The Canonical Genus of Whitehead Doubles of Pretzel Knots

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Goal of Talk

We prove that for an alternating pretzel knot, K, the canonical genus of its Whitehead doubles $W(K)$ is equal to the crossing number $c(K)$ of K, verifying a conjecture of Tripp in the case of these knots.

Knot

Definition

A knot is a closed curve in space with no self-intersections.

Definition

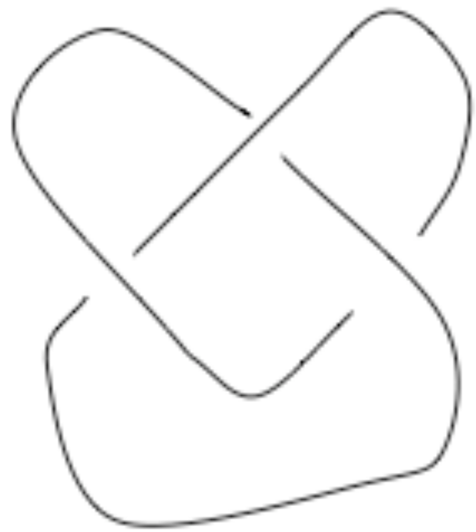
$K \subset S^3$ is a knot if it is a smooth $1-$manifold embedded in S^3.

Definition

A tame knot is a piecewise linear closed curve in $\mathbb{R}^3$.

Knots, cont.

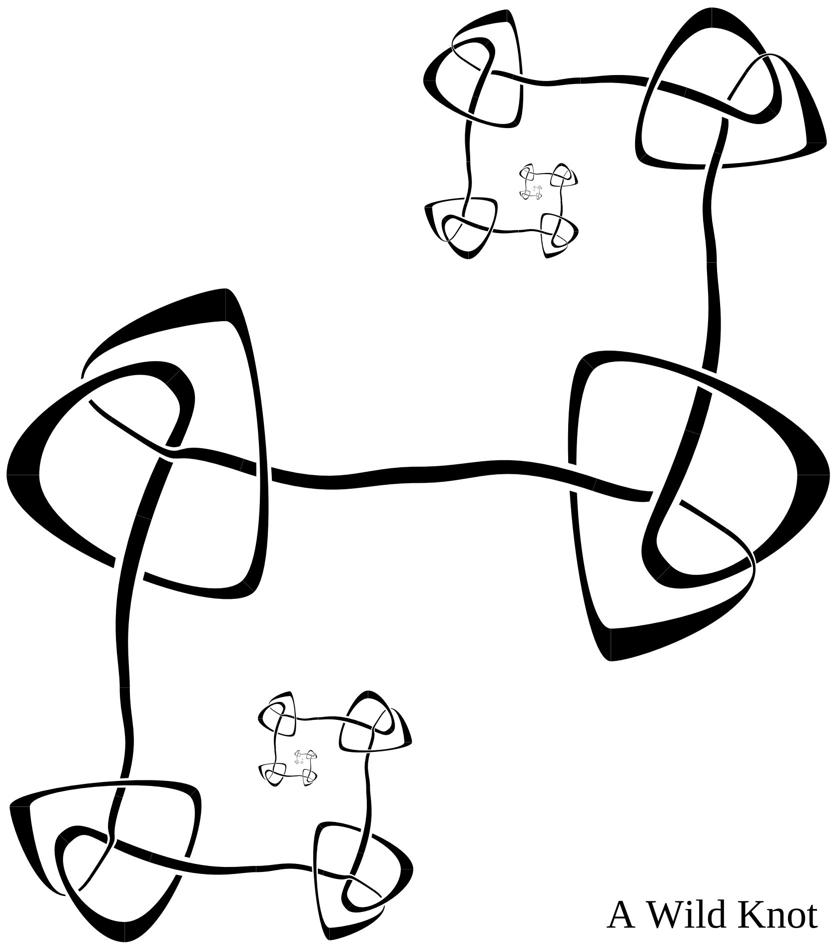
Include



as a knot.

Knots, cont.

Do not include wild knots:

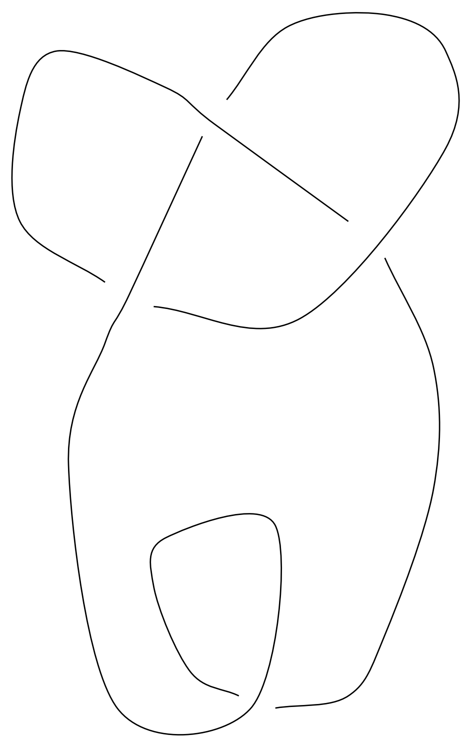


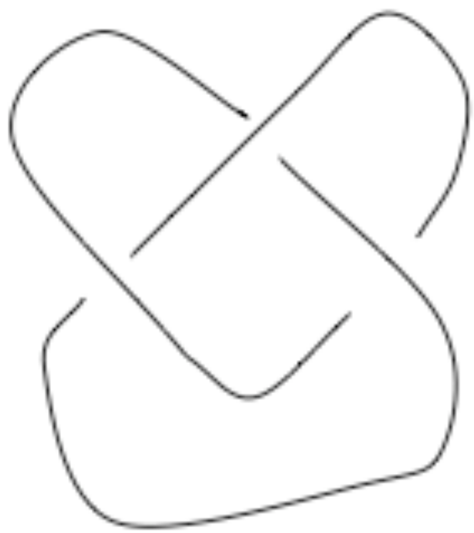
A Wild Knot

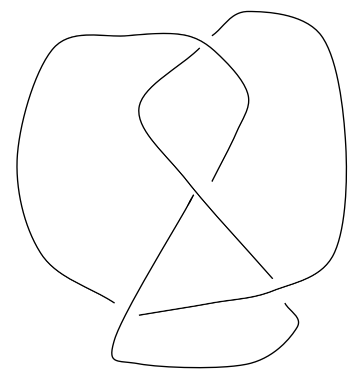
Crossing Number

Definition

The crossing number of a knot K, $c(K)$, is the smallest number of crossings in **any** projection of the knot.







$c(\textit{trefoil}) = 3$ $c(\textit{figure-8}) = 4$

Crossings	Number of Knots

Crossings	Number of Knots
0	1

Crossings	Number of Knots
0	1
3	1

Crossings	Number of Knots
0	1
3	1
4	1

Crossings	Number of Knots
0	1
3	1
4	1
5	2

Crossings	Number of Knots
0	1
3	1
4	1
5	2
6	3

Crossings	Number of Knots
0	1
3	1
4	1
5	2
6	3
7	7

Crossings	Number of Knots
0	1
3	1
4	1
5	2
6	3
7	7
8	21

Crossings	**Number of Knots**
0	1
3	1
4	1
5	2
6	3
7	7
8	21
9	49

Crossings	Number of Knots
0	1
3	1
4	1
5	2
6	3
7	7
8	21
9	49
10	165

Crossings	**Number of Knots**
0	1
3	1
4	1
5	2
6	3
7	7
8	21
9	49
10	165
11	552

Crossings	Number of Knots
0	1
3	1
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7	7
8	21
9	49
10	165
11	552
12	2176

Crossings	Number of Knots
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10	165
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12	2176
13	9988

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14	46,973

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11	552
12	2176
13	9988
14	46,973
15	253,293

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15	253,293
16	1,388,705

Crossings	Number of Knots
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10	165
11	552
12	2176
13	9988
14	46,973
15	253,293
16	1,388,705
17	open???

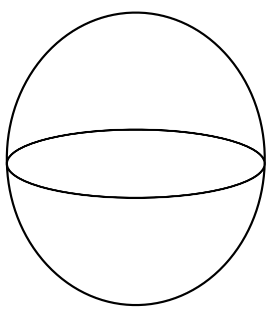
Genus

Definition

The genus of a surface is the largest number of nonintersecting simple closed curves that can be drawn on the surface without separating it.

Definition

The genus of an orientable surface is the number of holes in the surface.



Genus = 0

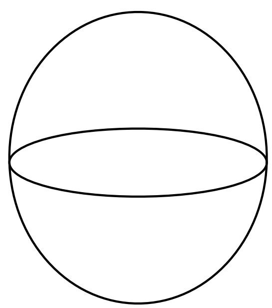
Genus

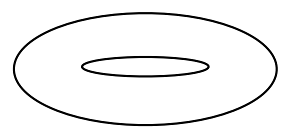
Definition

The genus of a surface is the largest number of nonintersecting simple closed curves that can be drawn on the surface without separating it.

Definition

The genus of an orientable surface is the number of holes in the surface.





Genus = 0 Genus =1

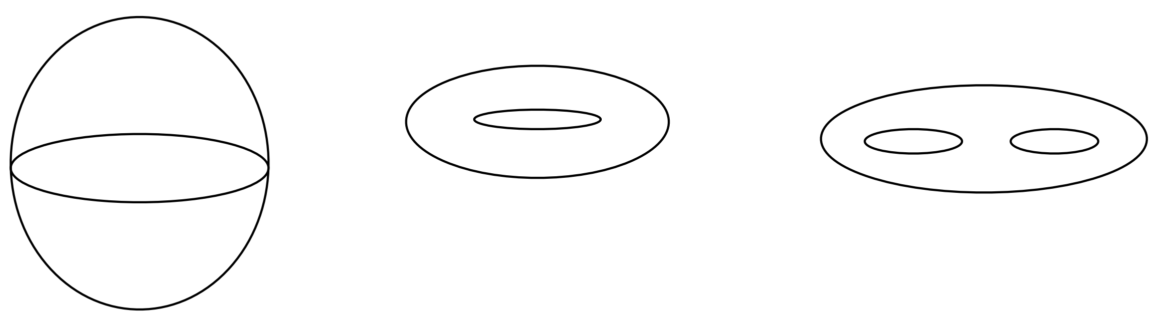
Genus

Definition

The genus of a surface is the largest number of nonintersecting simple closed curves that can be drawn on the surface without separating it.

Definition

The genus of an orientable surface is the number of holes in the surface.



Genus $= 0$ Genus $=1$ Genus$=2$

Definition

A Seifert surface of a knot K is a compact, connected, orientable surface $S \subset S^3$ such that the boundary of the surface is the knot (i.e., $\delta S = K$).

Definition

The genus of a knot K, denoted $g(K)$, is the minimal genus of all Seifert surfaces with boundary K.

Genus, cont.

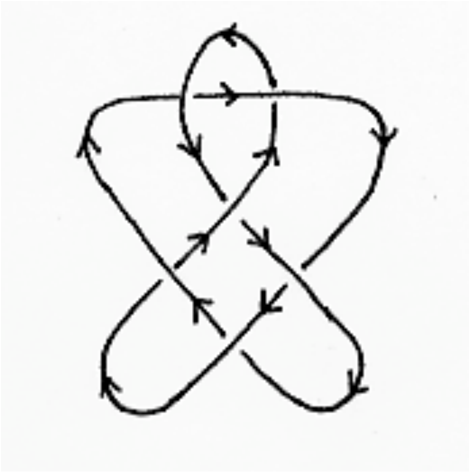
Definition

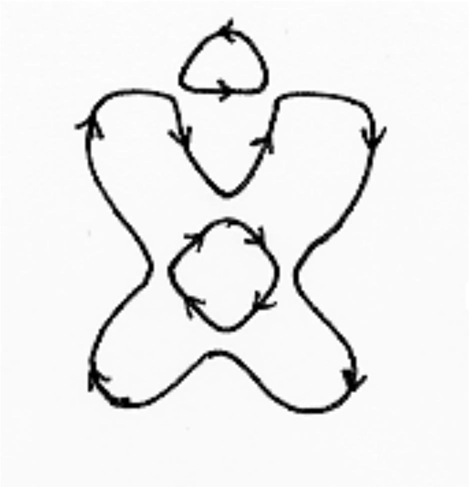
A Seifert surface is canonical if it is obtained from a diagram of K by applying Seifert's algorithm.

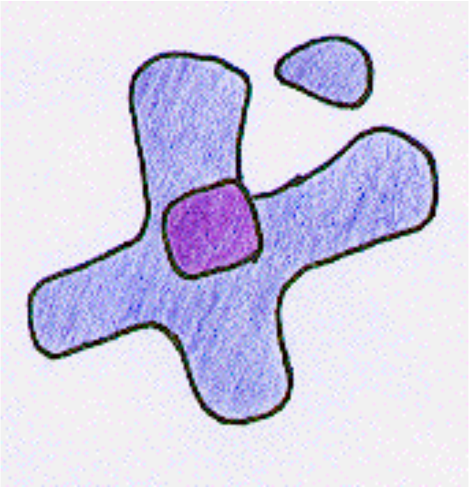
Definition

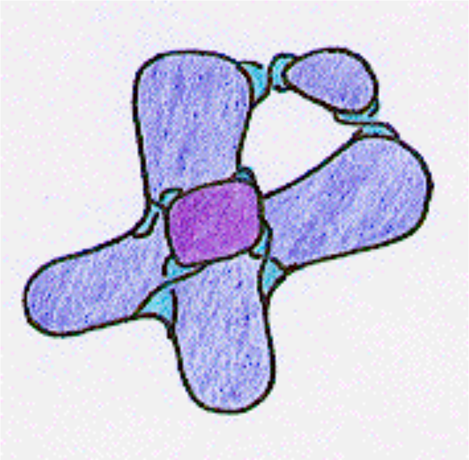
The canonical genus for K, $g_c(K)$, is the minimal genus among all canonical Seifert surfaces of K.

Example









Fact

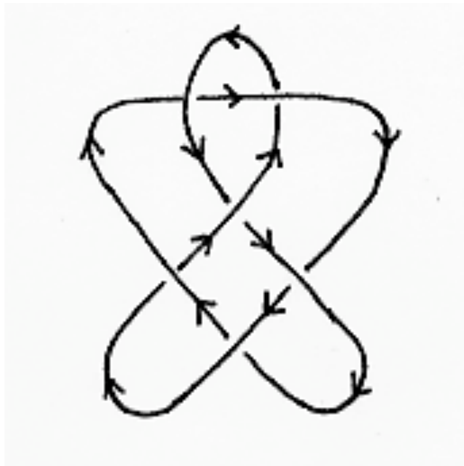
The genus of a surface is related to the Euler characteristic by

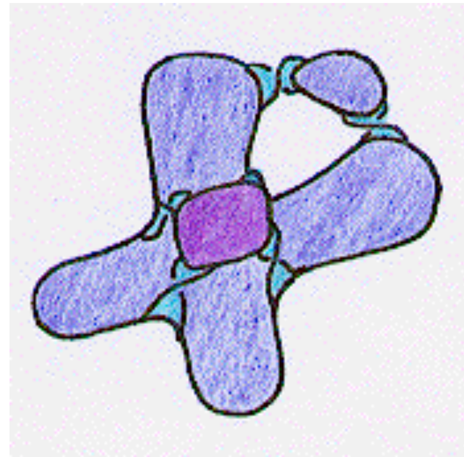
$$\chi = 2 - 2g.$$

Theorem

For a knot K with c crossings and s Seifert circles from Seifert's algorithm, the genus is

$$g = \frac{c - s + 1}{2}$$



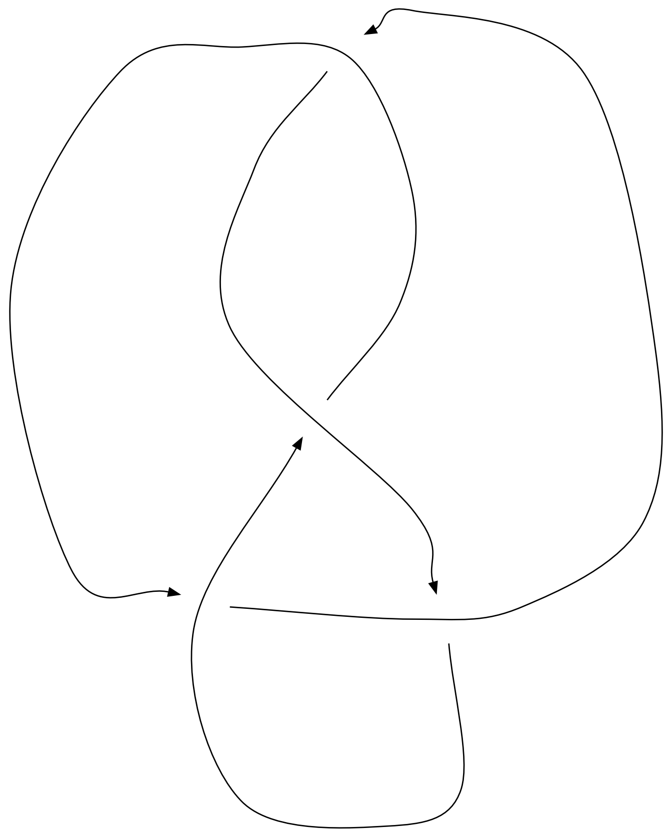


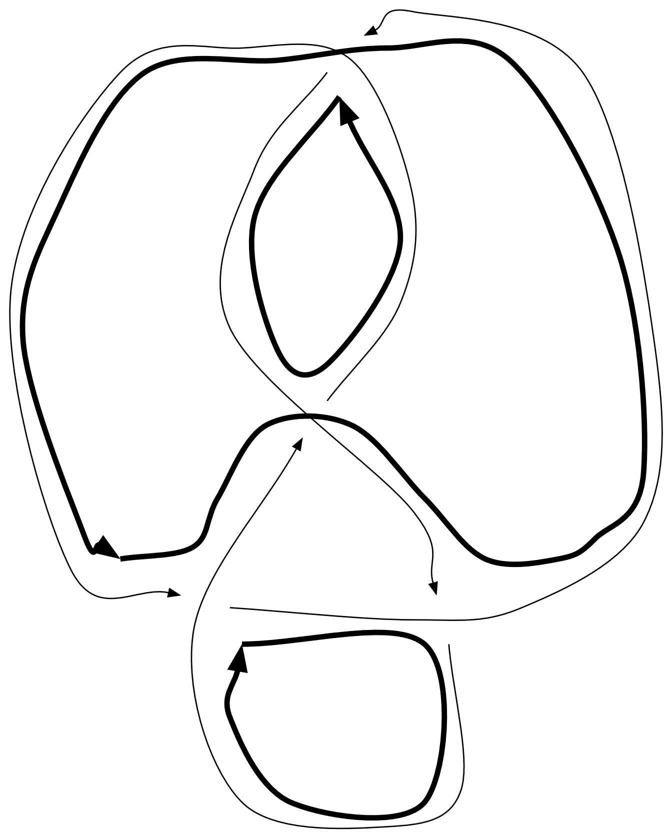
$$g = \frac{c - s + 1}{2}$$

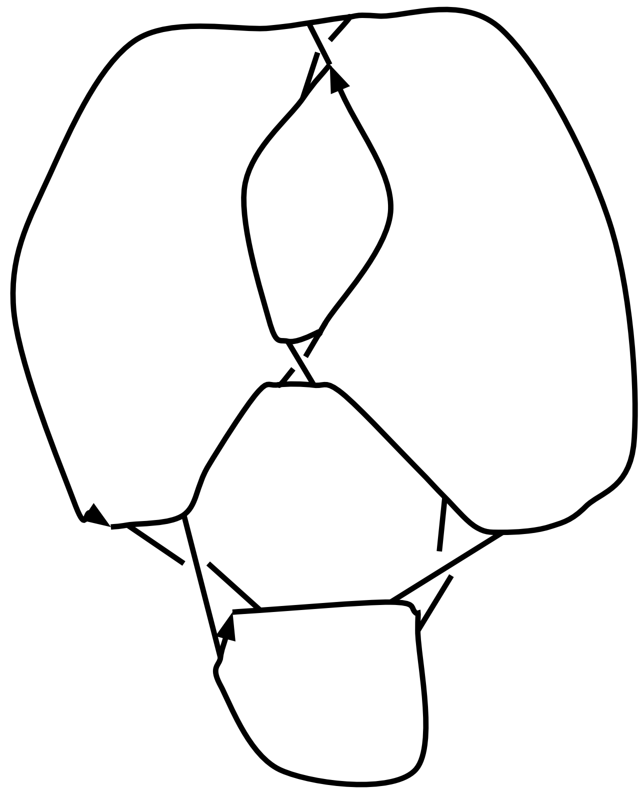
$$g = \frac{6 - 3 + 1}{2} = \frac{4}{2} = 2$$

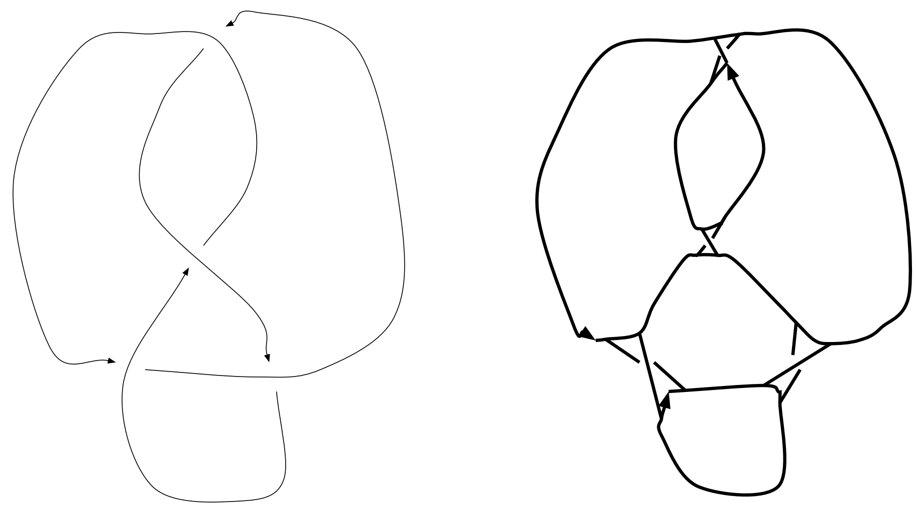
So $g_c(6_1) \leq 2$.

Example









$$g = \frac{c - s + 1}{2}$$

$$g = \frac{4 - 3 + 1}{2} = \frac{2}{2} = 1$$

Therefore $g_c(\text{figure-8}) \leq 1$.

Theorem
The unknot is the only knot which is genus 0.

Therefore $g_c(\text{figure-8}) = 1$.

Genera, cont.

Fact

Both genus and canonical genus are hard to calculate.

A Question

How are genus and canonical genus related?

Theorem

$g(K) \leq g_c(K)$

Theorem (Gabai, 1986, and others)

Seifert's algorithm gives a minimal genus Seifert surface when the algorithm is applied to an alternating projection of an alternating knot K.

Corollary

For alternating knots, $g_c(K) = g(K)$.

Ways to Compute Genus

- (Gabai, 1984) With a minimal genus surface candidate, sutured manifold theory can verify its minimality.
- (Morton, 1986) The $z-$degree of the HOMFLY polynomial, $P_K(v,z)$ of a knot K is at most twice the canonical genus, i.e., $\max \deg_z P_K(v,z) \leq 2g_c(K)$.

A canonical surface whose genus is half of the $z-$degree must therefore have genus equal to canonical genus. This condition cannot always be met, giving examples where the inequality is strict.
We have used the first technique to verify that we developed an infinite family of knots for which Morton's inequality is strict.

Both methods succeed for alternating knots. The second method has computed genera for knots up through 12 crossings (Stoimenow, 2002).

Known Results

- H. R. Morton (1986) – A twisted Whitehead double of the trefoil knot has canonical genus at least 3, while its genus is 1.

- A. Kawauchi (1994) – There exists a knot K so that $g_c(K) - g(K) = 2n$ for any $n \in \mathbb{N}$.

- J. J. Tripp (2002) – The canonical genus of a twisted Whitehead double of a $(2, n)-$torus knot is n, i.e., the canonical genus is equal to the crossing number of the original knot.

More Definitions

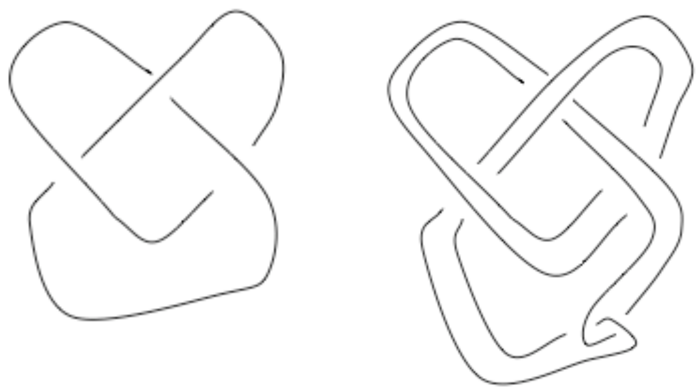
Definition (Whitehead double)

Let J be the Whitehead clasp in an unknotted solid torus, $S^1 \times B^2$. Let $h : S^1 \times B^2 \longrightarrow S^3$ be an embedding taking $S^1 \times \{0\}$ to a knot K. The knot $h(J)$ is the Whitehead double of K with n twists, where n is the linking number of the longitude of the torus with K.

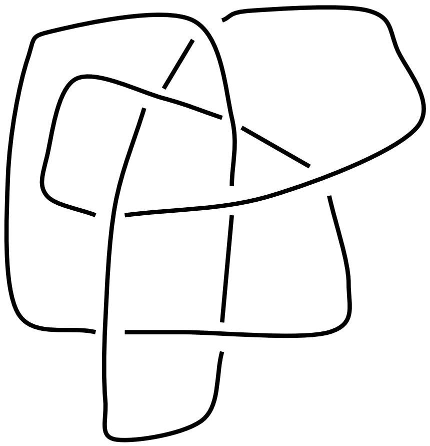
Fact

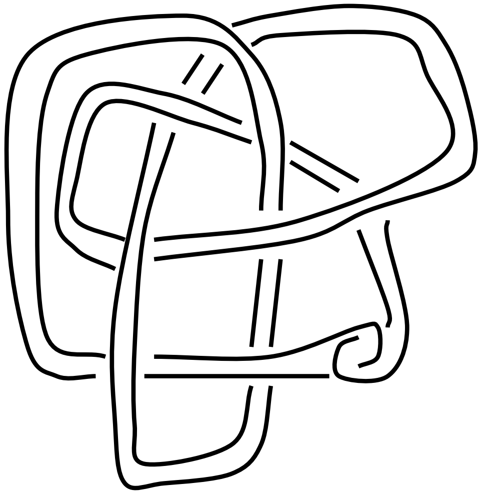
If $n = w(P)$ for the projection P of K, there is a standard projection of the Whitehead double which appears untwisted.

This is the Whitehead double that we study.



The 8_{19} Knot





The HOMFLY Polynomial

Definition

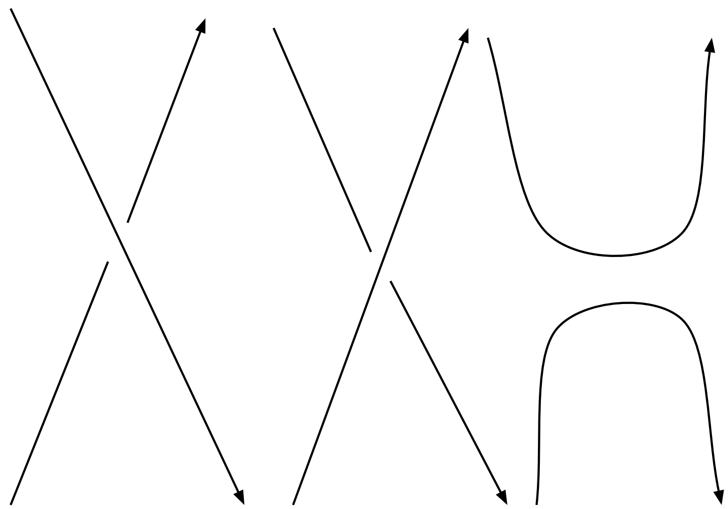
The HOMFLY polynomial (Freyd, Hoste, Lickorish, Millett, Ocneanu, Yetter, 1985) is a 2-variable Laurent polynomial defined to be the unique polynomial defined on link diagrams and invariant under the Reidemeister moves, satisfying

$$P_{\mathrm{unknot}}(v, z) = 1$$

and

$$v^{-1}P_{K_+} - vP_{K_-} = zP_{K_0}$$

where K_+, K_-, and K_0 are diagrams which all agree except at one crossing:



K_+ K_- K_0

The Conjecture

Conjecture (Tripp, 2002)

The crossing number of a knot is equal to the canonical genus of its Whitehead double.

Known Results

- Tripp (2002) – true for $(2, n)-$torus knots
- Nakamura (2004) – true for $2-$bridge knots

Their Methods

Theorem (Morton's Inequality, 1986)

The $z-$degree of the HOMFLY polynomial $P_K(v, z)$ of a knot K is at most twice the canonical genus, i.e.,

$$max\ deg_z P_K(v, z) \leq 2g_c(K)$$

Tripp and Nakamura show that the $z-$degree of the HOMFLY polynomial of the double is $2n = 2c(K)$ for the appropriate knot.

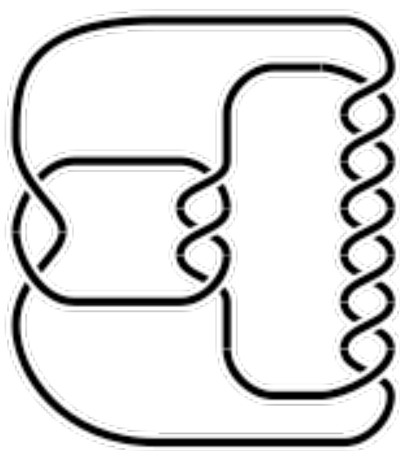
Definition

Definition

$P(k_1, k_2, \ldots k_n)$ is a pretzel link if it is the sum of tangles $k_1, k_2, \ldots, k_n$ with $k_i \geq 1$. It is a knot iff either n is odd or n is even and exactly one k_i is even.

Example

The $(-2, 3, 7)$ pretzel knot.

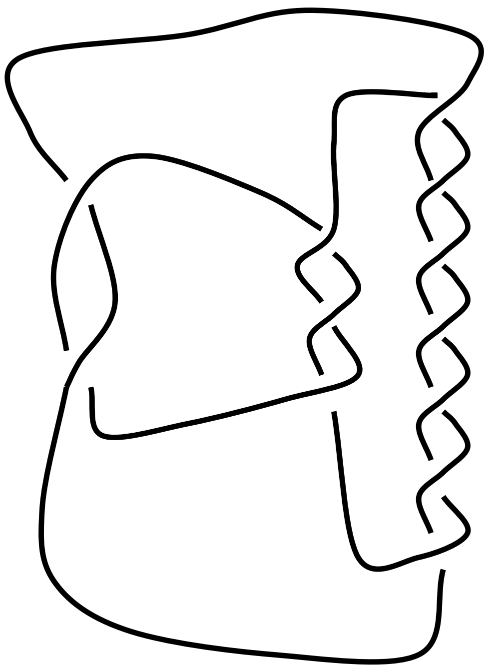


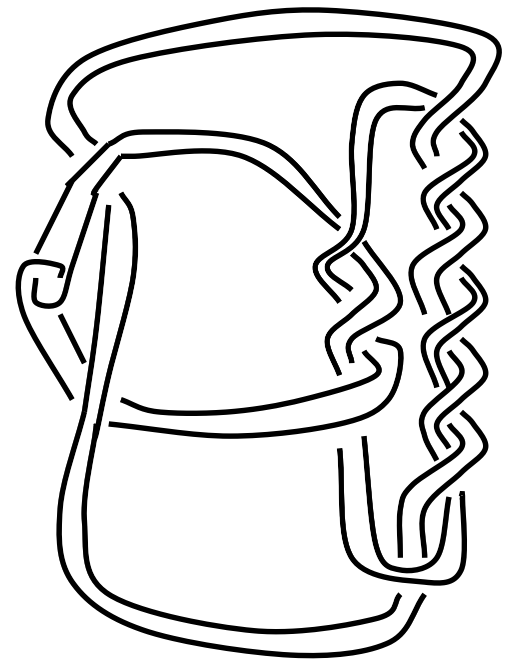
Our Result

Theorem (B-J, 2006)

If K is a pretzel knot $P(k_1, \ldots, k_n)$ with $k_1, \ldots, k_n \geq 1$ then $g_c(W(K)) = k_1 + \cdots + k_n = c(K)$.

Example - $P(2, 3, 7)$





$$g_c(W(K)) = 12 = c(K)$$

Proof

We show that if you begin with a knot diagram which satisfies

$$2c(K) = \max \deg_z P_{W(K)}(v, z)$$

you can make changes to the diagram which preserve this property.

If you begin with the trefoil knot, you can do this in such a way to get all pretzel knots.

$\square$

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- A. Stoimenow, *On the crossing number of positive knots and braids and braid index criteria of Jones and Morton-Williams-Franks*, Trans. Amer. Math. Soc. **354**, (2002), 3927–3954

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