Canonical genus and the Whitehead doubles of pretzel knots

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Definitions

Definition

$K \subset S^3$ is a knot if it is a smooth $1-$manifold embedded in S^3.

Definition

A Seifert surface of a knot K is a compact, connected, orientable surface $S \subset S^3$ such that $\delta S = K$.

Definition

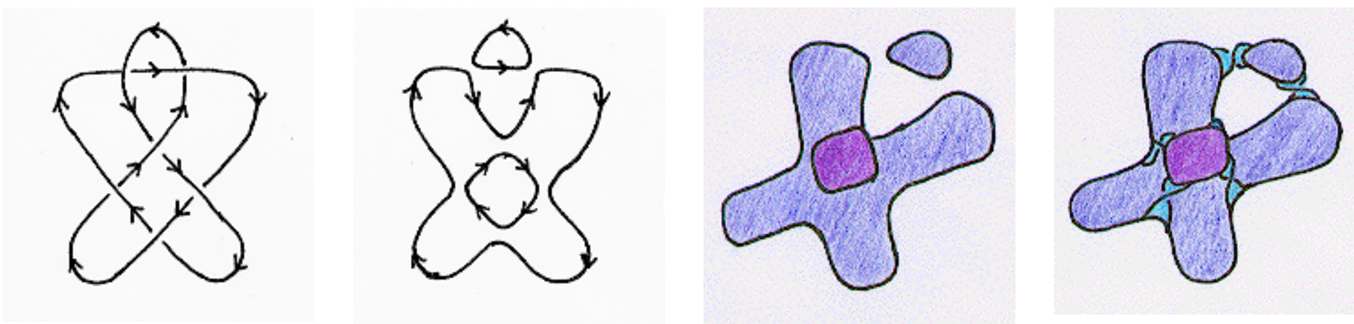
The genus of a knot K, denoted $g(K)$ is the minimal genus of all Seifert surfaces with boundary K.

Definitions, cont.

Definition

A Seifert surface is canonical if it is obtained from a diagram of K by applying Seifert's algorithm.

Example



Example from http://www.sgwater.org/math/knots/, referenced 12/27

Definitions, cont.

Definition

The canonical genus for K, $g_C(K)$ is the minimal genus among all canonical Seifert surfaces of K.

Fact

Both genus and canonical genus are hard to calculate.

A Question

How are genus and canonical genus related?

Fact

$g(K) \leq g_C(K)$

Theorem (Gabai, 1986, and others)

Seifert's algorithm gives a minimal genus Seifert surface when the algorithm is applied to an alternating projection of an alternating knot K.

Corollary

For alternating knots, $g_C(K) = g(K)$.

Ways to Compute Genus

- (Gabai, 1984) With a minimal genus surface candidate, sutured manifold theory can verify its minimality.
- (Morton, 1986) The $z-$degree of the HOMFLY polynomial, $P_K(v,z)$ of a knot K is at most twice the canonical genus, i.e., $\max \deg_z P_K(v,z) \leq 2g_C(K)$.

  A canonical surface whose genus is half of the $z-$degree must therefore have genus equal to canonical genus. This condition cannot always be met, giving examples where the inequality is strict.

Both methods succeed for alternating knots. The second method has computed genera for knots up through 12 crossings (Stoimenow, 2002).

Known Results

- H. R. Morton (1986) – A twisted Whitehead double of the trefoil knot has the canonical genus at least 3, while its genus is 1
- A. Kawauchi (1994) – There exists a knot K so that $g_C(K) - g(K) = 2n$ for any $n \in \mathbb{N}$.
- J. J. Tripp (2002) – The canonical genus of a twisted Whitehead double of a $(2, n)-$torus knot is n, i.e. it is equal to the crossing number of the original knot.

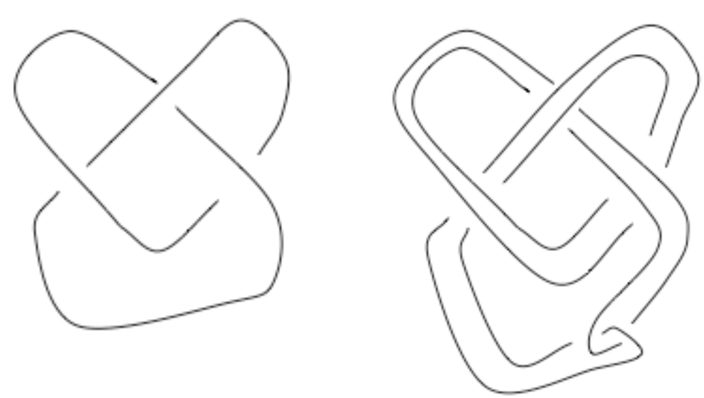
More Definitions

Definition (Whitehead double)

Let J be the Whitehead clasp in an unknotted solid torus, $S^1 \times B^2$. Let $h : S^1 \times B^2 \to S^3$ be an embedding taking $S^1 \times \{0\}$ to a knot K. The knot $h(J)$ is the Whitehead double of K with n twists, where n is the linking number of the longitude of the torus with K.

Fact

If $n = w(P)$ for the projection P of K, there is a standard projection of the Whitehead double which appears untwisted. This is the Whitehead double that we study.



The Conjecture

Conjecture (Tripp, 2002)

The crossing number of a knot is equal to the canonical genus of its Whitehead double.

Known Results

- Tripp (2002) – true for $(2, n)-$torus knots
- Nakamura (2004) – true for $2-$bridge knots

Morton's Inequality

Theorem (Morton, 1986)

The $z-$degree of the HOMFLY polynomial $P_K(v,z)$ of a knot K is at most twice the canonical genus, i.e.

$$max\ deg_z P_K(v,z) \leq 2g_C(K)$$

Tripp and Nakamura show that the $z-$degree of the HOMFLY polynomial of the double is $2n = 2c(K)$ for the appropriate knot.

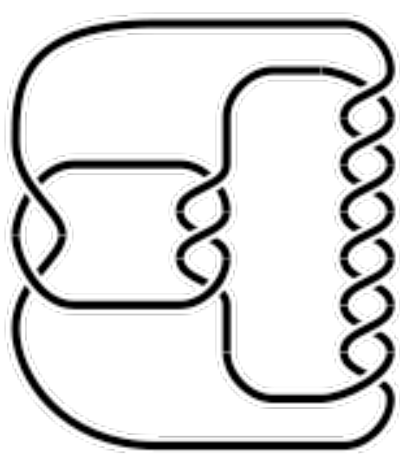
Definition

Definition

$P(k_1, k_2, \ldots k_n)$ is a pretzel link if it is the sum of tangles $k_1, k_2, \ldots, k_n$ with $k_i \geq 1$. It is a knot iff either n is odd or n is even and exactly one k_i is even.

Example

The $(-2, 3, 7)$ pretzel knot.



Our Results

Theorem (B-J, 2006)

If K is a pretzel knot $P(k_1, \ldots, k_n)$ with $k_1, \ldots, k_n \geq 1$ then $g_C(W(K)) = k_1 + \cdots + k_n = c(K)$.

Proof is similar to the techniques of Tripp and Nakamura.

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