equivalently,

$$\hat{\mu}_l = \chi_{[-(l+1)\pi, -l\pi)\cup[l\pi,(l+1)\pi)}, \quad l = 0, 1, 2, \cdots.$$

Write

$$\mu_{l,j,k}(t) = 2^{j/2}\mu_l(2^j t - k).$$

We have

$$\hat{\mu}_{l,j,k}(\omega) = e^{i2^{-j}k\omega}\chi_{[-2^j(l+1)\pi, -2^j l\pi)\cup[2^j l\pi, 2^j(l+1)\pi)}. \tag{20}$$

Define

$$U_j^l = \text{clos}_{L^2} \text{ span } \{2^{j/2}\mu_l(2^j t - k): \quad k \in \mathbb{Z}\}, \quad j \in \mathbb{Z}, \quad l \in \mathbb{Z}^+.$$

By (20), it follows that each function in U_j^l is a bandpass signal with lowest band $2^{j-1}l$ and highest band $2^{j-1}(l+1)$, and bandwidth 2^{j-1}. In addition, it also satisfies the sub-band coding condition (i). For any $n = 0, 1, 2, \cdots.$ we have

$$U_{j+1}^n = U_j^{2n} \oplus U_j^{2n+1}, \quad U_j^{2n} \perp U_j^{2n+1}, \quad j \in \mathbb{Z}.$$

Therefore, for any integers $j \geq 1$ and $k \geq 0$,

$$W_j = U_{j-k}^{2^k} \oplus U_{j-k}^{2^k+1} \oplus \cdots \oplus U_{j-k}^{2^{k+1}-1}.$$

Let $I_{j,l} = [2^j l\pi, 2^j(l+1)\pi]$, and let Λ and Γ be subsets of the integer set. Then the family $\{I_{j,l} : j \in \Lambda, l \in \Gamma\}$ is called a "dyadic partition" of $\mathbb{R}^+ := [0, \infty)$ if $\cup_{j\in\Lambda,l\in\Gamma} I_{j,l} = \mathbb{R}^+$ and $\text{mes}\,(I_{j,l} \cap I_{j',l'}) = 0, (j,l) \neq (j',l')$.

The following result is a direct consequence of Theorem 7.27 in [1].

Lemma 1. *Let $\{I_{j,l} : j \in \Lambda, l \in \Gamma\}$ be a dyadic partition of $\mathbb{R}^+$ as defined above. Then the family $\{\psi_{j,l,k} : j \in \Lambda, l \in \Gamma, k \in \mathbb{Z}\}$ is an orthonormal basis of L^2 and $L^2 = \oplus_{j\in\Lambda,l\in\Gamma} U_j^l$.*

Similar to the Shannon Sampling Theorem, we can establish the following.

Lemma 2. *If $f \in U_0^n$, that is, $supp^+ \hat{f} \subset I_{0,n} := [n\pi, (n+1)\pi]$, then*

$$f(t) = \sum_{k\in Z} f(k)\mu_{n,0,k}(t).$$

Proof. Let $\tilde{L}_{2\pi}^2$ be the space of 2π-periodic, square-integrable functions. Let $\widehat{f_p}(\omega)$ be the 2π-periodization of $\hat{f}(\omega)$. Then $\hat{f}_p \in \tilde{L}_{2\pi}^2$, and $\hat{f}_p(\omega)$ can be expanded as a Fourier series

$$\widehat{f_p}(\omega) = \sum C_k e^{-ik\omega},$$

where the coefficients of the Fourier series can be calculated by

$$C_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}_p(\omega)e^{ik\omega}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{ik\omega}d\omega = f(k).$$

Therefore, we have

$$\widehat{f_p}(\omega) = \sum f(k)e^{-ik\omega}, \quad \omega \in \mathbb{R}.$$

Since $\hat{f}(\omega) = \widehat{f_p}(\omega)$ for $\omega \in [-(n+1)\pi, -n\pi) \cup [n\pi, (n+1)\pi)$, and $\text{supp}\hat{f} \subset$

$[-(n+1)\pi, -n\pi] \cup [n\pi, (n+1)\pi]$, we have

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\omega)e^{it\omega}d\omega \\ &= \sum_{k\in Z} f(k)\frac{1}{2\pi}\left(\int_{-(n+1)\pi}^{-n\pi} e^{-ik\omega}e^{it\omega}d\omega + \int_{n\pi}^{(n+1)\pi} e^{-ik\omega}e^{it\omega}d\omega\right) \\ &= \sum_{k\in Z} f(k)((n+1)\ \text{sinc}\ ((n+1)(t-k)) - n\ \text{sinc}\ (n(t-k))) \\ &= \sum_{k\in Z} f(k)\mu_{n,0,k}(t). \end{aligned}$$

This completes the proof of Lemma 2.

By appropriate dilations, it is easy to generalize Lemma 2 to the functions in U_j^n. Hence, we have the following.

Lemma 3. *If $f \in U_j^n$, then $f(t) = \sum_k f(2^{-j}k)\mu_{n,j,k}(t)$.*

Recall that the Nyquist frequency of all functions in U_j^n is 2^j. Hence, Lemma 3 gives a formula for perfect recovery of functions in U_j^n sampled at the Nyquist rate.

We now proceed to complete the proof of Theorem 1. We first prove that there is a sub-band decomposition of $f(t)$ which provides sub-band coding rate less than λ_f. Let $\epsilon = \pi(\lambda_f - \sigma_f)$. There is a finite set $\{I_{j,l} : j \in \Lambda_f, l \in \Gamma_f\}$ such that $\text{mes}(I_{j,l} \cap I_{j',l'}) = 0$, $\text{supp}^+\hat{f} \subset \cup_{j\in\Lambda_f,l\in\Gamma_f} I_{j,l}$, and $\text{mes}\left(\sum_{j\in\Lambda_f,l\in\Gamma_f} I_{j,l}\right) - \pi\sigma_f < \epsilon$, which implies that $\text{mes}\left(\sum_{j\in\Lambda_f,l\in\Gamma_f} I_{j,l}\right)/\pi < \lambda_f$. Let $f_{j,l}(t)$ be the orthogonal projection of $f(t)$ on U_j^l. Then $f(t) = \sum_{j\in\Lambda_f,l\in\Gamma_f} f_{j,l}(t)$, where each $f_{j,l}(t)$ is a sub-band function with respect to f. Due to the packet structure, this decomposition satisfies the sub-band coding conditions (i) and (ii). Hence, it is a sub-band decomposition of $f(t)$. By Lemma 3, each $f_{j,l}(t)$ can be sampled at the sampling rate $2^j = \text{mes}(I_{j,l}/\pi)$. It follows that the function $f(t)$ can be decomposed as in (10) with the sub-band coding rate given by $\left(\sum_{j\in\Lambda_f,l\in\Gamma_f} \text{mes}(I_{j,l})\right)/\pi < \lambda_f$. On the other hand, let $f(t) = \sum_{k=1}^n f_k(t)$ be an arbitrary sub-band decomposition of f. Then each function f_k occupies a sub-band of f, say with the lowest band μ_k and the highest band ν_k. It is that $\text{supp}^+\hat{f}_k \subset [\mu_k, \nu_k]$. Then the sub-band coding rate of f is no less than $\frac{\text{mes}(\cup_{k=1}^n[\mu_k,\nu_k]}{\pi} \geq \sigma_f$. This completes the proof of Theorem 1.

4. Shannon Wavelet Library for Sub-band Coding

The Shannon wavelet packet introduced in the previous section has "primary band" 1/2. That is, the Shannon sampling function ϕ and the Shannon wavelet ψ both have bandwidth 1/2. Therefore, each function in the subspace U_j^l has the dyadic bandwidth 2^{j-1}. Of course we can construct the Shannon wavelet packets with primary band different from $2^j, j \in \mathbb{Z}$. Write $\Gamma^2 = \{2^j : j \in \mathbb{Z}\}$, consider a positive number $\nu \notin \Gamma^2$, and define

$$\phi^\nu(t) = (2\nu)^{1/2}\phi(2\nu t), \quad \psi^\nu(t) = (2\nu)^{1/2}\psi(2\nu t).$$

Then both $\phi^\nu(t)$ and $\psi^\nu(t)$ have bandwidth ν. Let

$$\phi_{j,k}^\nu(t) = 2^{j/2}\phi^\nu(2^j t - \frac{k}{2\nu}) = \left(2^{(j+1)/2}\nu^{1/2}\phi(2^{j+1}\nu x - t)\right)$$

and

$$V_j^\nu = \mathrm{span}_{L^2}\{\phi_{j,k}^\nu : \quad k \in \mathbb{Z}\}, \quad j \in \mathbb{Z}.$$

Then

$$\cdots \subset V_{-1}^\nu \subset V_0^\nu \subset V_1^\nu \subset \cdots$$

is an MRA in L^2. (Note that $\phi_{j,k}^\nu(t)$ is no longer an integer translate of $\phi^\nu(2^j t)$.)

Similarly, let

$$\psi_{j,k}^\nu(t) = 2^{j/2}\psi^\nu(2^j t - \frac{k}{2\nu}) = \left(2^{(j+1)/2}\nu^{1/2}\psi(2^{j+1}\nu t - k)\right).$$

Then $\{\psi_{j,k}^\nu : j, k \in \mathbb{Z}\}$ is an orthonormal basis of L^2. Using a similar argument as in the previous section, we can construct the wavelet packets with respect to ϕ^ν and ψ^ν. The new family of Shannon wavelet packet has primary band ν. Denote the Shannon wavelet packets with the primary band ν by $\mathcal{P}^\nu$. Two positive numbers ν and μ are said to be binarily similar if there exists an integer j such that $\nu = 2^j\mu$. Let $B \subset \mathbb{R}$ be the largest set of all numbers that are not binarily similar to each other. Then

$$\{\mathcal{P}^\nu : \quad \nu \in B\}$$

constitutes a "Shannon wavelet library."

We may now use the Shannon wavelet library for sub-band decomposition of a bandlimited signals. To do so, we can decompose a signal into sub-band signals from a single wavelet packet, say $\mathcal{P}^{\nu}$. In this case, the decomposition always satisfies the sub-band coding conditions (i) and (ii). In addition, all sub-band functions of a sub-band coding obtained in this way are synchronic, and therefore, no additional code is needed for synchronized transmission. We can also decompose a signal using several packets. For example, we may decompose a signal $f(t)$ into sub-band signals in a way that

$$f(t) = \sum_{l=1}^{m} \sum_{k=1}^{n_l} f_{lk}(t),$$

where all $f_{lk}(t), 1 \le k \le n_l$, have the same primary band, i.e., they are from the same wavelet packet family, say $\mathcal{P}^{\nu_l}$. But $f_{lk}(t)$ and $f_{l'k'}(t)$ have different primary bands if $l \neq l'$. Thus, the signal f is decomposed by using m different wavelet packets $\mathcal{P}^{\nu_l}, 1 \le l \le m$. Then the decomposition is a sub-band decomposition if all the ratios $\nu_k/\nu_{k'}, 1 \le k, k' \le m$, are rational numbers. In some cases, additional codes are needed for synchronized transmission.

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