

Life, Networks and Games

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About Me

- 1956: Born and grew up in Yazoo City, Mississippi
- 1978: BS in Math and Physics, MSU
 - Co-op student with Xerox Corp.
- 1983: MS in Applied Math, Northeastern U
 - Worked at MIT Lincoln Lab & MITRE Corp.
- 1897: PhD in Math, Vanderbilt U - Graph Theory
 - Worked at Telcordia (Bellcore) & Bell Labs
 - Data mining & network visualization
 - PBS series “Life by the Numbers”
- 1998: Worked at Rice U
- 2003: Professor and Chair of Math at TSU
- 2006: Professor at Texas State U.-San Marcos



Inspiration

- Life has puzzles and competitions
 - war, politics, romance, athletics, business, ...
- Proper strategy is unclear.
- Can we win? Draw? Avoid losing everything? How?
- Mathematics is a tool that might help us find some answers.



OUTLINE

Topics from Graph Theory

- Euler Trails
- Vertex Coloring
- Directed Graphs
- Neighbor Set, Independent Set, Absorbant Set, Kernel
- An Unsolved Problem

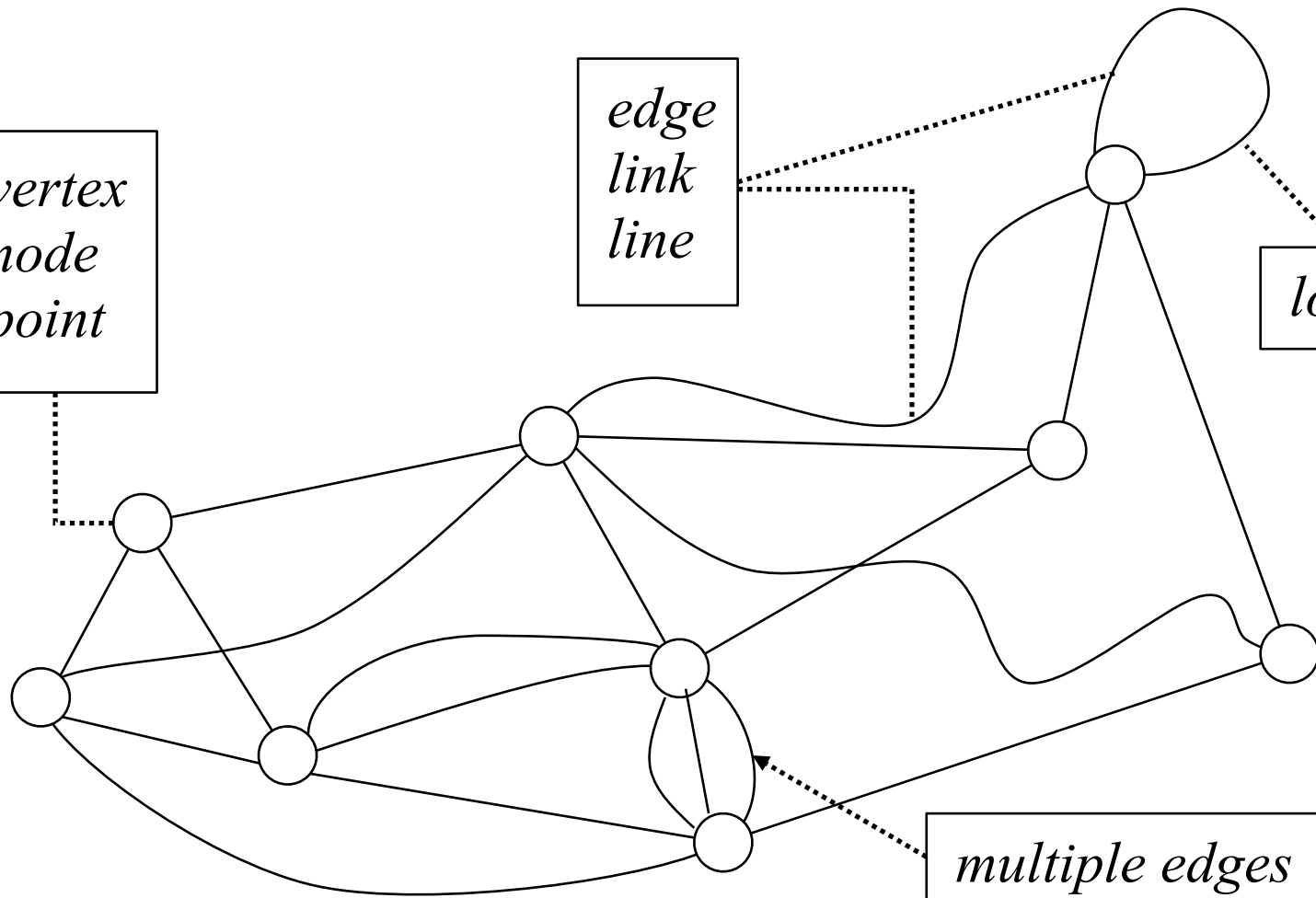
Graph (or Multigraph)

vertex
node
point

edge
link
line

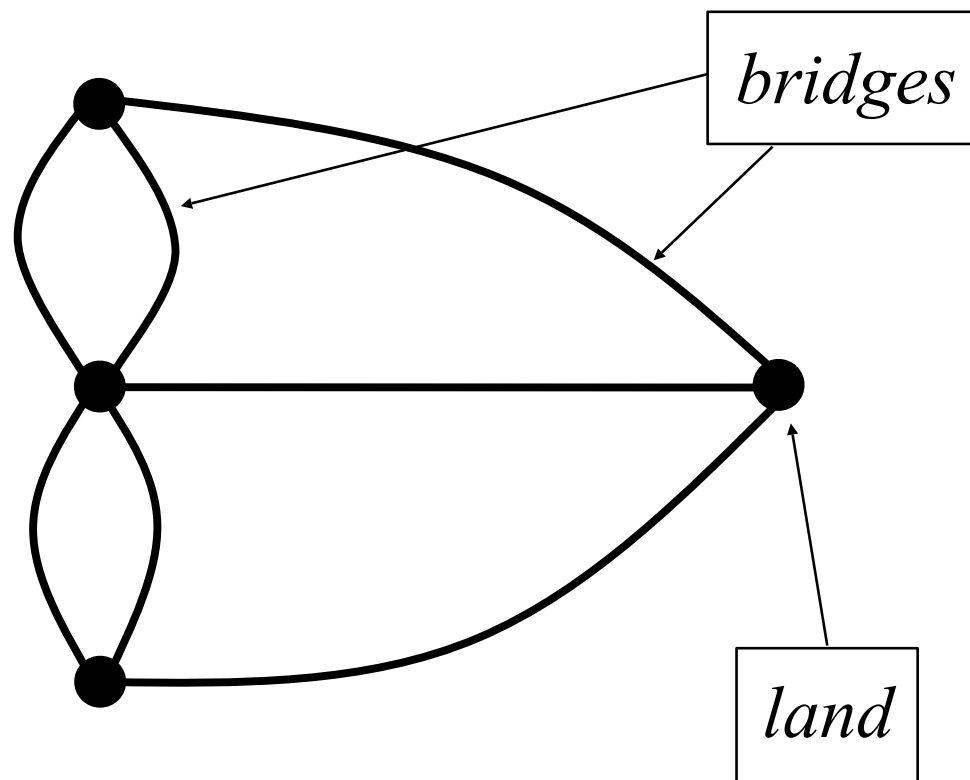
loop

multiple edges



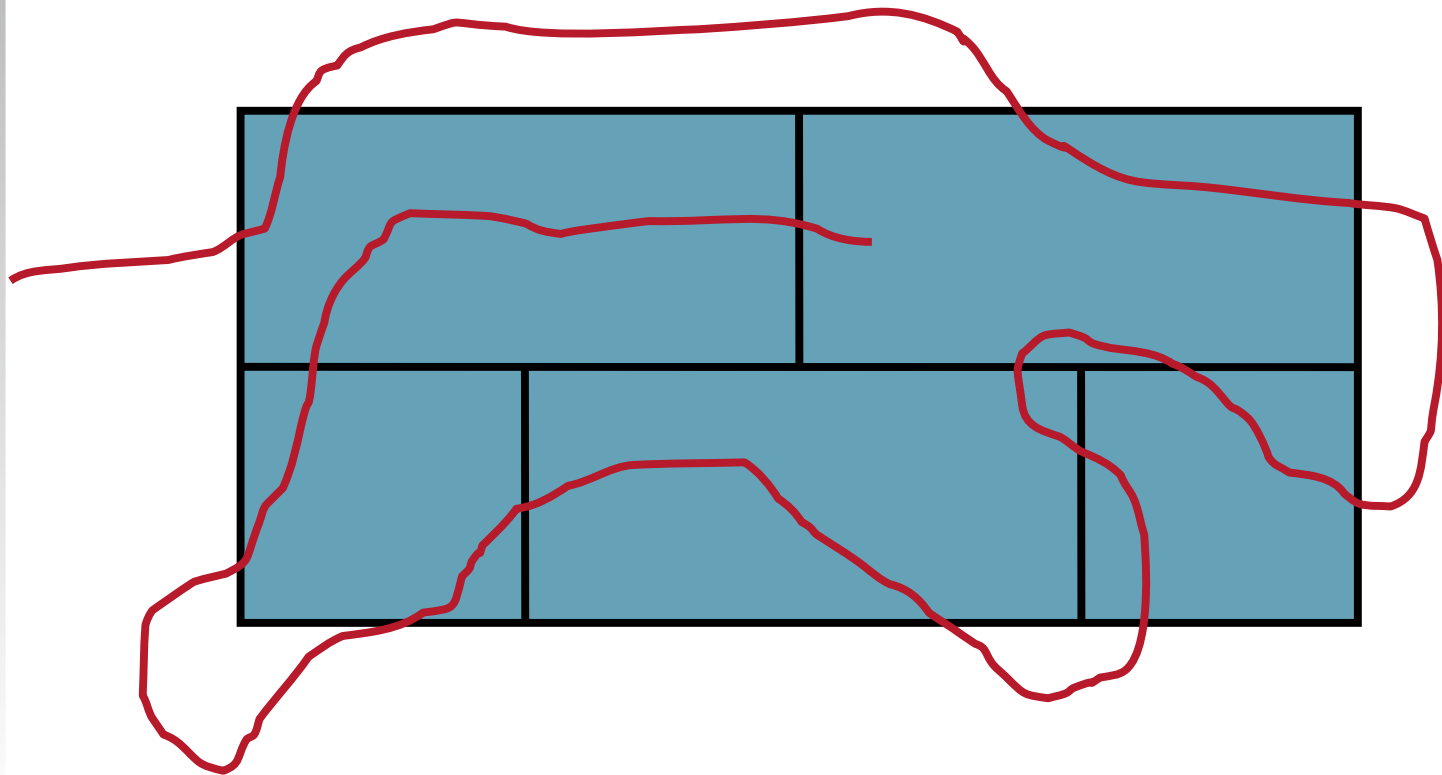
Bridges of Königsberg Problem

Can each bridge be traversed exactly once?



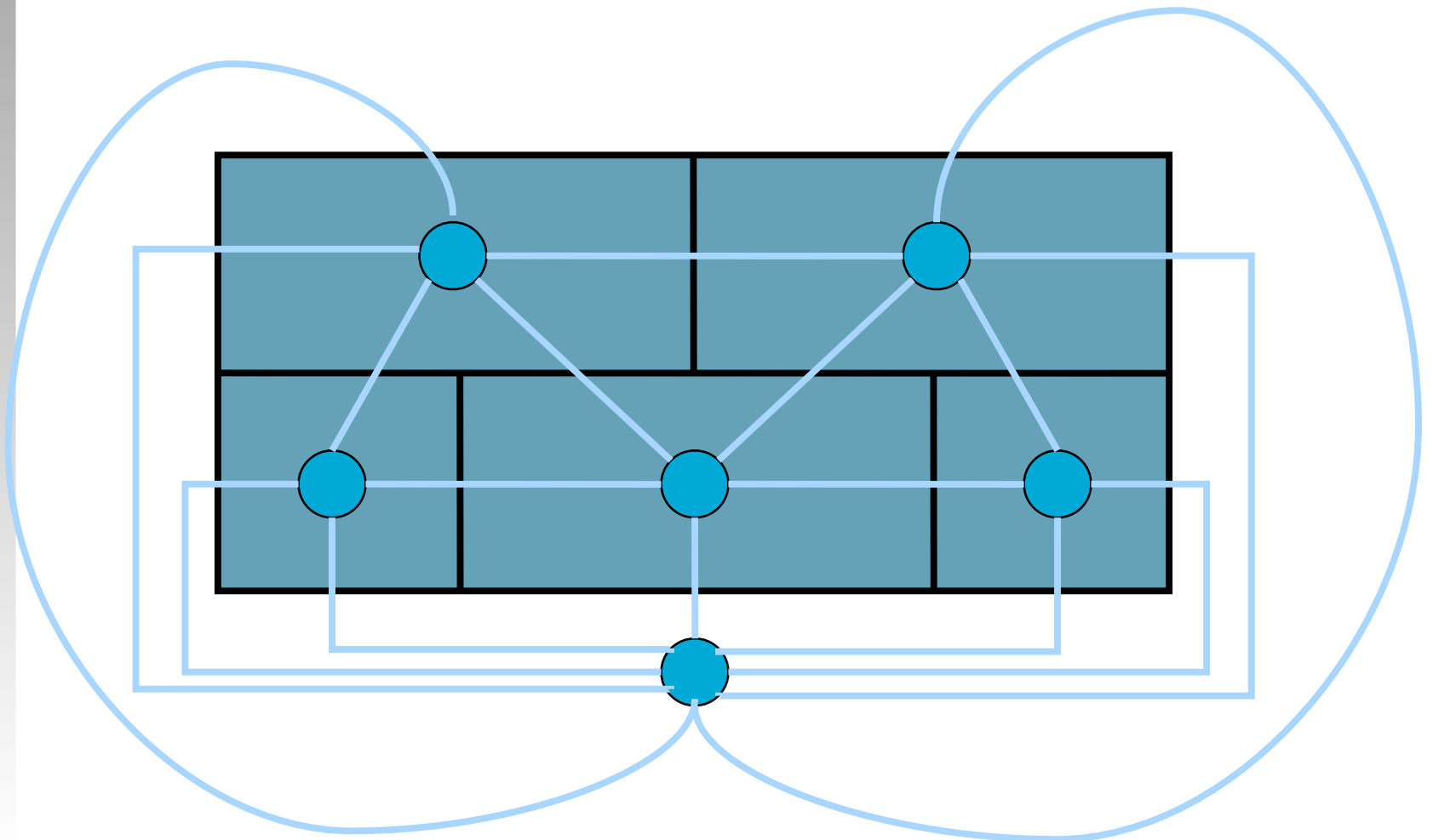
Puzzle

Draw a continuous curve crossing each line segment exactly once.



Solution: Construct the Dual

Traverse each edge of the dual exactly once.





No Solution

Definition

An **Euler trail** is a sequence $v_1, e_1, v_2, e_2, v_3, e_3, \dots$ of vertices and edges where **no edge is repeated** and v_i and v_{i+1} are the ends of e_i .

Theorem

A connected graph has an Euler trail iff it has **at most two vertices of odd degree**.

Conclusion

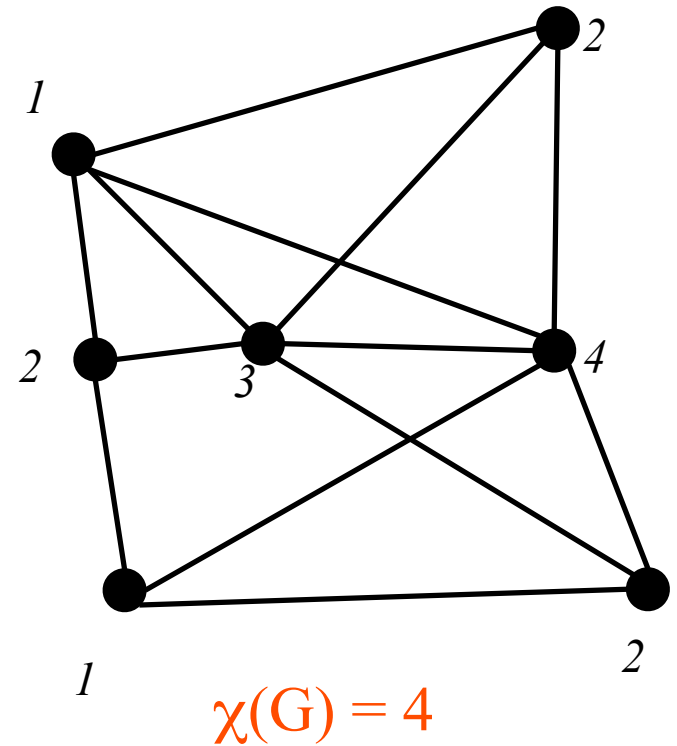
Since the graph has 4 vertices of odd degree, it has no Euler trail.

Party Problem

- People P_1, P_2, \dots, P_n meet for a party, but certain pairs are incompatible.
- *Goal:* Assign people to rooms so that no two people in the same room are incompatible.
- How many rooms are needed?

Vertex Coloring

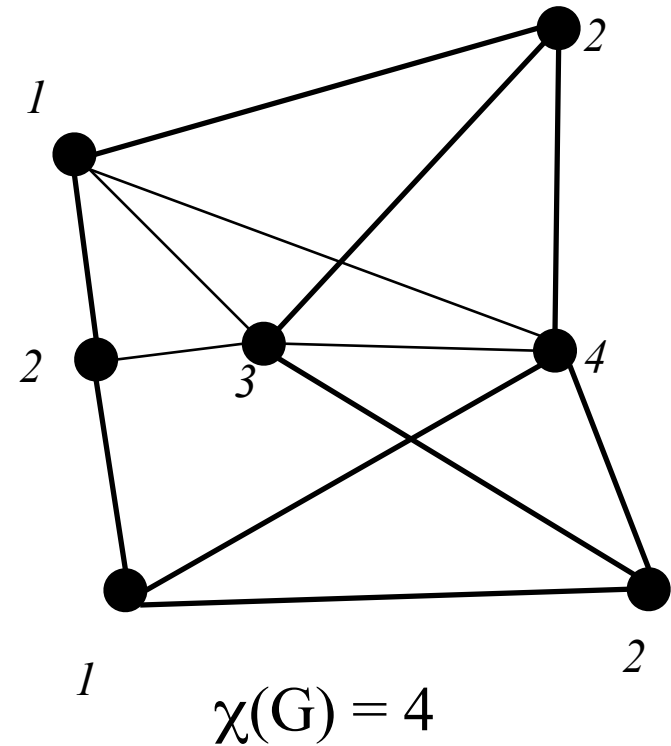
- A *k-coloring* is a labeling $f:V(G) \rightarrow \{1,2,\dots,k\}$.
- A k-coloring is *proper* if $xy \in E(G)$ implies $f(x) \neq f(y)$.
- G is *k-colorable* if it has a proper k-coloring.
- The *chromatic number* $\chi(G)$ is the smallest k such that G is k-colorable.



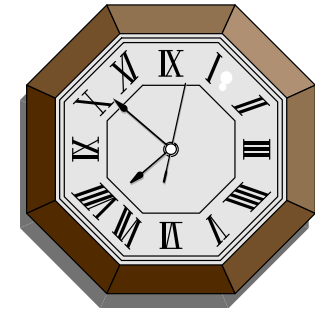
Solution to the Party Problem

Construct a conflict graph G .

- $V(G) = \{P_1, P_2, \dots, P_n\}$.
- $P_i, P_j \in E(G)$ iff P_i and P_j are incompatible.
- The **chromatic number** $\chi(G)$ is the least number of rooms.



Scheduling Problem

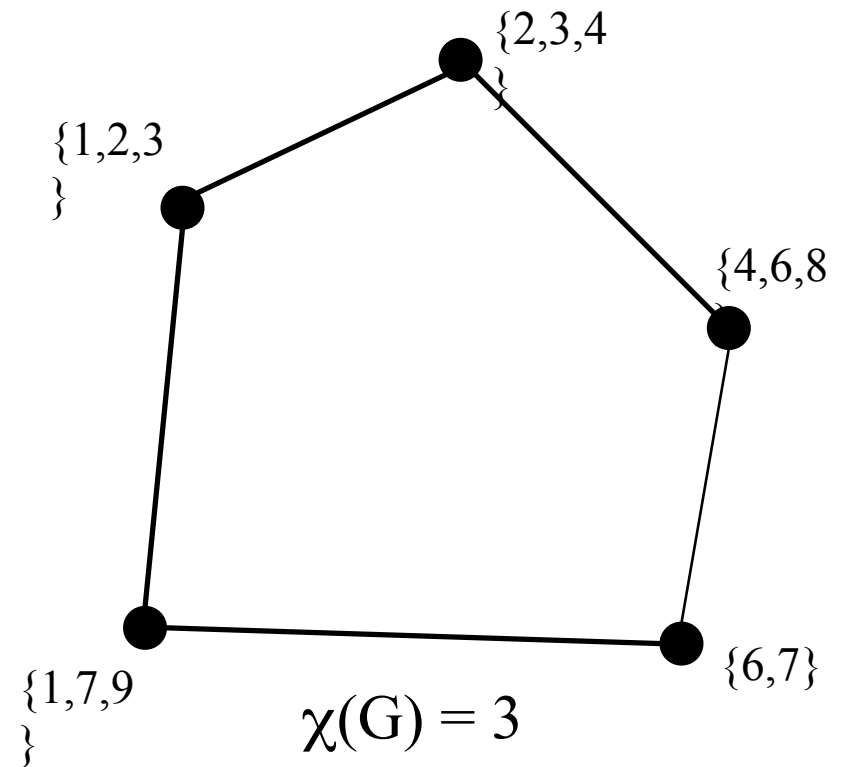


- Five different groups of students $\{1,2,3\}$, $\{6,7\}$, $\{1,7,9\}$, $\{4,6,8\}$, $\{2,3,4\}$ must take exams in the following Mathematics courses S_1 , S_2 , S_3 , S_4 , S_5 , respectively.
- *Goal:* Schedule the exams using a minimum number of time periods.

Solution to the Scheduling Problem

Construct a conflict graph G .

- $V(G) = \{S_1, S_2, S_3, S_4, S_5\}$.
- $S_i, S_j \in E(G)$ iff $S_i \cap S_j \neq \emptyset$.
- The **chromatic number** $\chi(G)$ is the minimum number of time periods.





Sam Houston Chemical Company (SHCC) Problem

- Wishes to manufacture a new batch of chemicals C_1, C_2, \dots, C_n , but for storage certain pairs are incompatible.
- *Goal:* Partition the warehouse into compartments storing incompatible chemicals in different compartments.
- What is the least possible number of compartments?



Subtraction Game

- Let S be a set (called the subtraction set) of positive integers.
- From a pile of n chips two players alternately remove $s \in S$ chips from the pile.
- The last player to move wins.
- Example: Let $n = 21$ and $S = \{1, 2, 3\}$.
 - *Survivor Thailand*

Subtraction Game Example

Player 1 removes		1		3		2		3		1	
Player 2 removes			2		3		2		1		3
Chips Remaining	2	2	1	1	1	1	8	5	4	3	0

Player 2 is the winner!!

What is a Combinatorial Game?

1. Clearly defined rules that specify the moves each player can make.

- not war, politics, romance, business



2. Exactly two players who alternate choosing a move.

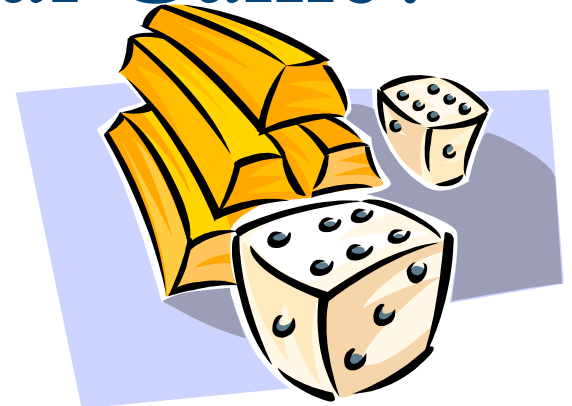
- not Solitaire, football



3. A player loses and the game ends when that player is unable to make a move.

- (no draw) not Tic-Tac-Toe, Chess

What is a Combinatorial Game?



4. It ends.

- not Monopoly

5. Both player have full information about the state of the game.

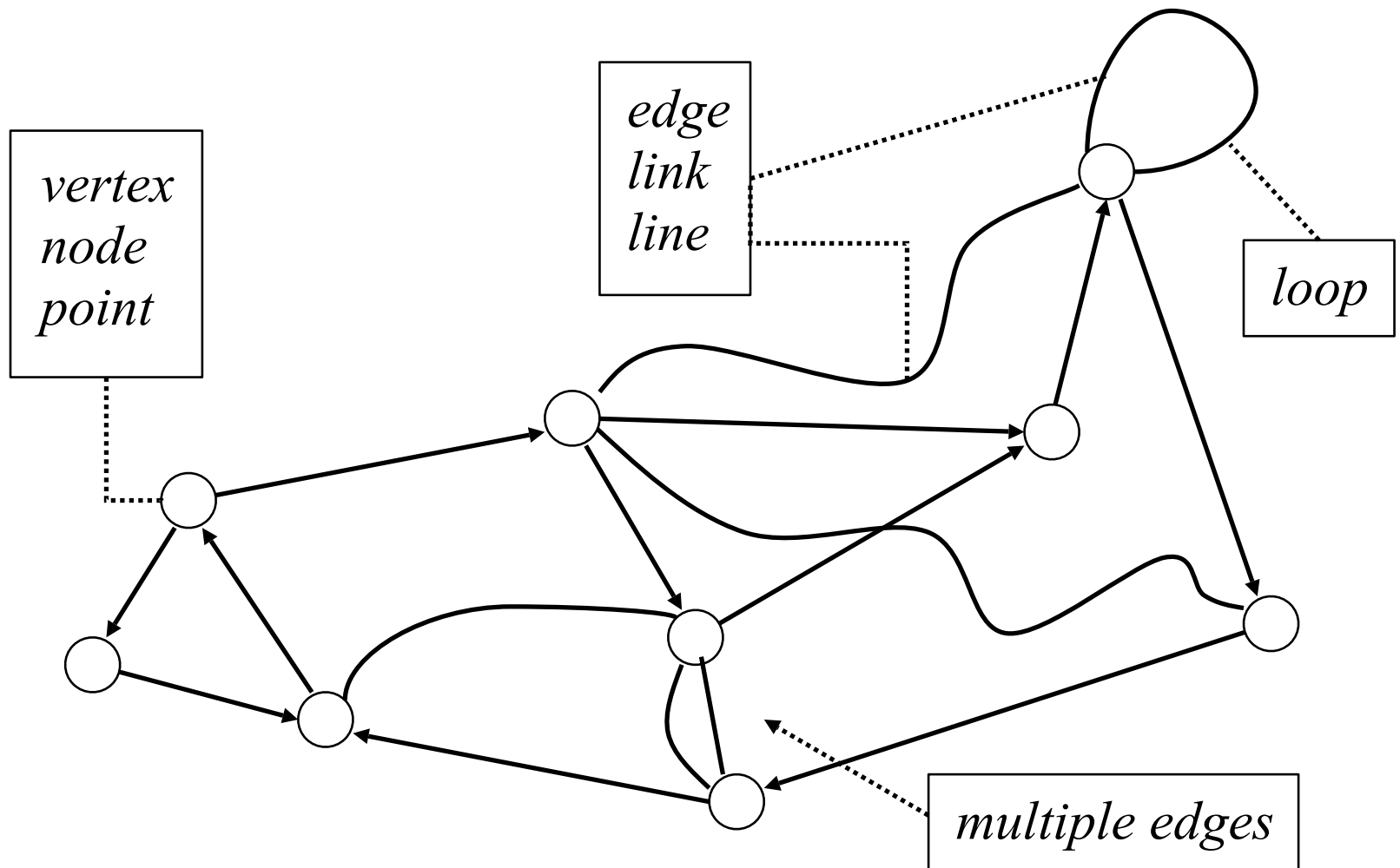
- not Poker, Scissors-Paper-Stone

6. There are no chance moves.

- not Poker, Backgammon, Monopoly

Digraph (Directed Graph)

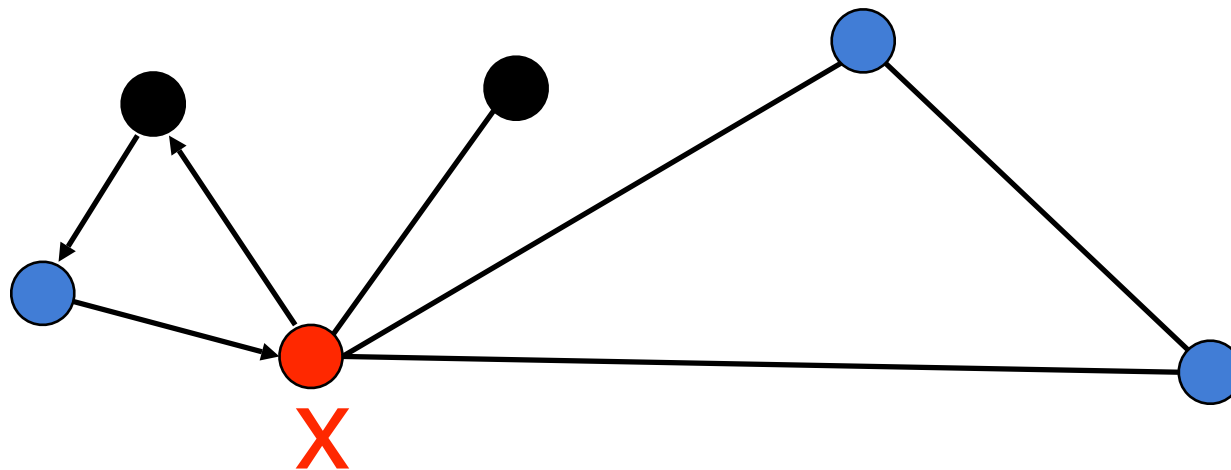
$$D = (V, A)$$



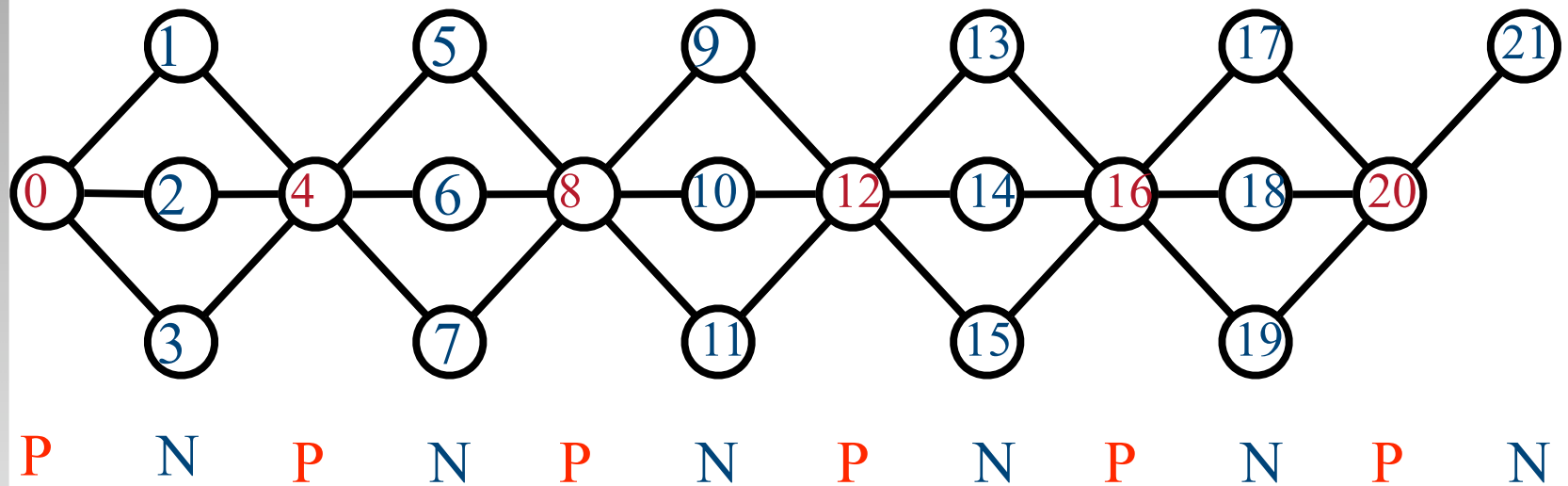
Neighbors of a Vertex

Outset $O(x) = \{v: xv \in A\}$

Inset $I(x) = \{w: wx \in A\}$



Solution to the Subtraction Game



P => win for the previous player

N => win for the next player



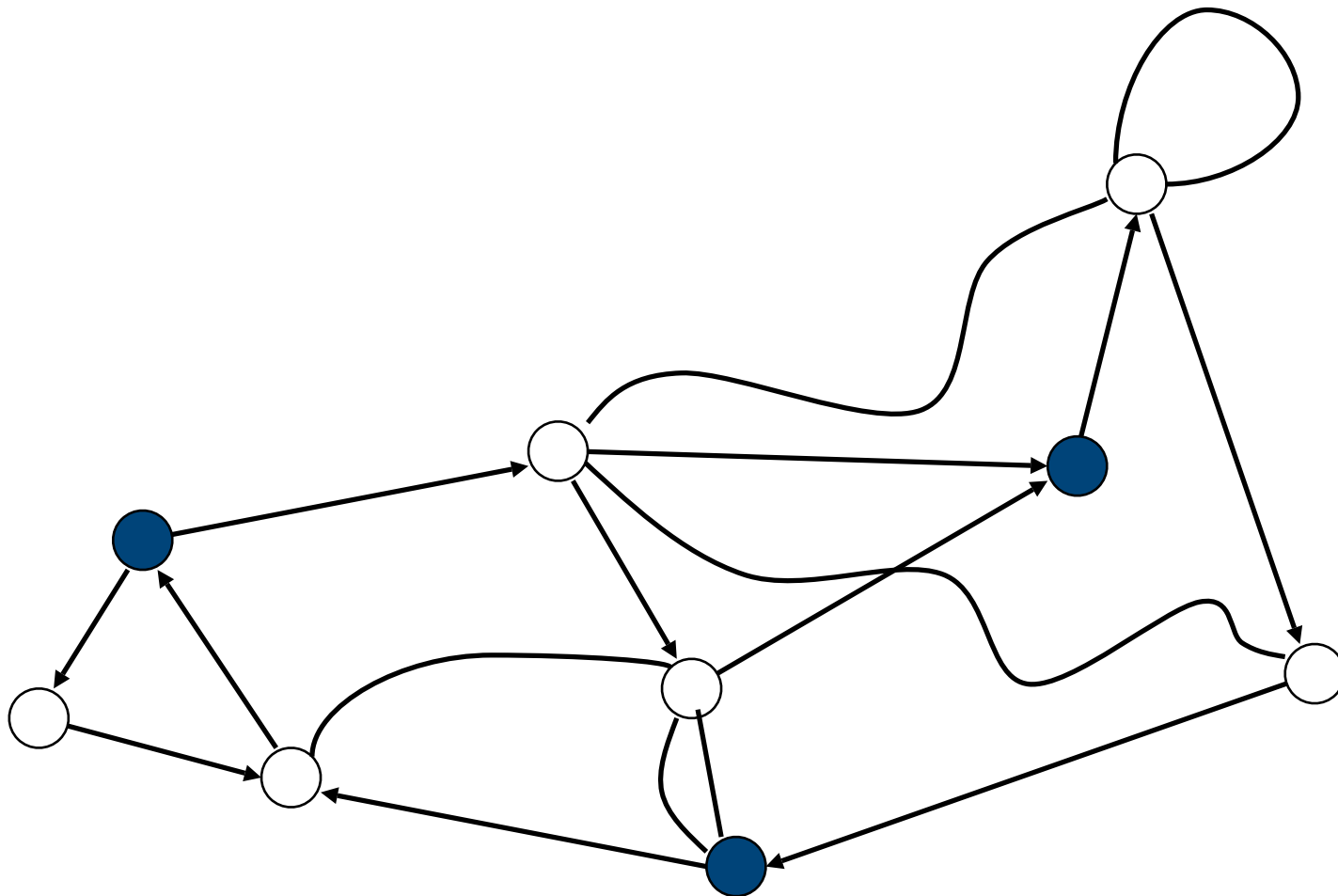
Algorithm

1. Label every terminal position as a **P-position**.
2. Label every position that can reach a labeled P-position in one move as an **N-position**.
3. If no new N-positions are found, STOP.
4. Label every position whose only moves are to labeled N-positions as a **P-position**.
5. If no new P-positions are found, STOP.
6. **Return to step 2**

The strategy of moving to P-positions wins.

Independent Set

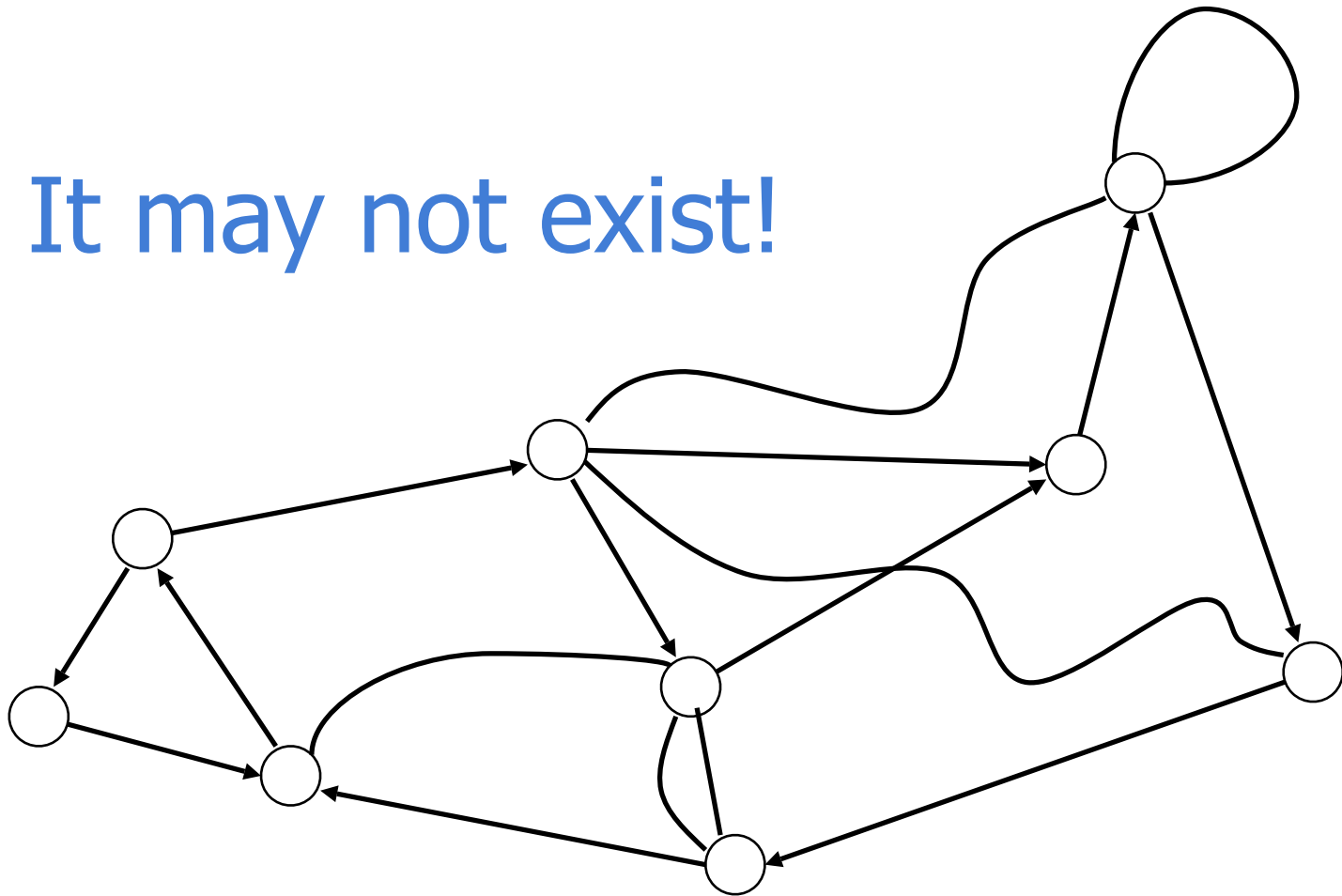
No two vertices are adjacent.



Kernel

Independent and Absorbant

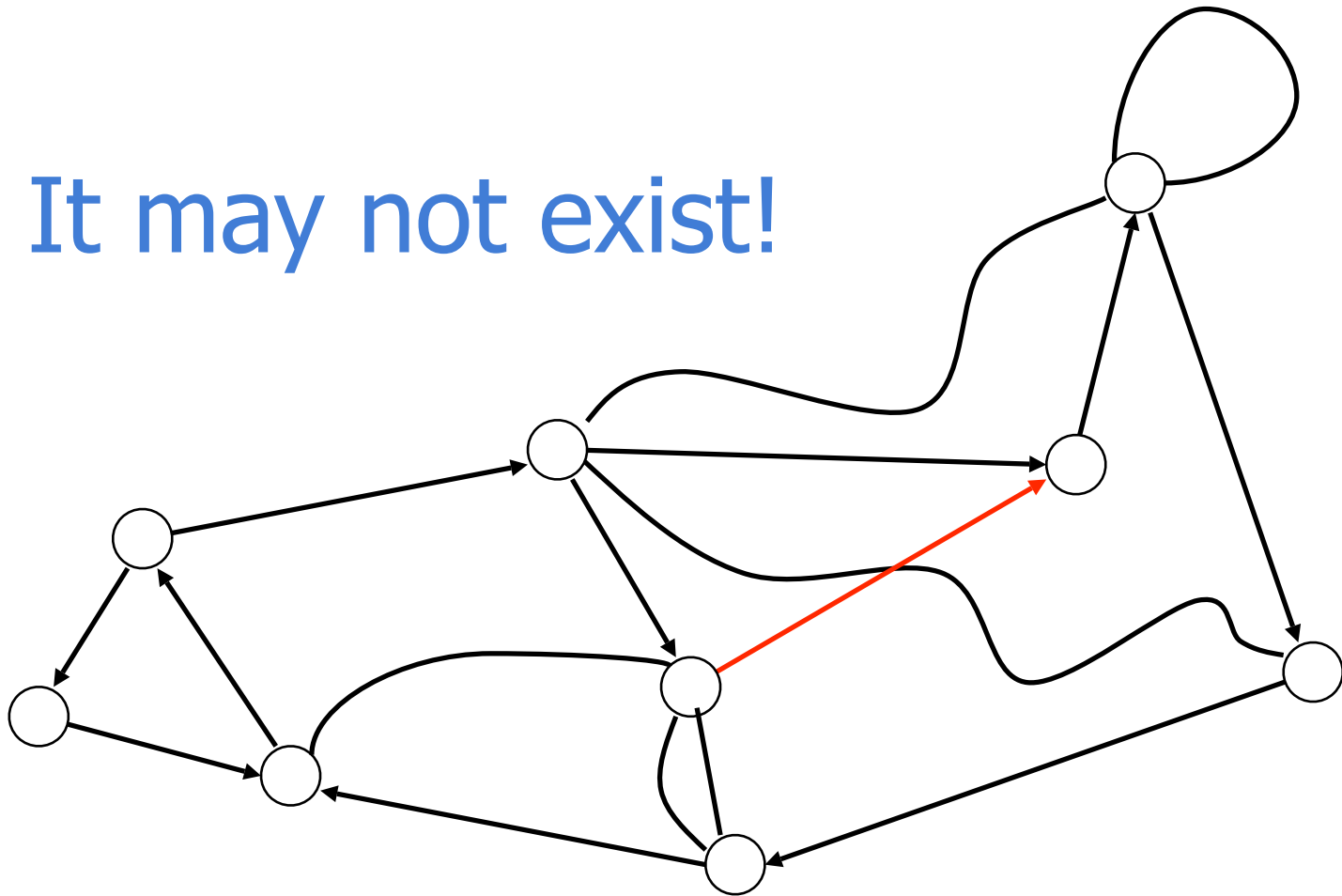
It may not exist!



Kernel

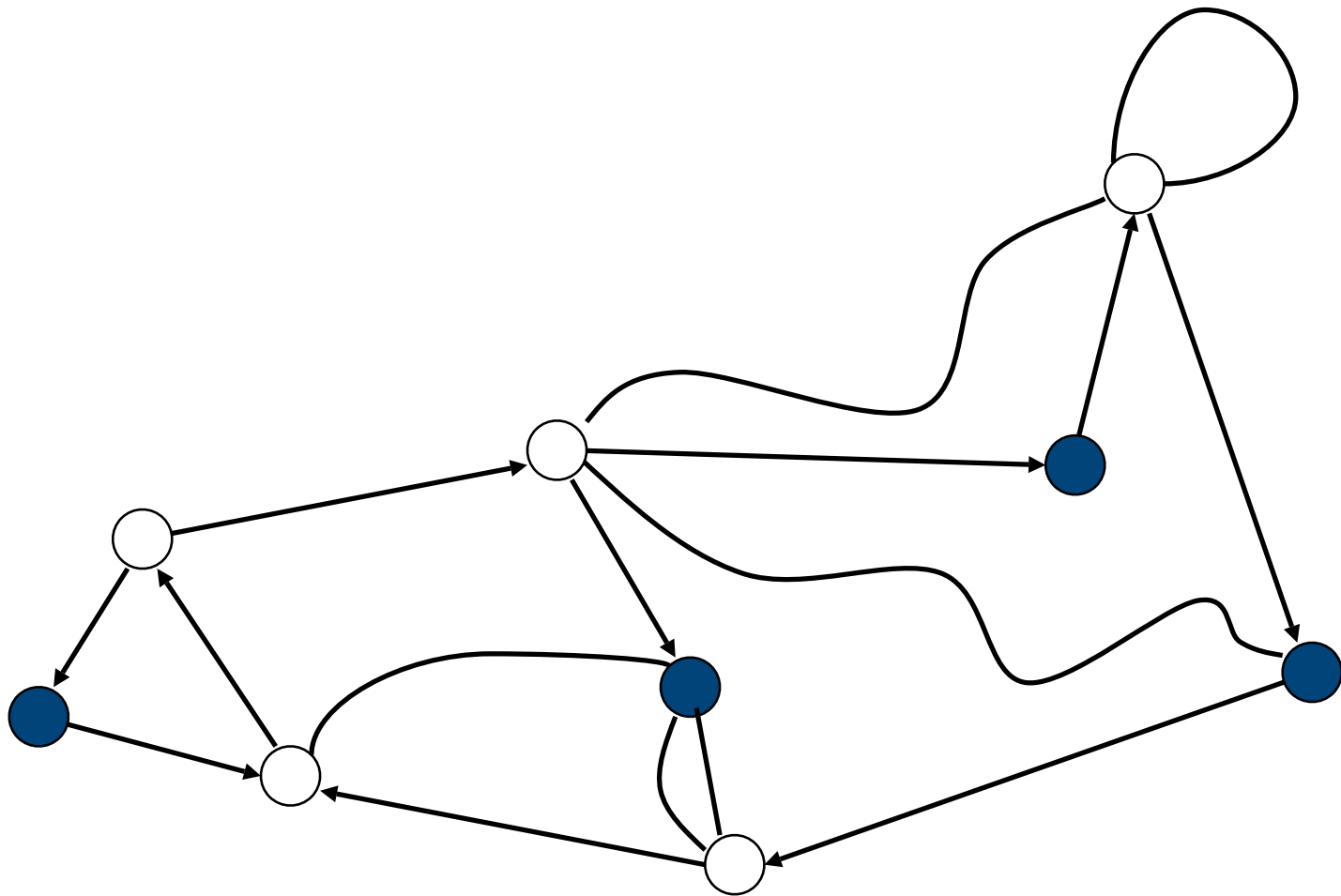
Independent and Absorbant

It may not exist!



Kernel

Independent and Absorbant





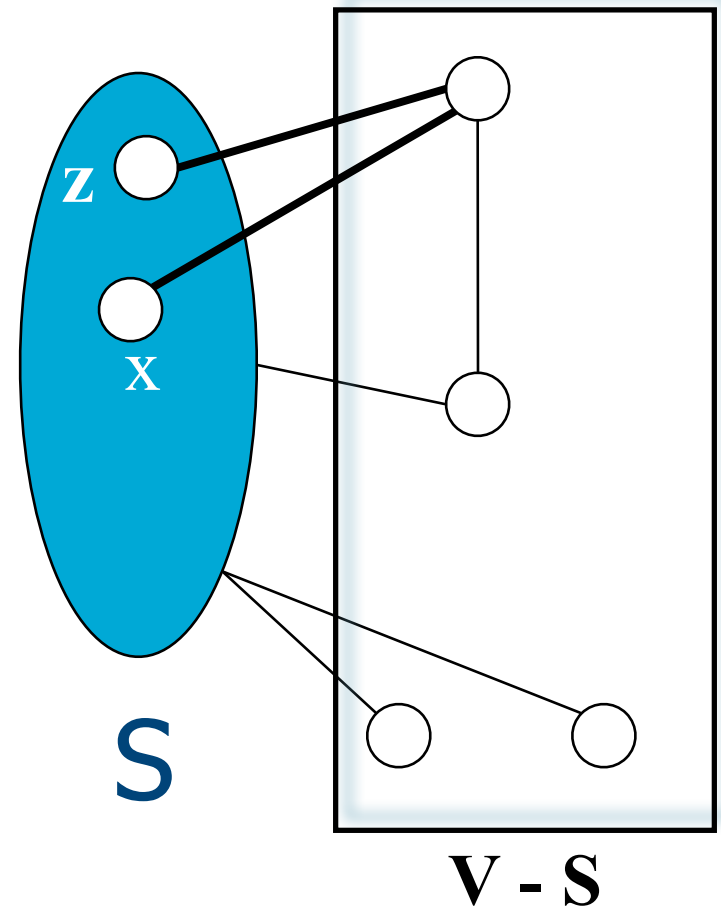
The Digraph Game

- Let $D = (V, A)$ be an acyclic digraph with a selected vertex x .
- Player *Blue* must select some outneighbor a of x , player *Red* must select some outneighbor b of a , and they continue alternating.
- To win means to reach any one of a set of terminal vertices for the first time.

Kernel S

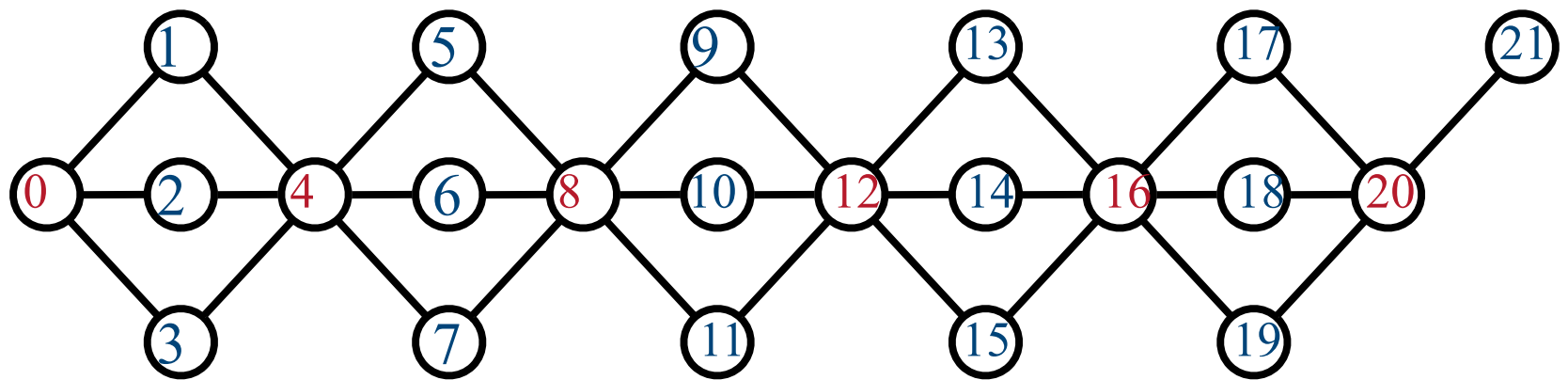
provides a winning strategy

- If Blue chooses x in S , either $O(x)$ is empty (Blue wins) or Red must choose a vertex in $V-S$.
- Blue can then choose a vertex z in S .
- Since terminal vertices have no outneighbors, they're in S .
- Since D is finite, the game must end.



Solution to the Subtraction Game

Some edges are not shown.



$$\text{Kernel} = \{0, 4, 8, 12, 16, 20\}$$



Subtraction Game in Disguise

The game of ChipsDown is played on a vertical strip of 22 squares where initially several chips are placed on square number 22 (the top).

- Two players alternately move any one of the chips either 1, 2, or 3 squares downward.
- No chip can be moved past square number 1.
- The last player to move wins.

If there is only one chip, who wins? How?
If there are $k > 1$ chips, who wins? How?



The Thirty-One Game

(Geoffrey Mott-Smith -1954)

From a deck of cards, take the Ace, 2, 3, 4, 5, and 6 of each suit. These 24 cards are laid out face up on a table.

The players alternate turning over cards and the sum of the turned over cards is computed as play progresses.

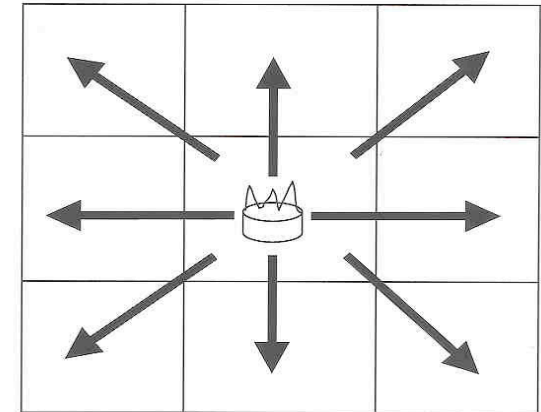
Each Ace counts as one. The player who first makes the sum go above 31 loses.

Who wins? How?

The Angel Problem

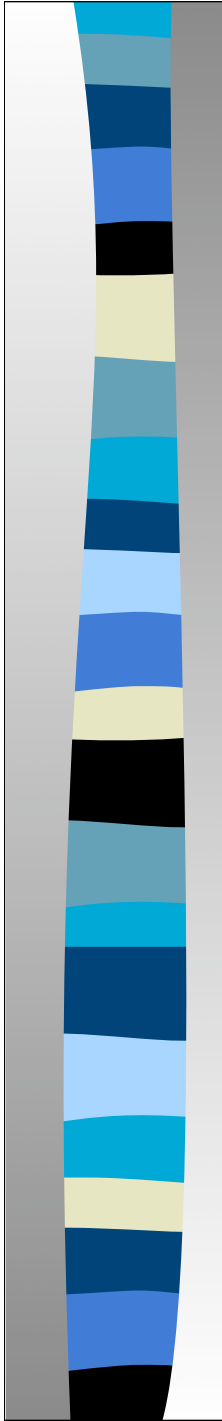
by John Conway

- Played on an infinite chessboard.
- Angel of power 100 can fly to any available square that could be reached within 100 King moves.
- Devil eats squares and has unlimited range.



CHESS KING

Can the Devil always trap the Angel?



Do You Remember?

Topics from Graph Theory

- Euler Trails – Drawing puzzles
- Vertex Coloring – Conflicts & Scheduling
- Directed Graph – Models games
- Neighbor Set, Independent Set, Absorbant Set, Kernel – Solves games
- Algorithms – every case
- **The Angel Problem** - UNSOLVED