Math 560
Fall 2005
Homework 8 Part 2, Partial Solutions
Assigned Monday, 24 October, 2005

1. (Problem #1, Section 2.5, p. 105-106) Let

$$f(x) = \begin{cases} 1 & x \neq 0 \\ 2 & x = 0 \end{cases}$$

$$g(x) = \begin{cases} 2 & x \neq 1, 2 \\ 3 & x = 1 \\ 4 & x = 2 \end{cases}$$

   (a) Verify that $\lim_{x\to 0} f(x) = 1$, $\lim_{x\to 1} g(x) = 2$, $\lim_{x\to 0} g(f(x)) = 3$, and $g(f(0)) = 4$.
   (b) Do the statements above still hold if

$$f(x) = \begin{cases} x+1 & x \neq 0 \\ 2 & x = 0 \end{cases}$$

2. Let f be defined on $\mathbb{R}^2$. If $\lim_{(x,y)\to(a,b)} f(x,y) = L$ and if the one-dimensional limits $\lim_{x\to a} f(x)$ and $\lim_{y\to b} f(y)$ both exist, prove that

$$\lim_{x\to a}\left(\lim_{y\to b} f(x,y)\right) = \lim_{y\to b}\left(\lim_{x\to a} f(x,y)\right) = L$$

3. Consider the function

$$f(x,y) = \begin{cases} \frac{\sin x - \sin y}{\tan x - \tan y} & \tan x \neq \tan y \\ \cos^3 x & \tan x = \tan y \end{cases}$$

   (a) Determine if $\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right)$ exists.
   Solution
   There are two cases:

$$\lim_{x\to 0}\left(\lim_{y\to 0} \frac{\sin x - \sin y}{\tan x - \tan y}\right)$$
$$= \lim_{x\to 0} \frac{\sin x}{\tan x}$$
$$= \lim_{x\to 0} \sin x \cdot \frac{\cos x}{\sin x}$$
$$= 1$$

$$\lim_{x\to 0}\left(\lim_{y\to 0} \cos^3 x\right) = \lim_{x\to 0} 1 = 1$$

(b) Determine if $\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right)$ exists.

Solution

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{\sin x-\sin y}{\tan x-\tan y}\right)$$
$$=\lim_{y\to 0}\frac{-\sin y}{-\tan y}$$
$$=\lim_{y\to 0}\sin y\cdot\frac{\cos y}{\sin y}$$
$$=1$$

(c) Determine if $\lim_{(x,y)\to(0,0)} f(x,y)$ exists.

(d) Explain the difference between the limits in (a) and (b) and the limit in (c).

The first two limits are along specific paths (namely horizontal and vertical paths) while the limit in (c) asks whether the limit exists regardless of the path.

4. Discuss the existence of $\lim_{|p|\to\infty}\frac{xy-z^2}{x^2+y^2+z^2}$

Solution

Consider spherical coordinates:

$$x=\rho\cos\theta\sin\varphi$$
$$y=\rho\sin\theta\sin\varphi$$
$$z=\rho\cos\varphi$$

$$\lim_{\rho\to\infty}\frac{\rho\cos\theta\sin\varphi-\rho^2\cos^2\varphi}{\rho^2}$$
$$=\lim_{\rho\to\infty}\frac{\cos\theta\sin\varphi-\rho\cos^2\varphi}{\rho}$$
$$=\lim_{\rho\to\infty}\frac{\cos\theta\sin\varphi}{\rho}-\cos^2\varphi$$
$$=-\cos^2\varphi$$

Therefore the limit dne.