A Study of Asymptotically Optimal Time-frequency Localization by Scaling Functions and Wavelets

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Abstract

The notions of stoplets and cowlets are introduced in this paper. We will call a scaling function a stoplet and its corresponding semi-orthogonal minimally supported wavelet a cowlet, if the two-scale sequence (or mask) of the caling function is a finite symmetric Pólya frequency sequence. The main result in this paper is that stoplets and cowlets are asymptotical Gaussians and modulated Gaussians, respectively, and provide asymptotically optimal (i.e. smallest) time-frequency lowpass and bandpass windows.

Key Words and Phrases: wavelets, Pólya frequency sequences, time-frequency windows, optimal time-frequency windows, uncertainty principle, stoplets, cowlets.

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1 Introduction

In signal processing, the wavelet transform is used to give localized time-frequency information of continuous-time signals (or functions) with finite energy. Hence, for good time-frequency localization, the size of the time-frequency windows must be reasonably small. This paper is devoted to a study of time-frequency windows of scaling functions and wavelets.

The most successful approach to constructing wavelets is perhaps the notion of multiresolution analysis (MRA) generated by some scaling function \(\phi\) \([12], [13]\). From the signal processing point of view, a scaling function \(\phi\) is a lowpass filter while its corresponding wavelet \(\psi\) is a bandpass filter. The time-frequency localization of a filter is measured by its time-frequency window, and the optimal time-frequency lowpass window is achieved only by the Gaussian functions. For a bandpass filter, however, since the zero frequency must be stopped, positive and negative frequency bands of a bandpass filter have to be treated separately. Hence, it is necessary to re-establish the uncertainty principle for their time-frequency windows. This will be done in Section 3. However, in contrast to lowpass filtering, it will be seen that the optimal lower bound of bandpass time-frequency windows can never be achieved. An objective of this paper is to show that this uncertainty limit can be achieved asymptotically by a family of bandpass filters. On the other hand, it is worthwhile to point out that since the uncertainty lower bound for lowpass time-frequency localization can be realized only by the Gaussian functions and these functions are not scaling functions, it is also important to identify families of scaling functions that give rise to asymptotically optimal time-frequency windows.

In [10], Goodman and Micchelli introduce the notion of riplets, which are scaling functions whose two-scale sequences (or masks) are Pólya frequency sequences (see [11] for the

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definition). Since a rplet is totally positive, and since we will be only concerned with symmetric scaling functions, we shall call symmetric rplets stoplets. Using a formula for constructing semi-orthogonal wavelets in [5], we will then formulate $B$-wavelets (i.e. semi-orthogonal wavelets with minimum supports) corresponding to the stoplets. This type of $B$-wavelets, with masks derived from alternately changing signs of some Pólya frequency sequences, will be called cowlets (to emphasize the notion of complete oscillation in [6]). The simplest stoplet is a cardinal polynomial $B$-spline, while the simplest cowlet is its corresponding semi-orthogonal $B$-wavelet with minimum support introduced in [4].

We will prove that the uncertainty lower bound for lowpass windows is asymptotically achieved by a sequence of stoplets with order of smoothness tending to infinity, while the uncertainty lower bound for bandpass windows is asymptotically achieved by its corresponding sequence of cowlets.

A brief history of this problem is outlined below. In the early 1970’s, I. J. Schoenberg already observed the limit property of dilated $B$-splines (cf. [14]). Recently Unser, Aldroubi, and Eden in [15] proved that the $\sqrt{n}$-dilated cardinal polynomial $B$-splines of order $n$ converge to a Gaussian function, both pointwise and in the $L^2$-norm as $n$ tends to infinity, while certain dilations and translations of the corresponding $B$-wavelets are asymptotically equal to some sine or cosine modulated Gaussians. These results are extended by Aldroubi and Unser [1] to scaling functions, under some strict conditions, generated by $n$-fold convolution products. We do not know of any results concerning the measure of time-frequency windows in the literature, although numerical experiments certainly suggest the convergence of the window sizes of cardinal polynomial $B$-wavelets (see, for example [3; pp.186-187]). And it

This paper is organized as follows. Main results are summarized in Section 2, and in Section 3, we prove the uncertainty principle for bandpass time-frequency windows. In Section 4, we establish several lemmas to be used in the proofs of Theorems 2 and 3. The main ingredients in these lemmas are inequalities for the Fourier transforms of stoplets and cowlets, which are of independent interest. Finally, the proofs of Theorems 2 and 3 constitute Section 5.

## 2 Preliminaries and Main Results

As usual (cf. [3; p.7]), time-frequency measurements are formulated as follows. Let $\phi \in L^2$ and $\hat{\phi}$ be its Fourier transform defined by

$$\hat{\phi}(\omega) = \int_{-\infty}^{\infty} \phi(x)e^{-i\omega x}dx.$$
If both

\[ t_\phi = \lim_{N \to \infty} \int_{-N}^{N} x |\phi(x)|^2 \, dx / \int_{-\infty}^{\infty} |\phi(x)|^2 \, dx \]

and

\[ w_\phi = \lim_{N \to \infty} \int_{-N}^{N} \omega |\hat{\phi}(\omega)|^2 \, d\omega / \int_{-\infty}^{\infty} |\hat{\phi}(\omega)|^2 \, d\omega \]

exist, then they will be called the time and frequency centers of \( \phi \), respectively, and the quantities

\[ \Delta_\phi = \left( \int_{-\infty}^{\infty} (x - t_\phi)^2 |\phi(x)|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{-\infty}^{\infty} |\phi(x)|^2 \, dx \right)^{\frac{1}{2}} \]

and

\[ \Delta_\phi = \left( \int_{-\infty}^{\infty} (\omega - w_\phi)^2 |\hat{\phi}(\omega)|^2 \, d\omega \right)^{\frac{1}{2}} \left( \int_{-\infty}^{\infty} |\hat{\phi}(\omega)|^2 \, d\omega \right)^{\frac{1}{2}} \]

are called the corresponding time and frequency localization radii of \( \phi \). If both \( \Delta_\phi \) and \( \Delta_\hat{\phi} \) are finite, we say that \( \phi \) is a window function and we denote the measure of its time-frequency window by

\[ M(\phi) = \Delta_\phi \Delta_\hat{\phi} . \]

Note that the area of the time frequency window is \( 4M(\phi) \). It is well known that if \( t_\phi(t) \) and \( \phi' \) are in \( L^2 \), then \( \phi \) is a window function. Observe that if \( \phi_r(\cdot) = c\phi(\cdot - \tau), c \neq 0, \) and \( \tau \in \mathbb{R} \), then \( \Delta_\phi = \Delta_{\phi_r} \) and \( \Delta_\hat{\phi} = \Delta_{\hat{\phi_r}} \). Also observe that \( \Delta_{\phi(\sigma \cdot)} = \sigma \Delta_\phi \) and \( \Delta_{\hat{\phi}(\sigma \cdot)} = \frac{1}{\sigma} \Delta_\hat{\phi} \).

For the measures of time-frequency windows, the following uncertainty principle is well known (see [3, Theorem 3.5] ).

**Theorem A.** If \( \phi \) is a window function, then

\[ M(\phi) \geq \frac{1}{2} \]

and equality holds if and only if

\[ \phi = kG_\sigma(x - \mu), \quad k \neq 0, \quad \sigma > 0, \quad \mu \in \mathbb{R}, \]

where

\[ G_\sigma(x) = \frac{1}{2\pi} e^{-\frac{(sx)^2}{2}} \]

is a Gaussian function.

For a Gaussian function, we have

\[ \hat{G}_\sigma(\omega) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{\omega^2}{2\sigma^2}} \]
so that

\[ \hat{G}_\sigma(\omega) = e^{-\frac{3\omega^2}{8\sigma^2}} \hat{G}_\sigma\left(\frac{\omega}{2}\right). \]

However, since \( \hat{G}_\sigma(\omega)/\hat{G}_\sigma(\omega/2) = e^{-\frac{3\omega^2}{8\sigma^2}} \) cannot be extended to be a \( 4\pi \)-periodic function, the Gaussian function \( G_\sigma \) is not a scaling function.

If a window function \( \psi \) satisfies \( \hat{\psi}(0) = 0 \), it is called a bandpass filter. As a bandpass filter, \( \hat{\psi} \) treats positive and negative frequency bands separately. Hence, the notion of the window of a bandpass filter \( \psi \) has to be modified. Of course we retain the definitions of the time center and localization radius of \( \psi \), while consider two frequency centers: the positive frequency center

\[ \omega^+ = \frac{\int_{0}^{\infty} \omega |\hat{\psi}(\omega)|^2 d\omega}{\int_{0}^{\infty} |\hat{\psi}(\omega)|^2 d\omega} \]

and the negative frequency center

\[ \omega^- = \frac{\int_{-\infty}^{0} \omega |\hat{\psi}(\omega)|^2 d\omega}{\int_{-\infty}^{0} |\hat{\psi}(\omega)|^2 d\omega} \]

for \( \psi \). Then the positive and negative frequency localization radii of \( \psi \) are defined, respectively, by

\[ \Delta^+ = \frac{\left( \int_{0}^{\infty} (\omega - \omega^+)^2 |\hat{\psi}(\omega)|^2 d\omega \right)^{\frac{1}{2}}}{\left( \int_{0}^{\infty} |\hat{\psi}(\omega)|^2 d\omega \right)^{\frac{1}{2}}} \]

and

\[ \Delta^- = \frac{\left( \int_{-\infty}^{0} (\omega - \omega^-)^2 |\hat{\psi}(\omega)|^2 d\omega \right)^{\frac{1}{2}}}{\left( \int_{-\infty}^{0} |\hat{\psi}(\omega)|^2 d\omega \right)^{\frac{1}{2}}} \]

When \( \psi \) is a real-valued function, it is clear that \(|\hat{\psi}|\) is an even function, so that \( \omega^- = -\omega^+ \) and \( \Delta^- = \Delta^+ \). In this paper, we will only consider real-valued bandpass filters, and can therefore ignore \( \omega^- \) and \( \Delta^- \).

A bandpass window function \( \psi \) with center \((t_\psi, \omega^+ \psi)\) and radii \( \Delta_\psi \) and \( \Delta^+ \) has time-frequency localization measurement

\[ M^+(\psi) := \Delta_\psi \Delta^+. \]

A function \( \psi \) is said to be symmetric (anti-symmetric) if there exists some \( b \) such that \( \psi(x) = \psi(b - x) \) \( (\psi(x) = -\psi(b - x)) \). Since symmetry is an important property for linear-phase filtering (see [3; p.156-167]), [8], we often require window functions to be symmetric or antisymmetric.
In Section 3, we will prove the following uncertainty principle for bandpass time-frequency windows.

**Theorem 1.** If \( \psi \in L^2 \cap L^1 \) is a real-valued symmetric or anti-symmetric function that satisfies \( t\psi(t) \in L^2, \psi' \in L^2, \) and \( \hat{\psi}(0) = 0 \), then
\[
M^+(\psi) > \frac{1}{2}.
\]
Furthermore, the lower bound \( \frac{1}{2} \) cannot be improved and cannot be attained.

According to Theorem A, no scaling function can achieve the optimal lower bound of the window measure either. This motivates our study of scaling functions and wavelets that asymptotically achieve the optimal bound of \( \frac{1}{2} \).

Recall that a real-valued sequence \( a \) is called a Pólya frequency sequence if all the minors of the bi-infinite matrix \( A \) with \((i, j)\)th entries given by
\[
A_{ij} = a_{j-i}, \quad i, j \in \mathbb{Z},
\]
( where \( a = \{a_j\} \) and \( a_n := 0 \) if \( n \) is not in the index set of \( a \) ) are non-negative, that is,
\[
A\left(i_1, \ldots, i_p; j_1, \ldots, j_p\right) = \det_{k,\ell=1,\ldots,p} A_{i_k j_\ell} \geq 0,
\]
for all integers \( p \geq 1 \) and \( i_1 < \cdots < i_p, \) \( j_1 < \cdots < j_p \). (see [11] for the properties of Pólya frequency sequences.)

This real sequence \( a \) is called symmetric (anti-symmetric) if there is an integer \( k \) such that \( a_j = a_{k-j} \) (\( a_j = -a_{k-j} \)) for all \( j \in \mathbb{Z} \). Here, \( \frac{k}{2} \) is called the symmetric center of \( a \). The \( z \)-transform \( a(z) = \sum_{j \in \mathbb{Z}} a_j z^j \) of \( a \) is called the symbol of \( a \). If \( a \) is a finite sequence, its length will be denoted by \( |a| \).

Now let \( \phi \in L^1 \) be a scaling function with two-scale sequence \( a \), namely,
\[
\phi(x) = 2 \sum_{j \in \mathbb{Z}} a_j \phi(2x - j) \quad \text{(2.1)}
\]
and
\[
\int_{-\infty}^{\infty} \phi(x)dx = 1.
\]

The autocorrelation of \( \phi \) is the function
\[
\Phi(x) = \int_{-\infty}^{\infty} \phi(x + y)\overline{\phi(y)}dy \quad \text{(2.2)}
\]
and the Euler–Frobenius Laurent series corresponding to \( \phi \) is defined by
\[
B_\phi(z) = \sum_{n \in \mathbb{Z}} \Phi(n)z^n. \quad \text{(2.3)}
\]
By the Poisson summation formula, we can rewrite (2.3) as
\[ E_\phi(\omega) := B_\phi \left( e^{-i\omega} \right) = \sum_{n \in \mathbb{Z}} \left| \hat{\phi}(\omega + 2n\pi) \right|^2. \] (2.4)

Now, for the scaling function \( \phi \) in (2.1), its corresponding semi-orthogonal \( B \)-wavelet \( \psi \) is defined by
\[ \hat{\psi}(\omega) = C_\psi e^{-i\frac{\omega}{2}} \left( -e^{-i\frac{\omega}{2}} \right) B_\phi \left( -e^{-i\frac{\omega}{2}} \right) \left( \frac{\omega}{2} \right), \] (2.5)
where \( C_\psi \) is a positive constant so chosen that \( \| \hat{\psi} \|_\infty = 1 \).

If \( a \) is a symmetric finite Pólya frequency sequence, we will call \( \phi \) a stoplet and \( \psi \) a cowlet, respectively. From (2.1) and (2.2), it is easy to see that any stoplet is a probability distribution function. We denote its standard deviation by
\[ \sigma = \left( \int_{-\infty}^{\infty} \phi(x) (x - t_\phi)^2 \, dx \right)^{\frac{1}{2}}. \] (2.6)

It is also known that if the symbol of the stoplet \( \phi \) is in the form \( a(z) = \sum_{j=0}^{m} a_j z^j, a_0 a_m \neq 0 \), then the time-frequency center of \( \phi \) is \((\frac{m}{2}, 0) \) (cf. [5]). On the other hand, the time center of a cowlet as defined by (2.5) is always \( \frac{1}{2} \) (cf. [7]).

Our main results are stated in the following theorems.

**Theorem 2.** For each \( n \), let \( a^n = \{ a^n_j \}_{j=0}^\infty \) be a finite symmetric Pólya frequency sequence with symbol
\[ a_n(z) = \left( 1 + \frac{z}{2} \right)^n p_n(z), \] (2.7)
for some polynomial \( p_n(z) \) that satisfies \( p_n(1) = 1, p_n(-1) \neq 0 \) and \( \deg p_n \leq Cn \), where \( C \) is a positive constant independent of \( n \). Let \( \phi_n \) be the stoplet determined by \( a^n \) as in (2.1)–(2.2) and \( \sigma_n \) be its standard variation. Then \( \sigma_n \to +\infty \) as \( n \to \infty \),
\[ \lim_{n \to \infty} \left\| \hat{\phi}_n \left( \frac{\omega}{\sigma_n} \right) e^{ik_n \omega} - e^{-\frac{\omega^2}{2}} \right\|_{L^p} = 0, \quad 1 \leq p < \infty, \] (2.8)
and
\[ \lim_{n \to \infty} \left\| \sigma_n \phi_n \left( \sigma_n x + \frac{k_n}{2} \right) - \frac{1}{2\pi} e^{-\frac{x^2}{2}} \right\|_{L^q} = 0, \quad 2 \leq q < \infty. \] (2.9)

Furthermore,
\[ \lim_{n \to \infty} \frac{1}{\sigma_n} \Delta \phi_n = \lim_{n \to \infty} \sigma_n \Delta \hat{\phi}_n = \frac{1}{\sqrt{2}}, \] (2.10)
so that
\[ \lim_{n \to \infty} M(\phi_n) = \frac{1}{2}. \] (2.11)

**Theorem 3.** Let \( \psi_n \) be the cowlets corresponding to the stoplets \( \phi_n \) as in Theorem 2. Then for each \( n \), there is a unique \( \omega_n \) in \((0, \infty)\), at which the function \( |\hat{\psi}_n(\omega)| \) attains
its absolute maximum value. Furthermore, $\pi \leq \omega_n \leq 2\pi$, and $\tau_n := \sqrt{\hat{\psi}_n'(\omega_n)} \to \infty$ as $n \to \infty$. In addition, we have

$$\lim_{n \to \infty} \left\| e^{\frac{i\omega}{\tau_n} \hat{\psi}_n} \left( \frac{\omega}{\tau_n} \right) - e^{-\frac{(\omega - \omega\tau_n)^2}{2}} \right\|_{L^p(0, +\infty)} = 0, \quad 1 \leq p < \infty,$$

(2.12)

so that for even $n$,

$$\lim_{n \to \infty} \left\| \tau_n \hat{\psi}_n \left( \tau_n x + \frac{1}{2} \right) - \frac{1}{2\pi} \cos (\tau_n \omega_n x) e^{-\frac{x^2}{2}} \right\|_{L^q} = 0, \quad 2 \leq q < +\infty;$$

(2.13)

and for odd $n$,

$$\lim_{n \to \infty} \left\| \tau_n \hat{\psi}_n \left( \tau_n x + \frac{1}{2} \right) - \frac{1}{2\pi} \sin (\tau_n \omega_n x) e^{-\frac{x^2}{2}} \right\|_{L^q} = 0, \quad 2 \leq q < +\infty.$$

(2.14)

Furthermore,

$$\lim_{n \to \infty} \frac{1}{\tau_n} \Delta \hat{\psi}_n = \lim_{n \to \infty} \tau_n \Delta_{\hat{\psi}_n} = \frac{1}{\sqrt{2}},$$

(2.15)

so that

$$\lim_{n \to \infty} M^+ (\psi_n) = \frac{1}{2}.$$

(2.16)

A polynomial cardinal $B$-spline and its corresponding $B$-wavelet (cf. [4]) are examples of stoplets and cowlets. Let $N_n$ be the cardinal $B$-spline of order $n$ and $\psi_{N_n}$ be its corresponding semi-orthogonal $B$-wavelet. As an application of Theorems 2 and 3, we have the following corollaries.

**Corollary 1.**

$$\lim_{n \to \infty} \left\| \hat{N}_n \left( \frac{\omega \sqrt{12}}{\sqrt{n}} \right) e^{\frac{\sqrt{3}n\omega i}{2}} - e^{-\frac{\omega^2}{2}} \right\|_{L^p} = 0, \quad 1 \leq p < \infty,$$

(2.17)

$$\lim_{n \to \infty} \left\| \frac{n}{12} N_n \left( \sqrt{\frac{n}{12}} x + \frac{n}{2} \right) - \frac{1}{2\pi} e^{-\frac{x^2}{2}} \right\|_{L^q} = 0, \quad 2 \leq q < \infty.$$

(2.18)

Furthermore,

$$\lim_{n \to \infty} \sqrt{\frac{12}{n}} \Delta_{N_n} = \lim_{n \to \infty} \sqrt{\frac{n}{12}} \Delta_{\hat{N}_n} = \frac{1}{\sqrt{2}},$$

(2.19)

so that

$$\lim_{n \to \infty} M (N_n) = \frac{1}{2}.$$

(2.20)

**Corollary 2.** Let $\psi_{N_n}$ be the semi-orthogonal $B$-wavelet corresponding to $N_n$. Let $\omega_0 (\approx 1.6367\pi)$ be the unique value in $(0, \infty)$, at which the function $C(\omega) := \frac{8(1 - \cos \omega)}{\omega (2\pi - \omega)^2}$ attains its absolute maximum value. Set $\alpha = \frac{1}{\sqrt{C'(\omega_0)}} |C''(\omega_0)| (\approx 0.3745)$. Then

$$\lim_{n \to \infty} \left\| e^{\frac{i\omega}{\alpha \sqrt{n}} \hat{\psi}_{N_n}} \left( \frac{\omega}{\alpha \sqrt{n}} \right) - e^{-\frac{(\omega - \omega\sqrt{n})^2}{2}} \right\|_{L^p(0, +\infty)} = 0, \quad 1 \leq p < \infty.$$

(2.21)
and therefore, for even $n$,

$$\lim_{n \to \infty} \left\| \alpha \sqrt{n} \psi_{N_n} \left( \alpha \sqrt{n} x + \frac{1}{2} \right) - \frac{1}{2\pi} \cos \left( \alpha \omega_0 \sqrt{n} x \right) e^{-\frac{x^2}{2}} \right\|_{L^2} = 0, \quad 2 \leq q < +\infty, \quad (2.22)$$

and for odd $n$,

$$\lim_{n \to \infty} \left\| \alpha \sqrt{n} \psi_{N_n} \left( \alpha \sqrt{n} x + \frac{1}{2} \right) - \frac{1}{2\pi} \sin \left( \alpha \omega_0 \sqrt{n} x \right) e^{-\frac{x^2}{2}} \right\|_{L^2} = 0, \quad 2 \leq q < +\infty. \quad (2.23)$$

Moreover,

$$\lim_{n \to \infty} \frac{1}{\alpha \sqrt{n}} \Delta_{\psi_{N_n}} = \lim_{n \to \infty} \alpha \sqrt{n} \Delta_{\psi_{N_n}} = \frac{1}{\sqrt{2}}, \quad (2.24)$$

so that

$$\lim_{n \to \infty} M^+(\psi_{N_n}) = \frac{1}{2}. \quad (2.25)$$

We remark that although (2.17), (2.18) and (2.21)–(2.23) were already established in [15], our approach and method are different.

In addition to cardinal polynomial $B$-splines and their corresponding semi-orthogonal $B$-wavelets, other stoplets have also been used in such applications as image processing. For example, P. Burt and E. Adelson [2] considered the sequences

$$\begin{aligned}
\begin{cases}
a_0 = \alpha, \\
a_1 = a_{-1} = \frac{1}{4}, \\
a_2 = a_{-2} = \frac{1}{8} - \frac{1}{2}b, \\
a_n = 0, \quad \text{otherwise,}
\end{cases}
\end{aligned}$$

where $\alpha \in \mathbb{R}$. Here, the symbol of $\mathbf{a}$ is given by

$$Q_2(\omega) = \left( \cos \frac{1}{2} \omega \right)^2 \left( 1 - b \sin^2 \frac{\omega}{2} \right),$$

with $b = 4(1 - 2\alpha)$. Note that for $0.375 \leq \alpha \leq 0.5$ (or $0 \leq b \leq 1$), $\mathbf{a}$ is a symmetric Pólya frequency sequence that generates a stoplet. In [2], the authors have pointed out that, when the Laplacian pyramid generated by this sequence $\mathbf{a}$ is used in the image encoding, the distribution of pixel gray level values at various stages of the encoding process depends on the values of parameter $\alpha$.

The Burt-Adelson filters can be easily extended from $m = 2$ to any order $m \geq 2$. For instance, for $m = 4$, we have

$$\begin{aligned}
\begin{cases}
a_0 = \frac{3}{2} - \frac{b}{4}, \\
a_1 = a_{-1} = \frac{1}{8} - \frac{b}{16}, \\
a_2 = a_{-2} = \frac{1}{8} + \frac{b}{8}, \\
a_3 = a_{-3} = \frac{b}{16}, \\
a_n = 0, \quad \text{otherwise,}
\end{cases}
\end{aligned}$$
For $0 \leq b \leq 1$, $a$ is a Pólya frequency sequence, and the symbol of $a$ is given by

$$Q_4(\omega) = \left( \cos \frac{1}{2} \omega \right)^4 \left( 1 - b \sin^2 \frac{\omega}{2} \right).$$

For odd integers $m$, however, a phase-shift by $e^{i\omega/2}$ of the symbol $Q_m(\omega)$ is needed. This corresponds simply to the integer indeces of the sequence $a$.

3 Uncertainty Principle for Bandpass Time-frequency Localization

The objective of this section is to establish Theorem 1.

**Proof of Theorem 1.** Without loss of generality, we assume that $\psi$ in Theorem 1 is symmetric (or anti-symmetric) with respect to the origin, i.e., $\psi(x) = \psi(-x)$ (or $\psi(x) = -\psi(-x)$). It follows that $\hat{\psi}(\omega) = \hat{\psi}(-\omega)$ (or $\hat{\psi}(\omega) = -\hat{\psi}(-\omega)$).

Setting $\hat{\psi}_+ = \chi_{[0, +\infty)} \hat{\psi}$, we have $\|\hat{\psi}_+\|_{L^2} = \frac{1}{2} \|\hat{\psi}\|_{L^2}$ and $\|\hat{\psi}'_+\|_{L^2} = \frac{1}{2} \|\hat{\psi}'\|_{L^2}$. Let us denote the inverse Fourier transforms of $\hat{\psi}_+$ by $\psi_+$. Then it is obvious that $\psi_+$ is also a window function. By Theorem A, $\Delta_{\psi_+} \geq \frac{1}{2}$ and the equality holds if and only if $\psi_+(x) = kG_\sigma(x - \mu)$. However, since $\text{supp} \hat{\psi}_+ = [0, +\infty)$, $\psi_+$ cannot be in the form of $kG_\sigma(x - \mu)$. Hence, any $\psi$ in Theorem 1 must satisfy

$$\Delta_{\psi_+} > \frac{1}{2}.$$  

(3.1)

We are now ready to prove that $M^+(\psi) > 1/2$ also. Note that since the time center of $\psi$ is zero and $\psi_+(x) = \psi_+(-x)$, the center of $\psi_+$ is also zero. Hence, we have

$$\Delta_{\psi} = \frac{\|x\psi(x)\|_{L^2}}{\|\psi\|_{L^2}} = \frac{\|\hat{\psi}'\|_{L^2}}{\|\hat{\psi}\|_{L^2}} = \frac{\|\hat{\psi}_+\|_{L^2}}{\|\psi_+\|_{L^2}} = \frac{\|x\psi_+(x)\|_{L^2}}{\|\psi_+\|_{L^2}} = \Delta_{\psi_+}.$$  

From the fact that $\Delta_{\psi}^+ = \Delta_{\hat{\psi}_+}$ and inequality (3.1), it follows that

$$M^+(\psi) := \Delta_{\psi} \Delta_{\hat{\psi}_+} = \Delta_{\psi_+} \Delta_{\hat{\psi}_+} > \frac{1}{2}.$$  

Next we will show that the lower band $\frac{1}{2}$ cannot be improved. For this purpose, we consider a sequence of functions $\psi_n$, with Fourier transform given by

$$\hat{\psi}_n(\omega) = \begin{cases} \left[ \left( \frac{1}{n} + n \right) \omega - \omega^2 \right] e^{-\left( \frac{\omega - \frac{1}{2}}{\frac{1}{n}} \right)^2}, & \omega \in \left[ 0, \frac{1}{n} \right], \\ e^{-\left( \frac{\omega - \frac{1}{2}}{\frac{1}{n}} \right)^2}, & \omega \in \left( \frac{1}{n}, +\infty \right), \\ -\hat{\psi}_n(-\omega), & \omega \in (-\infty, 0). \end{cases}$$
It can be verified that $\psi_n(x)$ satisfies the conditions in Theorem 1 and that
\[
\int_{-\infty}^{\infty} x |\psi_n(x)|^2 \, dx = 0.
\]
In addition, we also have
\[
\int_0^{\infty} \omega |\hat{\psi}_n(\omega)|^2 \, d\omega = n + o(1).
\]
Hence, it follows that $\Delta_{\psi_n} = \frac{1}{\sqrt{2}} + o(1)$ and $\Delta_{\hat{\psi}_n} = \frac{1}{\sqrt{2}} + o(1)$, so that
\[
\lim_{n \to \infty} \Delta_{\psi_n} \Delta_{\hat{\psi}_n} = \frac{1}{2}.
\]

4 Inequalities for stoplets and cowlets

The main task of this section is to establish certain inequalities for stoplets and cowlets that are needed in the proofs of Theorems 2 and 3. Let us start with the following fundamental fact about Pólya frequency sequences.

**Theorem B.** [11; Theorem 9.5, page 427]. A necessary and sufficient condition for $a = \{a_j\}$ to be a Pólya frequency sequence is that
\[
a(z) = rz^k \exp \left( sz + tz^{-1}\right) \frac{\prod_{j=1}^{\infty} (1 + \alpha_j z)}{\prod_{j=1}^{\infty} (1 + \beta_j z)} \frac{\prod_{j=1}^{\infty} (1 + \delta_j z^{-1})}{\prod_{j=1}^{\infty} (1 + \gamma_j z^{-1})},
\]
where $r > 0$, $s, t \geq 0$, $k \in \mathbb{Z}$, and the sequences $\alpha = \{\alpha_j\}$, $\beta = \{\beta_j\}$, $\delta = \{\delta_j\}$, and $\gamma = \{\gamma_j\}$ are all non-negative and summable.

If a Pólya frequency sequence $a$ is finite, then $s = t = 0$, $\beta = \gamma = 0$, and both $\alpha$ and $\delta$ are finite sequences. In addition, if the finite sequence $a$ is also symmetric and $\sum a_j = 1$, then $\alpha = \delta$ and $r = \prod (1 + a_j)^2$. Therefore, we have the following.

**Lemma 4.1.** If $a = \{a_j\}$ is a symmetric finite Pólya frequency sequence which satisfies $\sum a_j = 1$, then
\[
a(e^{-i\omega}) = e^{-i\frac{k \omega}{2}} \cos \frac{L \omega}{2} \prod_{j=1}^{m} \left( 1 - b_j \sin^2 \frac{\omega}{2} \right),
\]
where $L \in \mathbb{Z}^+$, $0 < b_j < 1$, $1 \leq j \leq m$, and $k = L + 2m = |a| - 1$. Moreover, if $\sum (-1)^j a_j = 0$, then $L \geq 1$.

Later we will use the notation
\[
A_L(\omega) = \cos \frac{L \omega}{2} \prod_{j=1}^{m} \left( 1 - b_j \sin^2 \frac{\omega}{2} \right).
\]
Then the Fourier transform of a stoplet $\phi_L$ with mask $a$ is given by

$$
\hat{\phi}_L(\omega) = e^{-i\frac{k\omega}{2}} \prod_{j=1}^{\infty} A_L \left( \frac{\omega}{2^j} \right).
$$

Note that the function $\hat{\phi}_L^c(x) := \phi_L(x + \frac{k}{2})$ has the same time-frequency window as that of $\phi_L$, but its center is at the origin. The advantage for introducing $\hat{\phi}_L^c$ is that its Fourier transform is a real-valued function given by

$$
\hat{\phi}_L^c(\omega) = \prod_{j=1}^{\infty} A_L \left( \frac{\omega}{2^j} \right).
$$

(4.3)

To simplify the notations, we sometimes omit the subscript “$L$” if there is no danger of any confusion. The following properties of $A(\omega)$ are obvious.

$$
A(\omega) > 0, \ |\omega| < \pi; \quad |A(\omega)| \leq 1, \ \omega \in \mathbb{R},
$$

and if $L \geq 1$,

$$
\left| \tan \frac{\omega}{2} A(\omega) \right| \leq 1, \ \omega \in \mathbb{R}. \quad (4.4)
$$

Hence, it follows that

$$
\hat{\phi}_L^c(\omega) > 0, \ |\omega| < 2\pi, \quad \text{and} \quad |\hat{\phi}_L^c(\omega)| \leq 1, \ \omega \in \mathbb{R}.
$$

On the other hand, we also have $A(0) = 1$, $A'(0) = 0$, and

$$
A''(0) = (\ln'' A(\omega))_{\omega=0} = - \left( \frac{L}{4} + \frac{1}{2} \sum_{j=1}^{m} b_j \right).
$$

(4.5)

Therefore,

$$
(\hat{\phi}_L^c)''(0) = \left( \sum_{j=1}^{\infty} \frac{1}{2^{2j}} \right) A''(0) = \frac{1}{3} A''(0) = - \left( \frac{L}{12} + \frac{1}{6} \sum_{j=1}^{m} b_j \right),
$$

(4.6)

and

$$
\sigma^2 = \left| (\hat{\phi}_L^c)''(0) \right| = \frac{L}{12} + \frac{1}{6} \sum_{j=1}^{m} b_j.
$$

(4.7)

The following lemma gives certain estimates for the derivatives of $A(\omega)$.

**Lemma 4.2.** Let $A(\omega)$ be defined as in (4.2). Then

(1) $A'(\omega) < 0$ for $0 < \omega < \pi$, and

$$
|A'(\omega)| \leq 2 \left| A''(0) \tan \frac{\omega}{2} A(\omega) \right|, \ \omega \in \mathbb{R}; \quad (4.8)
$$
\[ A''(0) \leq (\ln A)''(\omega) \leq A''(0) \cos \omega, \quad |\omega| \leq \frac{\pi}{2}; \quad (4.9) \]

\[ (\ln A)'''(\omega) \leq 2 \left| A''(0) \sec^5 \frac{\omega}{2} \sin \frac{\omega}{2} \right|, \quad \omega \in \mathbb{R}. \quad (4.10) \]

**Proof.** Since
\[ A'(\omega) = -A(\omega) \left( \frac{L}{2} \tan \frac{\omega}{2} + \frac{1}{2} \sum_{j=1}^{m} \frac{b_j \sin \omega}{1 - b_j \sin^2 \frac{\omega}{2}} \right), \]
we have \( A'(\omega) < 0 \), for \( 0 < \omega < \pi \). In addition, since
\[ \left| \frac{b_j \sin \omega}{1 - b_j \sin^2 \frac{\omega}{2}} \right| \leq 2b_j \left| \tan \frac{\omega}{2} \right|, \quad \omega \in \mathbb{R}, \]
we have, by applying (4.5), (4.8). To prove (4.9), we simply recall that
\[ (\ln A)''(\omega) = -\left( \frac{L}{4 \cos^2 \frac{\omega}{2}} + \sum_{j=1}^{m} \frac{b_j \left( 1 - (2 - b_j) \sin^2 \frac{\omega}{2} \right)}{2 \left( 1 - b_j \sin^2 \frac{\omega}{2} \right)^2} \right), \]
\[ \frac{b_j \cos \omega}{2} \leq \frac{b_j \left( 1 - (2 - b_j) \sin^2 \frac{\omega}{2} \right)}{2 \left( 1 - b_j \sin^2 \frac{\omega}{2} \right)^2} \leq b_j, \quad |\omega| \leq \frac{\pi}{2}, \]
and
\[ \frac{L}{4 \cos \omega} < \frac{L}{4 \cos^2 \frac{\omega}{2}} \leq \frac{1}{2} L. \]
Finally, assertion (4.10) follows from
\[ (\ln A)'''(\omega) = -\left( \frac{L \sin \frac{\omega}{2}}{4 \cos^3 \frac{\omega}{2}} + \sin \omega \sum_{j=1}^{m} \frac{b_j \left[ (3b_j - 2) - (2b_j - b_j^2) \sin^2 \frac{\omega}{2} \right]}{4 \left( 1 - b_j \sin^2 \frac{\omega}{2} \right)^3} \right) \]
and
\[ \left| \frac{(3b_j - 2) - (2b_j - b_j^2) \sin^2 \frac{\omega}{2}}{(1 - b_j \sin^2 \frac{\omega}{2})^3} \right| \leq \frac{2}{\cos^6 \frac{\omega}{2}}. \]

**Lemma 4.3.** Let \( \tilde{\phi}_L^{\epsilon} \) be defined as in (4.3). Then \( (\tilde{\phi}_L^{\epsilon})'(\omega) < 0, \omega \in (0, 2\pi), \)
\[ |(\tilde{\phi}_L^{\epsilon})'(\omega)| < \tilde{\phi}_L^{\epsilon}(\pi) \left| \frac{\pi}{\omega} \sin \frac{\omega}{2} \right| L, \quad |\omega| \geq \pi; \quad (4.11) \]
\[ |(\tilde{\phi}_L^{\epsilon})'(\omega)| \leq 6\omega^2 L \tilde{\phi}_L^{\epsilon-1}(\omega), \quad \omega \in \mathbb{R}; \quad (4.12) \]
\[ -\sigma_L^2 \leq (\ln \tilde{\phi}_L^{\epsilon})''(\omega) \leq -\sigma_L^2 \cos \frac{\omega}{2}, \quad |\omega| \leq \pi; \quad (4.13) \]
and there is a constant $C > 0$ such that

$$\left| \left( \ln \hat{\phi}_L^c \right)^{\prime\prime\prime}(\omega) \right| \leq C\sigma^2, \quad |\omega| \leq 3\pi/2. \quad (4.14)$$

**Proof.** The inequality $(\hat{\phi}_L^c)'(\omega) < 0, \omega \in (0, 2\pi)$, is trivial. To prove (4.11), we set

$$P(\omega) = \prod_{j=1}^m \left( 1 - b_j \sin^2 \frac{\omega}{2} \right)$$

and

$$T(\omega) = \prod_{\ell=1}^{\infty} P \left( \frac{\omega}{2^\ell} \right).$$

It is obvious that $|P(\omega)| \leq 1$ and $P(\omega)$ is strictly decreasing on $[0, \pi]$. Hence, $T(\omega)$ is also strictly decreasing on $[0, 2\pi]$ and $T(\pi) > |T(\omega)| \geq 0, \; |\omega| \in (\pi, 2\pi)$. Now for any $\omega, \; |\omega| > 2\pi$, let us choose $k \in \mathbb{Z}^+$ that satisfies $2^k \pi < |\omega| \leq 2^{k+1} \pi$, so that

$$|T(\omega)| = \left| \prod_{\ell=1}^k P \left( \frac{\omega}{2^\ell} \right) T \left( \frac{\omega}{2^k} \right) \right| \leq |T \left( \frac{\omega}{2^k} \right)| < T(\pi).$$

Since

$$\hat{\phi}_L^c(\omega) = \left( \frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} \right)^L T(\omega) \leq \left( \frac{\sin \frac{\pi \omega}{2}}{\frac{\omega}{2}} \right)^L \frac{2^L}{\pi} T(\pi) \leq \hat{\phi}_L^c(\pi) \left| \frac{\pi}{\omega} \right|^L \sin \left( \frac{\omega}{2} \right)^L,$$

It is clear that (4.11) holds.

Next, since

$$\hat{\phi}_L^c'(\omega) = \hat{\phi}_L^c(\omega) \sum_{j=1}^{\infty} 2^{-j} \left( \frac{d}{d\omega} \ln A_L \right) \left( \frac{\omega}{2^j} \right),$$

we have, by (4.8),

$$\left| \left( \frac{d}{d\omega} \ln A_L \right) \left( \frac{\omega}{2^j} \right) \right| \leq 6\sigma^2 \left| \tan \frac{\omega}{2^{j+1}} \right|.$$ 

Therefore, if follows from

$$\left| \tan \frac{\omega}{2^{j+1}} \right| = \left| 2^j \sin^2 \frac{\omega}{2^{j+1}} \csc \frac{\omega}{2} \prod_{\ell=1}^{j-1} \cos \frac{\omega}{2^{\ell+1}} \right| \leq 2^j \sin \frac{\omega}{2^{j+1}} \csc \frac{\omega}{2} \leq \left| \frac{\omega}{2} \csc \frac{\omega}{2} \right|$$

that

$$\left| (\hat{\phi}_L^c)'(\omega) \right| \leq 6\sigma^2 \left( \sum_{j=1}^{\infty} \frac{1}{2^j} \right) \left| \frac{\omega}{2} \csc \frac{\omega}{2} \hat{\phi}_L^c(\omega) \right| \leq 6\sigma^2 \left| \hat{\phi}_{L-1}^c(\omega) \right|, \quad \forall \omega \in \mathbb{R}.$$ 

This establishes (4.12).

Similarly, we have

$$-\sigma^2 \leq (\ln \hat{\phi}_L^c)''(\omega) = \sum_{k=1}^{\infty} \frac{1}{2^{2k}} (\ln A_L)''' \left( \frac{\omega}{2^k} \right) \leq -3\sigma^2 \left( \sum_{k=1}^{\infty} \frac{1}{2^{2k}} \cos \frac{\omega}{2^k} \right), \quad |\omega| \leq \pi,$$
and this implies (4.13).

Finally, (4.14) can be derived from the fact that
\[
\left| (\ln \hat{\varphi}_L)^\prime\prime\prime(\omega) \right| = \sum_{k=1}^{\infty} \left( \frac{1}{2^{3k}} (\ln A_L)^\prime\prime\prime(\frac{\omega}{2^k}) \right) \leq \sigma_L^2 \left| \frac{\omega}{4} \sin \frac{\omega}{4} \right|, \quad \omega \in \mathbb{R}.
\]

This completes the proof of the lemma.

Now we can establish certain estimates for the rates of decay of both \( \hat{\varphi}^c \) and \( (\hat{\varphi}^c)' \).

**Lemma 4.4.** Let \( L \geq C\sigma \) for some positive \( C \) independent of \( L \). Then there is a function \( g \in L^1 \cap L^\infty \) such that
\[
\left| \hat{\varphi}^c \left( \frac{\omega}{\sigma} \right) \right| \leq g(\omega), \quad |\omega\hat{\varphi}^c \left( \frac{\omega}{\sigma} \right)| \leq g(\omega), \quad \text{and} \quad \left| (\hat{\varphi}^c)' \left( \frac{\omega}{\sigma} \right) \right| \leq g(\omega).
\]

**Proof.** We first prove that
\[
\left| \hat{\varphi}^c \left( \frac{\omega}{\sigma} \right) \right| \leq e^{-\frac{\omega^2}{4}}, \quad |\omega| \leq \frac{2\pi\sigma}{3}.
\] (4.15)

For this purpose, we note that \( \ln \hat{\varphi}(0) = (\ln \hat{\varphi}^c)'(0) = 0 \), so that
\[
\ln \hat{\varphi}^c \left( \frac{\omega}{\sigma} \right) = \frac{\omega^2}{2\sigma^2} (\ln \hat{\varphi}^c)' \left( \frac{\theta\omega}{\sigma} \right), \quad 0 \leq \theta \leq 1.
\]

Hence, by (4.13), we have, for \( |\omega| \leq \frac{2\pi\sigma}{3} \),
\[
\left( \ln \hat{\varphi}^c \right)' \left( \frac{\theta\omega}{\sigma} \right) \leq -\sigma^2 \cos(\pi/3) \leq -\frac{1}{2}\sigma^2,
\]

so that
\[
\ln \hat{\varphi}^c \left( \frac{\omega}{\sigma} \right) \leq -\frac{1}{4}\omega^2
\]

and (4.15) follows.

On the other hand, for \( |\omega| \geq \frac{2\pi\sigma}{3} \), we have
\[
\left| \hat{\varphi}^c \left( \frac{\omega}{\sigma} \right) \right| \leq \left| \frac{2\sigma}{\omega} \sin \frac{\omega}{2\sigma} \right|^L T \left( \frac{\omega}{\sigma} \right) \leq \left| \frac{2\sigma}{\omega} \sin \frac{\omega}{2\sigma} \right|^L.
\]

Since \( \left| \frac{2\sigma}{\omega} \right| \leq \frac{\pi}{3} < 1 \) and \( L \geq C\sigma \), we have, for sufficiently large \( L \),
\[
\left| \hat{\varphi}^c \left( \frac{\omega}{\sigma} \right) \right| \leq \left| \frac{2\sigma}{\omega} \right|^C \sigma < \frac{1}{1 + |\omega|^3}, \quad |\omega| > \frac{2\pi\sigma}{3}.
\] (4.16)

Combining (4.15) and (4.16), we obtain
\[
\hat{\varphi}(\omega/\sigma) \leq e^{-\frac{1}{4}\omega^2} + \frac{1}{1 + |\omega|^3}
\]

and
\[
\omega \hat{\varphi}(\omega/\sigma) \leq \omega e^{-\frac{1}{4}\omega^2} + \frac{1}{1 + |\omega|^2}.
\]
Finally, since \( |\hat{\phi}^c(\frac{x}{\sigma})''|' = 1 \), we obtain, by applying (4.12), \( |(\hat{\phi}^c)'_L(\frac{x}{\sigma})| \leq |\hat{\phi}^c_{L-1}(\frac{x}{\sigma})| \leq e^{-\frac{1}{2}r^2} + \frac{1}{1+|\omega|^2}. \) This complete the proof of the lemma.

Now we start to deal with the estimates of \( \hat{\psi}^c \) and its derivatives. For convenience, we set \( \psi^c(x) = \psi(x + \frac{1}{2}) \). Then from (2.5), we see that \( \|\hat{\psi}^c\|_{\infty} = 1 \), and its Fourier transform is a real-valued function given by

\[
\hat{\psi}^c(\omega) = C_\psi A \left(\pi - \frac{\omega}{2}\right) E_\phi \left(\pi - \frac{\omega}{2}\right) \hat{\phi}^c \left(\frac{\omega}{2}\right). \quad (4.17)
\]

Next, we will establish some properties of \( E_\phi \). For this purpose, we recall a result from [10].

**Theorem C.** Let \( \phi(x) \) be a function with a Pólya frequency sequence mask. Then the determinants

\[
\det_{\ell,j=1,\ldots,p} \phi \left( x_\ell - i_j \right)
\]

are non-negative for any \( p \geq 1 \), any \( x_1 < \ldots < x_p \), and any integers \( i_1 < \ldots < i_p \).

As a consequence of Theorem C, we have the following.

**Lemma 4.5.** \( E_\phi \) takes on the form

\[
E_\phi(\omega) = \prod_{j=1}^{k} \left( 1 - r_j \sin^2 \frac{\omega}{2} \right), \quad (4.18)
\]

where \( 0 < r_j < 1 \), \( 1 \leq j \leq k \).

**Proof.** By (2.2), we have \( \hat{\Phi}(\omega) = |\hat{\phi}(\omega)|^2 \), so that

\[
\hat{\Phi}(\omega) = A \left(\frac{\omega}{2}\right)^2 \hat{\Phi} \left(\frac{\omega}{2}\right),
\]

which implies that the mask of \( \Phi \) is also a finite Pólya frequency sequence. By Theorem C, \( \det_{\ell,j=1,\ldots,p} \Phi(k_\ell - i_j) \) is non-negative for any integers \( k_1 < \ldots < k_p \) and \( i_1 < \ldots < i_p \). That is, \( \{\Phi(j)\} \) is a Pólya frequency sequence. Note that since \( \sum \Phi(j) = \sum |\hat{\phi}(2k\pi)|^2 = 1 \), \( \Phi(j) = \Phi(-j) \), and \( E_\phi(\pi) > 0 \), it follows by Lemma 4.1 that the function \( E_\phi(\omega) := \sum \Phi(j) e^{-ij\omega} \) must take on the form (4.18).

Since \( E_\phi \) takes on the form (4.18), we may apply Lemma 4.2 to establish the following.

**Lemma 4.6.** Let the integer \( L \) in the symbol \( A \) of the function \( \phi \) be at least \( 1 \). Then

\[
E_\phi(\omega) \leq 2 \left[ \hat{\phi}^c(\omega) \right]^2, \quad |\omega| \leq \frac{\pi}{2}; \quad (4.19)
\]

and

\[
-2\sigma^2 \leq (\ln E_\phi)'(\omega) \leq 0, \quad |\omega| \leq \pi/2. \quad (4.20)
\]
Proof. Since $\frac{\omega/2}{\sin(\omega/2)} \leq \frac{\pi}{2\sqrt{2}}$, for $|\omega| \leq \pi/2$, we have, by (4.11),

$$
|\hat{\varphi}^c(2k\pi + \omega)| \leq \hat{\varphi}^c(\pi) \left( \frac{\pi}{2|k|\pi - |\omega|} \right)^L \leq \hat{\varphi}^c(\omega) \left( \frac{\pi}{2|k|\pi - |\omega|} \right)^L \left( \frac{2\pi}{\sqrt{2}} \right)^L \left| \frac{\omega}{2\sin \frac{\omega}{2}} \right|^L
$$

so that

$$
E_\varphi(\omega) = \sum_{k \in \mathbb{Z}} \left| \hat{\varphi}^c(\omega + 2k\pi) \right|^2 \leq |\hat{\varphi}^c(\omega)|^2 \left( 1 + \sum_{k=1}^{\infty} \left( \frac{2L+1}{(4k-1)^2L} \right) \right) \leq 2|\hat{\varphi}^c(\omega)|^2.
$$

This establishes (4.19). To prove (4.20), we recall that $\hat{\varphi}^c(2k\pi) = \delta_{0k}$, which leads to

$$
E''_\varphi(0) = 2 \sum_{k \in \mathbb{Z}} \left( \hat{\varphi}''^c(2k\pi) \hat{\varphi}^c(2k\pi) + \left[ \hat{\varphi}'^c(2k\pi) \right]^2 \right) \geq 2\hat{\varphi}''^c(0) = -2\sigma^2.
$$

On the other hand, we have, by (4.18), $E''_\varphi(0) < 0$. Hence, applying (4.13) to the function $E_\varphi(\omega)$, we obtain (4.20).

Lemma 4.7. Let $L \geq 1$. Then

$$
0 \leq \hat{\psi}^c(\omega) \leq \hat{\psi}^c(2\pi) \left( \frac{8\sin(\omega/2)}{\pi^2} \right)^L, \quad 0 \leq \omega \leq \pi; \quad (4.21)
$$

$$
|\hat{\psi}^c(\omega)| \leq \hat{\psi}^c(2\pi) \left( \frac{2\pi \sin(\omega/4)}{|\omega|} \right)^L, \quad \omega \geq 2\pi; \quad (4.22)
$$

and

$$
-\frac{3}{2} \sigma^2 \leq \left( \ln \hat{\psi}^c \right)''(\omega) \leq -\frac{\sqrt{2}}{8} \sigma^2, \quad \pi \leq \omega \leq 2\pi. \quad (4.23)
$$

In addition,

$$
\left| \left( \ln \hat{\psi}^c \right)'''(\omega) \right| \leq C\sigma^2, \quad \pi \leq \omega \leq 3\pi, \quad (4.24)
$$

where $C$ is a constant independent of $L$.

Proof. By (4.19), we have, for $0 \leq \omega \leq \pi$,

$$
0 \leq \hat{\psi}^c(\omega) \leq 2C_\psi A(\pi - \omega/2)\hat{\varphi}^c(\pi - \omega/2)\hat{\varphi}^c(\omega/2) = 2C_\psi \hat{\varphi}^c(\omega/2)\hat{\varphi}^c(\pi - \omega/2)\hat{\varphi}^c(2\pi - \omega).
$$

Since $(\ln \hat{\psi}^c(\omega))'' < 0$, for $0 < \omega < \pi$, we see that $\ln \hat{\psi}^c(\omega)$ is concave downward on $(0, \pi)$, and therefore, we have

$$
\hat{\varphi}^c(\omega/2)\hat{\varphi}^c(\pi - \omega/2) \leq e^{2\hat{\varphi}^c(\pi/2)} \leq \left( \frac{2\sin \frac{\pi}{4}}{\pi} \right)^{2L} = \left( \frac{8}{\pi^2} \right)^L, \quad 0 \leq \omega \leq \pi.
$$
Now, from (4.17), we have $C_\psi \hat{\phi}^c(\pi) = \hat{\psi}^c(2\pi)$, and it follows from (4.11) that

$$
\hat{\phi}^c(2\pi - \omega) \leq \hat{\phi}^c(\pi) \left( \frac{\pi \sin(\omega/2)}{2(\pi - \omega/2)} \right) \leq C_\psi^{-1} \hat{\psi}^c(2\pi) \sin L(\omega/2), \quad \omega \in [0, \pi].
$$

This yields (4.21).

The inequality (4.22) can be derived directly from (4.11) by using the fact that

$$
|\hat{\psi}^c(\omega)| \leq C_\psi |\hat{\phi}^c(\omega/2)| \leq C_\psi \hat{\phi}^c(\pi) \left| \frac{2\pi(\sin \frac{\omega}{4})}{\omega} \right|^L, \quad \omega \geq 2\pi.
$$

Next, applying (4.9), (4.13), and (4.20) to $A$, $\hat{\phi}^c$, and $E_\phi$, respectively, we have, for $\pi < \omega \leq 2\pi$,

$$
-\frac{3}{4} \sigma^2 (= \frac{1}{4} A''(0)) \leq (\ln A(\pi - \omega/2))^\prime \leq 0,
-\frac{1}{2} \sigma^2 \leq (\ln E_\phi(\pi - \omega/2))^\prime \leq 0,
-\frac{1}{4} \sigma^2 \leq (\ln \hat{\phi}^c(\omega))^\prime \leq -\frac{1}{4} \sigma^2 \cos(\frac{\omega}{4}).
$$

A combination of these inequalities gives (4.23). Finally, (4.24) can be derived from (4.10) and (4.14).

The next lemma determines the location of the unique value, at which the function $|\hat{\psi}^c(\omega)|$, $\omega \in [0, \infty)$, attains its maximum value.

**Lemma 4.8.** Assume that $L \geq 1$ in (4.2). Then there exists a unique value $\omega_0$, at which the function $|\hat{\psi}^c(\omega)|$, $\omega \in [0, \infty)$, attains its maximum value 1. Moreover, $\omega_0 \in (\pi, 2\pi)$.

**Proof.** It is easy to see that $\hat{\psi}^c(\omega) \geq 0$, $\omega \in [\pi, 2\pi]$. Recall, from (4.23), that $\ln \hat{\psi}^c(\omega)$ is strictly concave downward on $[\pi, 2\pi]$. Therefore, there is a unique value $\omega_0 \in [\pi, 2\pi]$, at which the function $\ln \hat{\psi}^c(\omega)$, $\omega \in [\pi, 2\pi]$, (and hence the function $\hat{\psi}^c(\omega)$, $\omega \in [\pi, 2\pi]$,) attains its maximum value. To prove that $\omega_0$ is also the only position in $[0, \infty]$, at which the function $|\hat{\psi}^c(\omega)|$ attains its maximum value, we simply apply (4.21) and (4.22) to show that $|\hat{\psi}^c(\omega)| \leq \hat{\psi}^c(2\pi)$, for $\omega \in [0, \pi) \cup (2\pi, \infty)$. This completes the proof of the lemma.

Note that $\hat{\psi}^c(\omega_0) = 1$ and $(\hat{\psi}^c)'(\omega_0) = 0$. Hence, $(\ln \hat{\psi}^c)^\prime(\omega_0) = (\hat{\psi}^c)^\prime(\omega_0)$. Since $\omega_0 \in (\pi, 2\pi)$, it follows from (4.23) that

$$
\frac{\sqrt{3}}{4} \sigma \leq \tau := \sqrt{|\hat{\psi}^c(\omega_0)|} \leq \frac{\sqrt{6}}{2} \sigma. \quad (4.25)
$$

We are now ready to estimate the rates of decay of $\hat{\psi}^c$ and $(\hat{\psi}^c)'$. 

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**Lemma 4.9.** Let $L \geq C\sigma^2$, where $C > 0$ is a constant independent of $L$. Then there is a function $G \in L^1 \cap L^\infty$ such that $|\hat{\psi}^c(\omega + \omega_0)| \leq G(\omega)$, $|\omega \hat{\psi}^c(\omega + \omega_0)| \leq G(\omega)$, and $\left|\left(\hat{\psi}^c(\omega + \omega_0)\right)\right|' \leq G(\omega)$.

**Proof.** Similar to the proof of Lemma 4.4, we first give the estimates of $\hat{\psi}^c$ on different intervals.

(i) For $0 \leq \frac{\omega}{\tau} + \omega_0 < \pi$, since $\pi \leq \omega_0 \leq 2\pi$, we have $(\frac{\omega}{\tau})^2 \leq \pi^2$. Hence, by (4.21), we obtain

$$
|\hat{\psi}^c\left(\frac{\omega}{\tau} + \omega_0\right)| \leq \left(\frac{8}{\pi^2}\right)^L \left(\sin\left(\frac{\omega}{2\tau} + \frac{\omega_0}{2}\right)\right)^L \leq e^{-L\ln\frac{\pi^2}{2}} \left(\sin\left(\frac{\omega}{2\tau} + \frac{\omega_0}{2}\right)\right)^L.
$$

(4.26)

Note that $\frac{L}{\tau^2} \geq 2C/3$. Hence, by setting $s = \frac{3}{2C\pi^2} \ln\left(\frac{\pi^2}{8}\right)$, we have

$$
|\hat{\psi}^c\left(\frac{\omega}{\tau} + \omega_0\right)| \leq e^{-s\omega^2} \left|\sin\left(\frac{\omega}{2\tau} + \omega_0/2\right)\right|^L.
$$

(4.27)

(ii) For $\pi \leq \frac{\omega}{\tau} + \omega_0 \leq 2\pi + \delta$, where $\delta > 0$ is some constant to be determined later, since $\ln\hat{\psi}^c(\omega_0) = (\ln\hat{\psi})'(\omega_0) = 0$, we have, by Taylor expansion,

$$
\left(\ln\hat{\psi}\right)\left(\frac{\omega}{\tau} + \omega_0\right) = \frac{\omega^2}{8\tau^2} (\ln\hat{\psi})''\left(\frac{\theta\omega}{\tau} + \omega_0\right), \ 0 \leq \theta \leq 1.
$$

From (4.23) and (4.24), it follows that there is a sufficiently small $\delta > 0$, such that, for $\pi \leq \frac{\omega}{\tau} + \omega_0 \leq 2\pi + \delta$,

$$
\ln\hat{\psi}^c\left(\frac{\omega}{\tau} + \omega_0\right) \leq -\frac{\omega^2}{8\tau^2} C\sigma^2 \leq -\frac{C}{12}\omega^2,
$$

which is equivalent to

$$
0 < \hat{\psi}^c\left(\frac{\omega}{\tau} + \omega_0\right) \leq e^{-\frac{C}{12}\omega^2}, \ \pi \leq \frac{\omega}{\tau} + \omega_0 \leq 2\pi + \delta.
$$

(4.28)

(iii) For $\frac{\omega}{\tau} + \omega_0 > 2\pi + \delta$, we have $\hat{\psi}^c(2\pi) < 1$. Hence, we have, from (4.22),

$$
\left|\hat{\psi}^c\left(\frac{\omega}{\tau} + \omega_0\right)\right| \leq \left(\frac{2\pi}{\omega + \omega_0}\right)^L \left|\sin\left(\frac{\omega}{2\tau} + \frac{\omega_0}{2}\right)\right|^L
$$

(4.29)

which yields

$$
\left|\hat{\psi}^c\left(\frac{\omega}{\tau} + \omega_0\right)\right| < \frac{1}{|\omega|^3}, \ \frac{\omega}{\tau} + \omega_0 > 2\pi + \delta.
$$

(4.30)

Combining (4.27), (4.28), and (4.30), we obtain the first two estimates in the lemma.

To establish the last estimate in the lemma, we take the derivative of $\hat{\psi}^c$ in the form of (4.17). Then by (4.8) and (4.12), we have

$$
\left|\left(\frac{d}{d\omega}\hat{\psi}^c\right)(\omega + \omega_0)\right| \leq C\left|\omega \csc\left(\frac{\omega}{4\tau} + \frac{\omega_0}{4}\right)\hat{\psi}^c\left(\frac{\omega}{\tau} + \omega_0\right)\right|.
$$

Using (4.26), (4.28), and (4.29), we complete the proof of the lemma.
5 Proofs of Theorems 2 and 3

We are now ready to complete the proofs of Theorems 2 and 3. First it is obvious that Lemma 4.2 yields

\[ \frac{n}{12} \leq |\sigma_n^2| \leq \frac{1 + C}{6} n, \quad (5.1) \]

so that \( \lim_{n \to \infty} \sigma_n = +\infty \). Now let us prove that for any choice of \( M > 0 \), we have

\[ \lim_{n \to \infty} \max_{|\omega| \leq M} \left| \frac{\hat{\phi}_n^c (\omega \sigma_n)}{\sigma_n} - e^{-\frac{\omega^2}{2}} \right| = 0, \quad (5.2) \]

and

\[ \lim_{n \to \infty} \max_{|\omega| \leq M} \left| \left( \frac{\hat{\phi}_n^c (\omega \sigma_n)}{\sigma_n} \right)' - e^{-\frac{\omega^2}{2}} \right| = 0. \quad (5.3) \]

For this purpose, we set \( e_n(\omega) = \ln \hat{\phi}_n^c (\frac{\omega}{\sigma_n}) - (-\frac{\omega^2}{2}) \). Since \( \lim_{n \to \infty} \sigma_n = \infty \), \( e_n(\omega) \) is well defined on \([-M, M]\) for sufficiently large \( n \). Now, recall that \( e_n(0) = e'_n(0) = e''_n(0) = 0 \). Then we have

\[ |e_n(\omega)| = \left| \frac{\omega^3}{6} \right| \max_{|\theta| \leq |\omega|} |e''_n(\theta)| = \left| \frac{\omega^3}{6\sigma_n^3} \right| \max_{|\theta| < M} \left| \left( \frac{d^3}{d\omega^3} \ln \hat{\phi}_n^c \right) \left( \frac{\theta}{\sigma_n} \right) \right|. \]

Let \( n \) be so large that \( \frac{M}{\sigma_n} < \pi \). Then, it follows from (4.14) that

\[ \max_{|\omega| \leq M} |e_n(\omega)| \leq \frac{M^3}{6\sigma_n^3} \cdot 8\sigma_n^2 = \frac{4M^3}{3\sigma_n} \to 0, \quad n \to \infty, \]

and this gives (5.2).

Similarly,

\[ \left| \left( \frac{\hat{\phi}_n^c (\omega \sigma_n)}{\sigma_n} - e^{-\frac{\omega^2}{2}} \right)' \right| \leq \left| \omega \left( \frac{\hat{\phi}_n^c (\omega \sigma_n)}{\sigma_n} - e^{-\frac{\omega^2}{2}} \right) \right| + |\hat{\phi}_n^c(\omega)e'_n(\omega)| \leq M|e_n(\omega)| + |e'_n(\omega)|, \]

and \( |e'_n(\omega)| \leq \frac{\omega^3}{2} |e''_n(\theta)| \). Hence, we have

\[ \max_{|\omega| \leq M} |e'_n(\omega)| \leq \frac{M^2}{2\sigma_n^3} 8\sigma_n^2 = \frac{4M^2}{\sigma_n} \to 0, \quad n \to \infty, \]

which gives (5.3). From (5.2) and Lemma 4.4, we now obtain (2.8) and (2.9).

Note that we also have \( \lim_{n \to \infty} \| \hat{\phi}_n \left( \frac{\omega}{\sigma_n} \right) \|_{L^2} = \| e^{-\frac{\omega^2}{2}} \|_{L^2} = 1 \) and

\[ \lim_{n \to \infty} \left\| \omega \hat{\phi}_n \left( \frac{\omega}{\sigma_n} \right) e^{-i\frac{\omega^2}{2\sigma_n} - \omega e^{-\frac{\omega^2}{2}}} \right\|_{L^2} = 0, \]

which implies that

\[ \lim_{n \to \infty} \left\| \omega \hat{\phi}_n \left( \frac{\omega}{\sigma_n} \right) \right\|_{L^2} = \lim_{n \to \infty} \| \omega e^{-\frac{\omega^2}{2}} \|_{L^2} = \frac{1}{\sqrt{2}}. \]
Hence, we may conclude that
\[ \lim_{n \to \infty} \sigma_n \Delta \hat{\phi}_n = \frac{1}{\sqrt{2}}. \]

On the other hand, since
\[ \int_{-\infty}^{\infty} x^2 f^2(x) dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \hat{f}'(\omega) \right|^2 d\omega, \]
we have
\[ \int_{-\infty}^{\infty} \left( x - \frac{t_{\phi_n}}{\sigma_n} \right)^2 \sigma_n^2 \phi_n^2(\sigma_n x) dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \hat{\phi}_n \left( \frac{\omega}{\sigma_n} \right) \right|^2 d\omega, \]
so that it follows from (5.3) and Lemma 4.5 that
\[ \lim_{n \to \infty} \frac{1}{\sigma_n} \Delta \phi_n = \frac{1}{\sqrt{2}}. \]

This completes the proof of Theorem 2 and the proof of Theorem 3 is similar.

In the following, we give the proofs of Corollaries 1 and 2.

Let \( N_n^c \) denote the center \( B \)-spline of order \( n \), that is \( \hat{N}_n^c(\omega) = \left( \frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} \right)^n \). It is easy to see that \( \frac{d^2}{d\omega^2} \hat{N}_n^c(\omega) \big|_{\omega=0} = -\frac{n}{12} \). Hence, Corollary 1 holds.

To prove Corollary 2, we denote by \( \omega_n \) the unique value in \( [0, \infty] \), at which the function \( \hat{\psi}_N^c(\omega) \) attains its maximum value. By Lemma 4.8, we have that \( \omega_n \in (\pi, 2\pi) \). Now we set \( D(\omega) = dC(\omega) \), where \( d \) is a constant so chosen that \( \| D(\omega) \|_\infty = 1 \). It is easy see that \( n\alpha^2 = \left| \left( \frac{d^2}{d\omega^2} D^n \right)(\omega_0) \right| \). Then by Theorem 3, it is clear that Corollary 2 follows, provided that \( \lim_{n \to \infty} \omega_n = \omega_0 \) and
\[
\lim_{n \to \infty} \left( \frac{\psi_N^c}{(D^n)^\prime}(\omega_n) \right) = 1.
\]

We first prove that \( \lim_{n \to \infty} \omega_n = \omega_0 \). Assume, on the contrary, that this is not true. Then there exists a subsequence \( \{ \omega_{n_j} \} \) of \( \{ \omega_n \} \), such that
\[
\lim_{j \to \infty} \omega_{n_j} = \omega_\infty \neq \omega_0.
\]

By the estimate of \( \left( \frac{\psi_N^c}{\omega_n} \right)'(\omega_n) \) in Lemma 4.9, we have
\[
\lim_{j \to \infty} \left( \hat{\psi}_{N_{n_j}}^c(\omega_\infty) - \hat{\psi}_{N_{n_j}}^c(\omega_{n_j}) \right) = \lim_{j \to \infty} \hat{\psi}_{N_{n_j}}^c(\omega_\infty) - 1 = 0.
\]

On the other hand, if we set
\[
G_n(\omega) = C_{\psi_N^c} \hat{N}_n^c \left( \frac{\omega}{2} \right) \hat{N}_n^c \left( \frac{\omega}{2} - \pi \right) \hat{N}_n^c (\omega - 2\pi)
\]
and
\[
G_{n,k}(\omega) = C_{\psi_N^c} \hat{N}_n^c \left( \frac{\omega}{2} \right) \hat{N}_n^c \left( \frac{\omega}{2} - \pi - 2k\pi \right) \hat{N}_n^c (\omega - 2\pi - 4k\pi),
\]

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then $G_n(\omega) = C_{\psi_Nn}C^n(\omega)$ and
\[
G_{n,k}(\omega) = G_n(\omega) \frac{(\pi - \frac{\omega}{2})^{2n}}{(2k\pi + \pi - \frac{\omega}{2})^{2n}}.
\]
Therefore, since $\omega_{\infty} \in (\pi, 2\pi)$, we have
\[
\sum_{k \neq 0} G_{n,k}(\omega_{\infty}) = o(G_n(\omega_{\infty})), \quad n \to \infty.
\]
Also, from
\[
\hat{\psi}_{Nn}(\omega) = \sum_{k \in \mathbb{Z}} G_{n,k}(\omega),
\]
we see that
\[
\lim_{j \to \infty} G_{n_j}(\omega_{\infty}) = \lim_{j \to \infty} \hat{\psi}_{Nn_j}(\omega_{\infty}) = 1. \tag{5.4}
\]
Now, since the maximum value of the function $C(\omega)$, $\omega \in [0, \infty)$, is uniquely attained at some $\omega_0 \in (\pi, 2\pi)$ and $\omega_0$ is unique, we see that if $\omega_{\infty} \neq \omega_0$, then
\[
\lim_{j \to \infty} \frac{G_{n_j}(\omega_{\infty})}{G_{n_j}(\omega_0)} = \lim_{j \to \infty} \left[ \frac{C(\omega_{\infty})}{C(\omega_0)} \right]^{n_j} = 0.
\]
This contradiction proves that $\omega_{\infty} = \omega_0$.

Similarly, we can also verify that
\[
\left( \sum_{k \neq 0} G_{n,k} \right)''(\omega) = o \left( \left( G_n \right)''(\omega) \right), \quad n \to \infty,
\]
for any $\omega \in (\pi, 2\pi)$. Note that $G_n(\omega) = \frac{C_{\psi_n}}{d^n} D^n(\omega)$, so that by (5.4), we have
\[
\lim_{n \to \infty} \frac{C_{\psi_n}}{d^n} = \lim_{n \to \infty} \frac{G_n(\omega_0)}{D^n(\omega_0)} = \lim_{n \to \infty} G_n(\omega_0) = 1,
\]
and this implies that
\[
\lim_{n \to \infty} \left( \frac{\psi_{Nn}}{D^n} \right)''(\omega_0) = \lim_{n \to \infty} \left( \frac{\psi_{Nn}}{D^n} \right)''(\omega_n) = 1.
\]
This completes the proof of Corollary 2.

References


