Math 560
Fall 2005
Homework 8 - Partial Solutions
Assigned Monday, 24 October, 2005

1. (Section 2.4, #6, p. 97) Use the intermediate value theorem to prove that any polynomial of odd degree with real coefficients has at least one real root.

   Proof. We know about the end behavior of odd degree polynomials. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with n odd and $a_n \neq 0$. Then if $a_n > 0$ we have $\lim_{x\to\infty} p(x) = \infty$ and $\lim_{x\to-\infty} p(x) = -\infty$. If $a_n < 0$ then we have $\lim_{x\to\infty} p(x) = -\infty$ and $\lim_{x\to-\infty} p(x) = \infty$. Since polynomials are continuous, we can apply the intermediate value theorem to see that since there is a x_1 so that $p(x_1) > 0$ and there is a x_2 so that $p(x_2) < 0$ (or vice versa) with $x_1 < x_2$ there is an $x_3 \in (x_1, x_2)$ so that $p(x_3) = 0$. □

2. (Section 2.4, #10, p. 97) Give a mathematical argument to show that a heated wire in the shape of a circle must always have two diametrically opposite points with the same temperature.

   Proof. Let $f(x)$ be the temperature at point $x \in S^1$. Since f is defined on S^1, we have that $f(x) = f(2\pi)$. We want to see that $f(x) = f(x+\pi)$. Notice that f is continuous. Define $g(x) = f(x) - f(x+\pi)$. Notice that g is continuous since f is. It suffices to show that $g(x) = 0$ for some x. Assume $g(x) \neq 0$. Then $g(x) > 0$ or $g(x) < 0$ for all x. If $g(x) > 0$ for all x, that means that $f(x) > f(x+\pi)$ for all x. In particular this means that $f(0) > f(\pi) > f(2\pi)$ which is a contradiction since $f(0) = f(2\pi)$. Therefore there must be a place where $g(x) < 0$. If $g(x) < 0$ for all x then $f(0) < f(\pi) < f(2\pi)$ which is also a contradiction. Therefore there must be at least one place where $g(x) > 0$ and at least one place where $g(x') < 0$. Therefore by IVT, there exists a place x_0 where $g(x_0) = 0$, i.e. $f(x_0) = f(x_0 + \pi)$, as desired.

   □

3. (Section 2.4, #8, p. 97) Show that any function that is locally constant on an open connected set D is in fact constant on D.